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Le Yan,[1] Marco Baity-Jesi,[2, 3, 4] Markus Müller,[5, 6] and Matthieu Wyart[1]

[1] Center for Soft Matter Research, Department of Physics, New York University,
4 Washington Place, New York, 10003, NY
[2] Departamento de Física Teórica I, Universidad Complutense, 28040 Madrid, Spain
[3] Dipartimento di Fisica, La Sapienza Università di Roma, 00185 Roma, Italy
[4] Instituto de Biocomputación y Física de Sistemas Complejos (BIFI), 50009 Zaragoza, Spain
[5] The Abdus Salam International Center for Theoretical Physics,
Strada Costiera 11, 34151 Trieste, Italy
[6] Department of Physics, University of Basel, Klingelbergstrasse 82, CH-4056 Basel, Switzerland
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Marginal stability is the notion that stability is achieved, but only barely so. This property constrains the ensemble of configurations explored at low temperature in a variety of systems, including spin, electron and structural glasses. A key feature of marginal states is a (saturated) pseudo-gap in the distribution of soft excitations. We examine how such pseudo-gaps appear dynamically by studying the Sherrington-Kirkpatrick (SK) spin glass. After revisiting and correcting the multi-spin-flip criterion for local stability, we show that stationarity along the hysteresis loop requires soft spins to be frustrated among each other, with a correlation diverging as $C(\lambda) \sim 1/\lambda$, where λ is the stability of the more stable spin. We explain how this arises spontaneously in a marginal system and develop an analogy between the spin dynamics in the SK model and random walks in two dimensions. We discuss analogous frustrations among soft excitations in short range glasses and how to detect them experimentally. We also show how these findings apply to hard sphere packings.

Introduction: In glassy materials with sufficiently long-range interactions, stability at low temperature imposes an upper bound on the density of soft excitations [1]. In electron glasses [2–7] stability towards hops of individual localized electrons requires that the density of states vanishes at the Fermi level, exhibiting a so-called Coulomb gap. Likewise, in mean-field spin glasses [8–15] stability towards flipping several "soft" spins implies that the distribution of local fields vanishes at least linearly. In hard sphere packings the distribution of forces between particles in contact must vanish analogously, preventing that collective motions of particles lead to denser packings [16–18]. Often, these stability bounds appear to be saturated [6, 9, 15, 17, 19, 20]. Such *marginal stability* can be proven for dynamical, out-of-equilibrium situations under slow driving at zero temperature [1] if the effective interactions do not decay with distance. This situation occurs in the Sherrington-Kirkpatrick (SK) model (see Eq. (1) below), but also in finite-dimensional hard sphere glasses, where elasticity induces non-decaying interactions [21]. Marginality is also found for the ground state or for slow thermal quenches by replica calculations for spin glass [10, 22] and hard sphere systems [23, 24], assuming infinite dimension.

The presence of pseudo-gaps strongly affects the physical properties of these glasses. The Coulomb gap alters transport properties in disordered insulators [2, 3], while its cousin in spin glasses suppresses the specific heat and susceptibility. It was recently proposed that the singular rheological properties of dense granular and suspension flows near jamming are controlled by the pseudo-gap exponents in these systems [25]. More generally, an argument of Ref. [1] shows that a pseudo-gap implies avalanche-type response to a slow external driving force, so-called crackling [26], for a range of applied forcing. Such behavior is indeed observed in these systems [6, 9, 27] and in the plasticity of crystals [28], and contrasts with depinning or random field Ising models where crackling occurs only at one specific value of forcing [29–31]. Despite the central role of pseudo-gaps, it has not been understood how they emerge dynamically, even though some important elements of the athermal dynamics of the SK spin glass have been pointed out in earlier works [11, 12].

In this Letter we identify a crucial ingredient that was neglected in previous dynamical approaches, and also in considerations of multi-spin stability: Soft spins are strongly frustrated among each other, a correlation that becomes nearly maximal for spins in the weakest fields. We expect analogous correlations in short range spin glasses, which can be probed experimentally. These correlations require revisiting earlier multi-spin stability arguments that assumed opposite correlations. We then argue, assuming stationarity along the hysteresis loop, that the correlation $C(\lambda)$ between the softest spins and spins in local fields of magnitude λ must follow $C(\lambda) \sim 1/\lambda^{\gamma}$, with $\gamma = 1$. Using this in a Fokker-Planck description of the dynamics we predict the statistics of the number of times a given spin flips in an avalanche.

