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Using a covariant formalism, we construct a chiral kinetic theory Lorentz invariant to order $\mathcal{O}(\hbar)$ which includes collisions. We find a new contribution to the particle number current due to the side jumps required by the conservation of angular momentum during collisions. We also find a conserved symmetric stress-energy tensor as well as the H-function obeying Boltzmann's H-theorem. We demonstrate their use by finding a general equilibrium solution and the values of the anomalous transport coefficients characterizing chiral vortical effect.

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Introduction.—The role of chiral anomalies in the collective dynamics has attracted considerable attention recently. It has been known for some time [1, 2] that a chiral medium in a magnetic field or in rotation can respond by a current along the field or the rotation axis – the chiral magnetic or chiral vortical effects (CME and CVE). More recently, the effects of anomalies in medium have come up in several different experimental and theoretical contexts. Charge-dependent correlations which may be driven by the CME [3] have been observed in heavy-ion collisions. The CVE in hydrodynamics has been discovered using gauge/gravity duality [4, 5]. Later, both CME and CVE were shown to be universally required by the second law of thermodynamics [6]. The recent discovery of "3D graphene" [7, 8] and the possible observation [9] of the CME-induced negative magnetoresistance [10] have opened a new experimental frontier for investigating physical consequences of anomalies.

Despite the recent progress, the role of anomaly in kinetic theory has not been completely understood. Kinetic theory is essential for the understanding of nonequilibrium dynamics and is applicable when external fields are weak and collisions are rare, so that each particle moves along its classical trajectory most of the time. Recent literature focuses on the kinetic theory without collisions. It was shown [11] that anomaly is encoded in the momentum-space Berry curvature, and the action for such a motion has been derived microscopically [12, 13]. Although the action and the equations of motion are not manifestly relativistic, a hidden Lorentz invariance, involving nontrivial modifications of Lorentz transformations, has been found up to order $\mathcal{O}(\hbar)$ [14]. Such modifications lead to side jumps necessary to ensure angular momentum conservation in collisions. However, the corresponding modifications to the collision term have not been found so far.

In this Letter we supply this so far missing important piece of the theory. First, we introduce a simple covariant formalism allowing us to demonstrate Lorentz invariance in an elegant and straightforward manner. We then discover that the side jumps not only make the collision integral nonlocal, but also require nontrivial contributions to the particle number, energy-momentum and entropy currents. We prove the validity of Boltzmann's H-theorem, guaranteeing relaxation to equilibrium. We determine the values of the CVE transport coefficients from the kinetic theory. With the goal of understanding the CVE, we focus on the physics of collisions without external electromagnetic fields, which will be considered elsewhere.

Spin and relativity of particle worldline.—First of all we need to generalize the side jump found in Ref. [14] to *finite* Lorentz transformations. Let us consider the angular momentum tensor of a relativistic spinning particle,

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu}, \tag{1}$$

where $S^{\mu\nu}$ is the spin. In relativistic classical mechanics the separation between orbital motion and internal rotation as well as the definition of the center of mass are ambiguous. One can shift x^μ by Δ^μ and, simultaneously, $S^{\mu\nu}$ by $\Delta^\nu p^\mu - \Delta^\mu p^\nu$ without changing $J^{\mu\nu}$. To define unambiguously the particle position x^μ, one needs to impose a gauge-fixing condition on $S^{\mu\nu}$. For a *massless* particle ($p \cdot p = 0$), the only Lorentz-covariant condition $p_\mu S^{\mu\nu} = 0$ is not sufficient – leaving residual shifts Δ satisfying $\Delta \cdot p = 0$. To fix the gauge completely one chooses an arbitrary frame and uses its 4-velocity n to impose

$$n_\nu S^{\mu\nu} = 0, \tag{2}$$

i.e., one requires that $S^{\mu\nu}$ has only spatial components in the frame n. Together the two conditions $p_\mu S^{\mu\nu} = n_\nu S^{\mu\nu} = 0$ fix $S^{\mu\nu}$ in terms of n and p up to an overall factor

$$S_n^{\mu\nu} = \lambda \frac{\epsilon^{\mu\nu\alpha\beta} p_\alpha n_\beta}{p \cdot n}\,. \tag{3}$$

If $n^\mu = (1, \mathbf{0})$, then $S_n^{ij} = \lambda \epsilon^{ijk} p^k / |\boldsymbol{p}|$ and thus λ is the helicity of the particle.