Model: We consider the SK model with N Ising spins ($s_i = \pm 1$) in an external field h:

$$\mathcal{H} = -\frac{1}{2} \sum_{i \neq j} J_{ij} s_i s_j - h \sum_{i=1}^{N} s_i. \qquad (1)$$

All spins are coupled to each other by a symmetric matrix

J_{ij}, whose elements are *i.i.d.* Gaussian random variables with zero mean and variance $1/N$. The total magnetization is $M \equiv \sum_i s_i$. We define the local field h_i and the local stability λ_i of spin i by

$$h_i \equiv -\frac{\partial \mathcal{H}}{\partial s_i} = \sum_{j \neq i} J_{ij} s_j + h, \quad \lambda_i = h_i s_i. \quad (2)$$

The spin s_i is called stable when it aligns with the local field, i.e. if $\lambda_i > 0$, and unstable otherwise. The energy to flip the spin $s_i \to -s_i$ (and hence $\lambda_i \to -\lambda_i$) is:

$$\Delta \mathcal{H}_1(i) \equiv \mathcal{H}(-s_i) - \mathcal{H} = 2s_i(\sum_{j \neq i} J_{ij} s_j + h) = 2\lambda_i. \quad (3)$$

As in Ref. [9], we consider the hysteresis loop at zero temperature obtained by quasi-statically increasing the field, as shown in Fig. 1(a). When a spin turns unstable, we apply a greedy Glauber dynamics that relaxes the system in an avalanche-like process towards a new one-spin-flip stable state by sequentially flipping the most unstable spin. Such hysteretic field ramping has also been used to find approximate ground states [32, 33]. Those states empirically exhibit a pseudo-gap in the distribution of the λ_i [9, 11, 15],

$$\rho(\lambda) = A\lambda^\theta + O(N^{-\theta/(1+\theta)}), \quad (4)$$

with $\theta = 1$ for $\lambda \ll 1$, as shown in Fig. 1(b), but with a slope A significantly larger than in equilibrium [10, 12, 34]. The avalanche size is power-law distributed [9]:

$$D(n) = n^{-\tau} d(n/N^\sigma)/\Xi(N), \quad (5)$$

where n is the number of flips in an avalanche. The scaling function $d(x)$ vanishes for $x \gg 1$. N^σ is the finite size cutoff, and $\Xi(N)$ is a size dependent normalization if $\tau \leq 1$. Numerical studies of the dynamics of the SK model indicate that $\tau = \sigma = 1$ and $\Xi = \ln N$ [9, 15], as shown by the finite size collapse in Fig. 1(c). While one can argue that $\theta = 1$ along the hysteresis curve [1], the exponents τ and σ have not been derived theoretically for the dynamics (unlike for "equilibrium avalanches", for which $\tau = 1$ has been obtained analytically [13, 14]).

Below we present a theoretical analysis of the dynamics. We assume that the average number of times a spin flips along the hysteresis loop diverges with N for any finite interval of applied field $[h, h + \Delta h]$ if $h = O(1)$. This assures that a stationary regime is reached rapidly. (For $\tau = 1$ this condition simply reads $\sigma + 1/(1 + \theta) > 1$) [35]. We further rely on $\theta < \infty$. This implies a diverging number of avalanches in the hysteresis loop, each contributing a subextensive amount of dissipation [35]. The latter rules out avalanches running into strongly unstable configurations, with an extensive number of spins with negative stability $|\lambda| = O(1)$. Thus, the lowest local stability encountered in an avalanche, λ_0, must satisfy $\lambda_0 \to 0$ as $N \to \infty$, as we confirm numerically in Fig. 2(a).
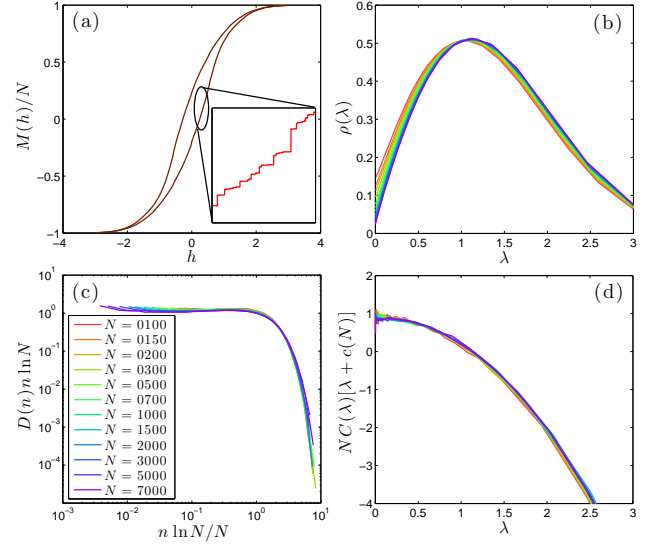

FIG. 1. (a) Hysteresis loop: Magnetization M under a periodic quasi-static driving of the external field h. Inset: magnified segment of the hysteresis loop of a finite size system. (b) Distribution of local stabilities, $\rho(\lambda)$, in locally stable states along the hysteresis loops for different system sizes N. (c) Finite size scaling of the avalanche size distribution $D(n)$ confirms $\tau = \sigma = 1$ up to logarithmic corrections. (d) Correlation $C(\lambda)$ between the least stable spin and spins of stability λ in locally stable states along the hysteresis loop. The data for different system sizes collapses, implying $C(\lambda \ll 1) \sim 1/\lambda$ in the thermodynamic limit.
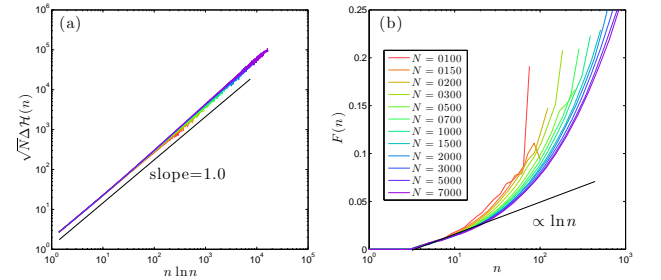

FIG. 2. (a) The average dissipated energy $\Delta \mathcal{H}$ in avalanches of size n scales as $\Delta \mathcal{H} \sim n \ln n/\sqrt{N}$. $-\Delta \mathcal{H}/n$ is a measure of the typical value of the stability of most unstable spins, $\lambda_0(n)$. Thus, in the thermodynamic limit, $\lambda_0 \sim \ln n/\sqrt{N} \ll 1$ even for very large avalanches. (b) The average number of times, $F(n)$, spins active in avalanches of size n re-flip later on in the avalanche.

Multi-spin stability criterion: A static bound for the pseudogap exponent θ is obtained by considering two of the softest spins i, j (with stabilities $\lambda_{\min} \sim 1/N^{1/(1+\theta)}$) [1, 36, 37]. Their simultaneous flip costs an energy $2(\lambda_i + \lambda_j - 2s_i s_j J_{ij})$. The last term scales as $1/\sqrt{N}$ and is negative if the two spins are *unfrustrated*. If this occurs with finite probability, a strong enough pseudogap, $\theta \geq 1$, is necessary to prevent the last term from overwhelming

the stabilizing terms. The extension of this argument to multi-spin stability reveals its subtle nature. Flipping a set $\mathcal{F}$ of m spins in a one-spin flip stable state costs

$$\Delta\mathcal{H}(\mathcal{F}) = 2\sum_{i\in\mathcal{F}}\lambda_i - 2\sum_{i,j\in\mathcal{F}}J_{ij}s_is_j. \quad (6)$$

The initial state is unstable to multi-flip excitations if $\Delta\mathcal{H} < 0$ for some $\mathcal{F}$. Refs. [36, 37] considered just the set of the m softest spins. Extremal statistics and the assumption of Eq. (4) implies the scaling of the maximal stabilities $\lambda(m) \sim (m/N)^{1/(1+\theta)}$, and thus $\sum_{i\leq m}\lambda_i \sim m\lambda(m)$. The term $\sum_{i\leq m}J_{ij}s_is_j \sim m(m/N)^{1/2}$ was erroneously argued to be positive *on average*, which yielded the bound $\theta \geq 1$ to guarantee $\Delta\mathcal{H}_m > 0$. However, numerically we find that on average $\sum_{i\leq m}J_{ij}s_is_j$ is negative for soft spins. More precisely, the correlation $C(\lambda) = -2\langle Jss\rangle$ between a spin of stability λ and the softest spin in the system is positive for small λ, as shown in Fig. 1(d). Postulating that:

$$C(\lambda) \sim \lambda^{-\gamma}N^{-\delta}, \quad (7)$$

it is straightforward to estimate that $\langle -\sum_{i\leq m}J_{ij}s_is_j\rangle \sim m^2C(\lambda(m)) \sim m^{2-\gamma/(1+\theta)}N^{\gamma/(1+\theta)-\delta}$. A more complete characterization of correlations is given in the Supplemental Material [38], Sec. A and B.