The frame dependence of the spin tensor $S_n^{\mu\nu}$ in Eq. (3) implies that the particle position x also depends on the frame choice, so that the total angular momentum in Eq. (1) does not. This means that if one changes the frame from n to n' the position shifts, $x' = x + \Delta_{nn'}$, so that

$$S^{\mu\nu}_{n'} - S^{\mu\nu}_{n} = p^\mu \Delta^\nu_{nn'} - p^\nu \Delta^\mu_{nn'} \,. \quad (4)$$

Dotting this equation with n_ν and choosing the point on the shifted world line where the shift is spatial in frame n, $\Delta_{nn'} \cdot n = 0$, we find

$$\Delta^\mu_{nn'} = -\frac{S^{\mu\nu}_{n'} n_\nu}{p\cdot n} = \lambda\, \frac{\epsilon^{\mu\alpha\beta\gamma} p_\alpha n_\beta n'_\gamma}{(p\cdot n)(p\cdot n')} \,. \quad (5)$$

This is the finite generalization of the infinitesimal side jump found in Ref. [14]. Finite side jumps have been also recently considered in Refs. [15, 16].

Collisionless current.—We now consider kinetic theory, where the system is characterized by the phase space particle density f. As the particle positions depend on the frame, so will f. Let us first ignore collisions, in which case f is constant along the world lines. Assuming f and f' in two frames n and n' are related by $f'(x') = f(x)$, we find to linear order in $\hbar$, with $\lambda \sim \mathcal{O}(\hbar)$,

$$f'(x) - f(x) = -\Delta \cdot \partial f \qquad \text{(collisionless)}, \quad (6)$$

where $\Delta \equiv \Delta_{nn'}$.

The kinetic equation expresses conservation of the phase-space current j^μ, $\partial_\mu j^\mu = 0$, in the collisionless case. According to Eq. (6), the naive phase-space current $p^\mu f$ is not a Lorentz vector, since f is not a scalar field (its value at a given point depends on the frame). One part of the solution was found in Ref. [14]: the covariant current must include a magnetization contribution which, in the classical picture, is caused by the intrinsic rotation of the particles. In our covariant notations

$$j^\mu = p^\mu f + S^{\mu\nu}\partial_\nu f \qquad \text{(collisionless)}, \quad (7)$$

where $S^{\mu\nu} = S^{\mu\nu}_n$. Both f and $S^{\mu\nu}$ transform nontrivially under the frame change $n \to n'$ according to Eqs. (4) and (6) but, after cancellations,

$$j'^\mu - j^\mu = -\Delta^\mu (p\cdot\partial f) \qquad \text{(collisionless)}. \quad (8)$$

Thus the current in Eq. (7) is frame independent in *collisionless* kinetic theory where $p\cdot\partial f = 0$.

Collisions will make the current in Eq. (7) frame dependent. To solve this problem, we have to step back and try to understand what contribution to the current we may have missed. Equation (8) hints that it is related to the side jump and is proportional to the collision rate.

Collisions and jump current.—Let us look at the collisions more closely and, for simplicity, consider elastic $2\to 2$ collisions. From the classical point of view, such collisions involve 2 incoming and 2 outgoing world lines. It is convenient to think of incoming particles as being annihilated and outgoing particles as being created in that process. For particles without spin, we can assume that all 4 annihilation/creation events happen at the same spacetime point x. The continuity of the particle current is obvious in this case.

However, for a spinning particle, this cannot remain true in all frames, because that would contradict conservation of angular momentum [14]. We assume here that for each given collision kinematics there is a special frame — the "no-jump frame" $\bar n$ — in which all four particle worldlines converge to one spacetime point as in the spinless case. The natural choice for this special frame is the center of mass frame: $\bar n = (p_A + p_B)/\sqrt{s}$, where p_A and p_B are the momenta of the incoming particles. To ensure continuity of the current in a given (lab) frame $n \neq \bar n$ we must include a "jump current" associated with the spacelike motion of each participant particle between the common collision spacetime point x and the particle's annihilation/creation point in the lab frame, $x + \bar\Delta$, where from Eq. (5)

$$\bar\Delta^\mu \equiv \Delta^\mu_{\bar n n} = \lambda\, \frac{\epsilon^{\mu\alpha\beta\gamma} p_\alpha \bar n_\beta n_\gamma}{(p\cdot n)(p\cdot \bar n)} \,. \quad (9)$$

This tunneling-like motion of the particle during the collision would be from $x + \bar\Delta$ to x if the particle is incoming, or the reverse if it is outgoing. Weighing by the probability of the collision with each given kinematics, we are led to consider the current

$$j^\mu = p^\mu f + S^{\mu\nu}\partial_\nu f \; + \int_{BCD} C_{ABCD}\, \bar\Delta^\mu, \quad (10)$$

where we introduced short-hand notations for the usual Lorentz invariant integration over the phase space of the particles B, C, and D:

$$\int \frac{d^4 p_B}{(2\pi)^3} 2\delta(p_B\cdot p_B)\theta(n\cdot p_B) \equiv \int_{p_B} \equiv \int_B, \quad (11)$$

etc. and for the collision kernel

$$C_{ABCD} \equiv W_{CD\to AB} - W_{AB\to CD} \,, \quad (12)$$

where W is the rate of collisions with given momenta $p_A \equiv p$, p_B, p_C and p_D. The signs of the two terms reflect the directions of the jump depending on whether A is incoming or outgoing.

Let us now check Lorentz covariance of the current j^μ in Eq. (10) by considering a different frame n', as we did before in Eq. (8). Comparing 4-vectors j'^μ and j^μ we find this time

$$j'^\mu - j^\mu = p^\mu (f' - f + \Delta\cdot\partial f) - \Delta^\mu (p\cdot\partial f) + \int_{BCD} C_{ABCD}\, (\Delta^\mu_{\bar n n'} - \Delta^\mu_{\bar n n}). \quad (13)$$

The last term can be transformed using Eqs. (5) and (4)

$$\Delta^\mu_{\bar{n}n'} - \Delta^\mu_{\bar{n}n} = -\frac{(S'-S)^{\mu\nu}\,\bar{n}_\nu}{p\cdot\bar{n}} = \Delta^\mu_{nn'} - p^\mu\,\frac{\Delta_{nn'}\cdot\bar{n}}{p\cdot\bar{n}}\,. \tag{14}$$

The meaning of Eq. (14) is straightforward: the jump from $\bar{n}$ to n' equals the jump from $\bar{n}$ to n plus the jump from n to n' (Δ) up to a shift along the world line [the last term in Eq. (14)].

Substituting into Eq. (13) we observe that Δ^μ is independent of the integration variables p_B, etc., and thus can be taken outside of the integration. The remaining integral coincides with the collision rate

$$\mathcal{C}(x;p\equiv p_A) = \int\limits_{BCD} C_{ABCD}\,, \tag{15}$$

and, since by kinetic equation $p\cdot\partial f = \mathcal{C}[f] + \mathcal{O}(\hbar)$, this term cancels the $\Delta^\mu(p\cdot\partial f)$ term in Eq. (13) to order $\hbar$.

To cancel the last term in Eq. (14) substituted into Eq. (13) the distribution function must transform under the Lorentz transformation (in addition to the shift of the argument by Δ in Eq. (6)) as

$$f' - f = -\Delta\cdot\partial f \;+ \int\limits_{BCD} C_{ABCD}\,\frac{\Delta\cdot\bar{n}}{p\cdot\bar{n}}. \tag{16}$$

The additional term in Eq. (16) compared to Eq. (6) accounts for the colliding particles undergoing the side jumps. Thus we verified that the phase-space current j^μ in Eq. (10) is Lorentz covariant provided f transforms as Eq. (16) and C_{ABCD} is Lorentz invariant.

Collision kernel.—Using Eq. (10) and $n\cdot\bar{\Delta}=0$ we see that $f = n\cdot j/n\cdot p$, i.e., naturally, the time component of the current, j^0, divided by the particle energy in the frame n. Since the collision probability $W_{AB\to CD}$ must be a Lorentz scalar, i.e., independent of n, the frame-dependent f cannot directly determine $W_{AB\to CD}$ as in $|M|^2 f_A f_B(1-f_C)(1-f_D)$. Instead, we must use the distribution function in a frame, associated with the collision itself. The most natural choice is the "no-jump" frame $\bar{n}$

$$\bar{f} = \frac{\bar{n}\cdot j}{\bar{n}\cdot p}\,. \tag{17}$$

Now, with the n-independent distribution function in Eq. (17) we can write [17]

$$W_{AB\to CD}[\bar{f}] = \frac{1}{2!}|M(s,t)|^2(2\pi)^4\delta^4(p_A+p_B-p_C-p_D) \times \bar{f}_A\bar{f}_B(1-\bar{f}_C)(1-\bar{f}_D), \tag{18}$$

where factor $1/2!$ accounts for the indistinguishability of the outgoing particles. Using the Lorentz covariant j^μ and $\mathcal{C}$, we can write a Lorentz invariant chiral kinetic theory with collisions

$$\partial\cdot j = \mathcal{C}[\bar{f}], \tag{19}$$

where j^μ is given by Eq. (10) and $\mathcal{C}$ is given by Eqs. (15), (12), (18) with $\bar{f}$ from Eq. (17).

Using Eqs. (10), (15) and the transformation of f in Eq. (16) we can also rewrite Eq. (19) as

$$p\cdot\partial f = \int\limits_{BCD} C_{ABCD}[f] \times\left(1 - \int\limits_{B'C'D'}\frac{\partial}{\partial f}C_{AB'C'D'}\frac{\bar{\Delta}\cdot\bar{n}'}{p\cdot\bar{n}'}\right), \tag{20}$$

where $\bar{n}'$ is the no-jump frame of the collision $AB'\leftrightarrow C'D'$. In the form of Eq. (20) Lorentz invariance is not manifest as in Eq. (19), but the collision kernel is expressed solely in terms of the distribution function f in the lab frame. Equations. (19) and (20) are equivalent to linear order in $\hbar$.

Conserved currents.—Since the underlying quantum theory of Weyl fermions is invariant under CPT, we must take into account antiparticles, which also participate in collisions. These can be easily incorporated by considering the particle charge $q=\pm1$ as an additional discrete index of the distribution function $f(x,p,q)$, indices A, B, etc. as composite indices $A=(p_A,q_A)$, etc. and accompanying integration over p by summation over q. CP invariance implies $\lambda = q|\lambda|$. The net current of q is given by

$$J^\mu_q = \sum_q\int_p q j^\mu, \tag{21}$$

and its conservation, $\partial_\mu J^\mu_q = 0$, follows from Eq. (19) and the charge conservation in a collision: $\sum_q\int_p q\mathcal{C}=0$.

Similarly, one can show that the following covariant symmetric (and traceless) tensor

$$T^{\mu\nu} = \sum_q\int_p\frac{1}{2}(\,p^\mu j^\nu + p^\nu j^\mu\,) \tag{22}$$

is conserved $\partial_\nu T^{\mu\nu}=0$ due to the energy-momentum and angular momentum conservation in the collisions.

Entropy current and H-theorem.—An important property of kinetic theory is the existence of the entropy—a functional of f which does not decrease with time. This is known as the H-theorem, which guarantees that the system relaxes to equilibrium. To prove the H-theorem we need to find the corresponding covariant current H^μ whose divergence is non-negative.

First let us generalize current j^μ to a current describing advection of a generic, for now, quantity $\mathcal{H}$ which is a function of the distribution function f. Following the same steps as in Eqs. (10)–(16) we find that the following current

$$\mathcal{H}^\mu = p^\mu\mathcal{H} + S^{\mu\nu}\partial_\nu\mathcal{H} \;+ \int\limits_{BCD} C_{ABCD}\,\bar{\Delta}^\mu\,\frac{\partial\mathcal{H}}{\partial f} \tag{23}$$

does not depend on the choice of the frame n to linear order in $\hbar$. With $\mathcal{H} \to f$, Eq. (23) becomes Eq. (10) and the meaning of the last term is again the contribution of the particles undergoing side jumps.

Using Eq. (20) one can also show that

$$\partial_\mu \mathcal{H}^\mu = \int\limits_{BCD} C_{ABCD}[\bar{f}] \, \frac{\partial \mathcal{H}}{\partial \bar{f}} \,. \tag{24}$$

Furthermore, using the $AB \leftrightarrow CD$ symmetry of the amplitude $|M|$ in Eq. (18) we can write Eq. (12) as

$$C_{ABCD} = W_{AB\to CD}\,(r-1), \tag{25}$$

where

$$\frac{W_{CD\to AB}}{W_{AB\to CD}} = \frac{\bar{f}_C \bar{f}_D (1-\bar{f}_A)(1-\bar{f}_B)}{\bar{f}_A \bar{f}_B (1-\bar{f}_C)(1-\bar{f}_D)} \equiv r, \tag{26}$$

and express the current $H^\mu \equiv \int_p \mathcal{H}^\mu$ as

$$\partial_\mu H^\mu = \int\limits_{ABCD} W_{AB\to CD}\,(r-1)\,\frac{\partial \mathcal{H}}{\partial \bar{f}_A} \,. \tag{27}$$

Now, choosing $\mathcal{H}$ so that $\partial \mathcal{H}/\partial f = \ln[(1-f)/f]$, i.e.,

$$\mathcal{H} = f \ln \frac{1}{f} + (1-f) \ln \frac{1}{1-f} \,, \tag{28}$$

and (as in Ref. [18]) using the $A \leftrightarrow B$ and $C \leftrightarrow D$ symmetry of $|M|$ in Eq. (18) and the fact that $r \to 1/r$ under $AB \leftrightarrow CD$ according to Eq. (26) we can write for the divergence of the entropy current H^μ in Eq. (27)

$$\partial_\mu H^\mu = \frac{1}{4} \int\limits_{ABCD} W_{CD\to AB}\,(r-1)\,\ln r \geq 0. \tag{29}$$

The rate of entropy production $\partial \cdot H$ vanishes when $r = 1$.

Equilibrium.— Let us denote, for convenience,

$$g(f) \equiv \ln \frac{1-f}{f}, \tag{30}$$

i.e., $f(g) = 1/(\exp g + 1)$. In terms of $\bar{g} \equiv g(\bar{f})$, the ratio r in Eq. (26) is given by

$$r = \exp(\bar{g}_A + \bar{g}_B - \bar{g}_C - \bar{g}_D). \tag{31}$$

The collision kernel in Eq. (25) vanishes if $r = 1$ (detailed balance), which happens if $\bar{g}$ is a linear combination of quantities conserved in the collision (energy, momentum, angular momentum and charge), i.e.,

$$g(\bar{f}_{\rm eq}) = p \cdot \bar{U} + \frac{1}{2} \bar{S}^{\alpha\beta} \bar{\Omega}_{\alpha\beta} - q\bar{Y}, \tag{32}$$

where $\bar{S} = S_{\bar{n}}$ is the spin tensor in the no-jump frame (orbital momentum is zero), q is the charge and $\bar{U}$, $\bar{Y}$ and $\bar{\Omega}_{\alpha\beta} = -\bar{\Omega}_{\beta\alpha}$ are coefficients (possibly x-dependent).

The distribution function $f_{\rm eq}$ in another frame n unrelated to the collision kinematics can be obtained by transformation (16), according to which $g = \bar{g} - (\bar{\Delta}\cdot\partial)\bar{g}$ (since $C_{ABCD}[\bar{f}_{\rm eq}] = 0$). Also expressing $\bar{S}$ in Eq. (32) in terms of S using Eq. (4) we can then write for $f_{\rm eq}$:

$$g(f_{\rm eq}) = p \cdot U + \frac{1}{2} S^{\mu\nu} \Omega_{\mu\nu} - qY, \tag{33}$$

where $U_\alpha = \bar{U}_\alpha + (\bar{\Omega}_{\alpha\beta} - \partial_\alpha \bar{U}_\beta))\bar{\Delta}^\alpha$, $\Omega = \bar{\Omega}$ and $Y = \bar{Y} - (\bar{\Delta}\cdot\partial)\bar{Y}$. The dependence on the collision kinematics via vector $\bar{n}$ (in $\bar{\Delta}$ according to Eq. (9)) drops out, as it must, if $\partial_\alpha \bar{U}_\beta = \bar{\Omega}_{\alpha\beta}$ and $\bar{Y} = \text{const}$, which also means $U = \bar{U}$ and $Y = \bar{Y}$. The distribution in Eq. (33) describes a rotating (shear-free) fluid. It is easy to check that $f_{\rm eq}$ given by Eq. (33) solves kinetic equation (19). In the conventional notations $U = \beta u$, where $\beta = \sqrt{U \cdot U}$ and $Y = \beta\mu$.

Chiral vortical effect.—Now, for the rotating distribution in Eq. (33), we can calculate the number current J_q^μ in Eq. (21) using Eq. (10). It is convenient to express the distribution in the local comoving frame, i.e., choose $n = u$ (we have not relied on n being coordinate independent). To linear order in gradients we find

$$J_q^\mu = n_q u^\mu + \xi \omega^\mu, \tag{34}$$

with $\omega^\mu \equiv \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma} u_\alpha \partial_\beta u_\gamma$, $n_q \equiv \sum_q \int_p (p\cdot u) q f_0$ and

$$\xi \equiv \beta \sum_q \int_p (p \cdot u) q \lambda \left(-\frac{df_0}{dg}\right) = \frac{\mu^2}{4\pi^2} + \frac{T^2}{12} \,, \tag{35}$$

where $-df_0/dg = f_0(1-f_0)$, $T = 1/\beta$ and f_0 is the Fermi-Dirac distribution to zeroth order in gradients, i.e., $g(f_0) = \beta(p\cdot u - q\mu)$.

Similarly, for the stress energy tensor in Eq. (22) we find

$$T^{\mu\nu} = w u^\mu u^\nu - p g^{\mu\nu} + \xi_T (\omega^\mu u^\nu + \omega^\nu u^\mu), \tag{36}$$

where w and p are the usual expressions for the enthalpy and pressure of the Weyl gas and

$$\xi_T = \frac{2}{3} \beta \sum_q \int_p (p\cdot u)^2 \lambda \left(-\frac{df_0}{dg}\right) = \frac{\mu^3}{6\pi^2} + \frac{\mu T^2}{6} \,, \tag{37}$$

which is twice the result found in Ref. [1] due to the contribution of the spin coupling to vorticity. For the entropy current in Eq. (23) we find

$$H^\mu = s u^\mu + \xi_H \omega^\mu, \tag{38}$$

where $s = \beta(w - \mu n)$ and

$$\xi_H = \frac{3}{2} \beta\, \xi_T - \beta\mu\, \xi = \frac{\mu T}{6} \,. \tag{39}$$

One can check that these results agree with the general form found in Ref. [19] required by the second law of thermodynamics.

Since CVE currents are non-dissipative and occur in equilibrium the collision kernel does not appear in these results directly. However, it is essential for determining the equilibrium distribution function which must obey $\mathcal{C}[f_{\rm eq}] = 0$. The $\mathcal{H}$-theorem we proved ensures that such a state exists.

In summary, we formulated a chiral kinetic theory Lorentz invariant to $\mathcal{O}(\hbar)$ *with collisions*, where we discovered a new contribution to currents due to the colliding particles' side jumps and found the corresponding nontrivial Lorentz transformation of the distribution function. We proved that the theory obeys Boltzmann's H-theorem and determined the values of the transport coefficients of the chiral vortical effect. An important question is how the theory couples to external electromagnetic fields and it will be considered elsewhere. Another natural next step is to apply the theory to study the dissipative transport or non-hydrodynamic response (see, e.g, Refs.[20, 21]) where collision terms play more direct role. These and other open questions we leave to future research.

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