It follows that the average r.h.s. of Eq. (6) is always positive. We argue that the stability condition nevertheless leads to a non-trivial constraint, because the last term of Eq. (6) can have large fluctuations. Indeed, consider all sets $\mathcal{F}$ of m spins belonging to the $m' > m$ softest spins, and for definiteness we choose $m' = 2m$ here. To determine the probability that the optimal set leads to a negative $\Delta\mathcal{H}$ in Eq. (6), we use an approximate estimate akin to the random energy model [39]. The variance of the fluctuation $X \equiv \sum_{i,j\in\mathcal{F}}J_{ij}s_is_j - \langle\sum_{i,j\in\mathcal{F}}J_{ij}s_is_j\rangle$ is of order $m/\sqrt{N}$. Since there are 2^{2m} sets $\mathcal{F}$, the number density having fluctuation X follows $\mathcal{N}(X) \sim \exp[2m\ln(2) - X^2N/m^2]$. The most negative fluctuation $X_{\min}$ is determined by $\mathcal{N}(X_{\min}) \sim 1$, leading to $X_{\min} \sim -m^{3/2}/\sqrt{N}$. Correlations neglected by this argument should not affect the scaling. The associated energy change is thus, according to Eq. (6) and the subsequent estimates of each term:

$$\Delta\mathcal{H}(\mathcal{F}_{\min}) = m^{(2+\theta)/(1+\theta)}/N^{1/(1+\theta)}+$$
$$m^{2-\gamma/(1+\theta)}N^{\gamma/(1+\theta)-\delta} - m^{3/2}/\sqrt{N}. \quad (8)$$

Multi-spin stability requires that for large N and fixed m this expression be positive. This yields the conditions:

$$\theta \geq 1, \quad \text{or} \quad \gamma/(1+\theta) - \delta \geq -1/2. \quad (9)$$

However, the correlation in Eq. (7) cannot exceed the typical coupling among spins, $C \lesssim 1/\sqrt{N}$, which requires $\gamma/(1+\theta) - \delta \leq -1/2$. Thus, if $\theta < 1$, stability imposes

the equality $\gamma/(1+\theta) - \delta = -1/2$, while the scaling with $m \gg 1$ additionally requires $2 - \gamma/(1+\theta) \geq 3/2$; or in other words, $\gamma \leq (1+\theta)/2 \leq 1$ and $\delta \leq 1$. In the relevant states, all three exponents θ, γ, and δ turn out to equal 1 and thus satisfy these constraints as exact equalities. We will now show how to understand this emergent marginal stability from a dynamical viewpoint.

Fokker-Planck equation: Consider an elementary spin flip event in the greedy relaxation dynamics, cf. Fig. 3. The stability of the flipping spin 0 (red) changes from λ_0 to $-\lambda_0$ as the spin flips from s_0 to $-s_0$. Due to the coupling J_{0j}, the stability of all other spins j (green or blue) receives a kick, $\lambda_j \rightarrow \lambda'_j = \lambda_j - 2J_{0j}s_0s_j$. Using an expansion in $1/N$, we can describe the dynamics of the distribution of local stabilities $\rho(\lambda, t)$ by a Fokker-Planck equation, similarly as in Refs. [11, 12]:

$$\partial_t\rho(\lambda, t) = -\partial_\lambda\left[v(\lambda, t) - \partial_\lambda D(\lambda, t)\right]\rho(\lambda, t)$$
$$- \delta(\lambda - \lambda_0(t)) + \delta(\lambda + \lambda_0(t)), \quad (10)$$

where t counts the number of *flips per spin*. The drift $v(\lambda, t) \equiv -2N\langle J_{0i}s_0s_i\rangle_{\lambda_i=\lambda} \equiv NC(\lambda, t)$ is the average positive kick received by a spin of stability λ. The diffusion constant $D(\lambda, t) \equiv 2N\langle J_{0i}^2\rangle_{\lambda_i=\lambda} = 2$ is the mean square of those kicks, where we have assumed that the random parts of successive kicks are uncorrelated, as our numerics support. For the dynamics to have a non-trivial thermodynamic limit the scaling $\langle J_{0i}s_0s_i\rangle \sim 1/N$ must hold, i.e., $\delta = 1$ in Eq. (7). We further recall that $\lambda_0(t) \rightarrow 0$ as $N \rightarrow \infty$. We may thus replace the δ-functions in Eq. (10) by a reflecting boundary condition at $\lambda = 0$,

$$\left[v(\lambda, t) - \partial_\lambda D(\lambda, t)\right]\rho(\lambda, t)|_{\lambda=0} = 0. \quad (11)$$

Since we assume that spins flip many times along the hysteresis loop, finite intervals on the loop correspond to diverging times $\Delta t \rightarrow \infty$. At those large times a dynamical steady state (ss) must be reached. In such a state the flux of spins must vanish everywhere:

$$v_{ss}(\lambda) = D\partial_\lambda\rho_{ss}(\lambda)/\rho_{ss}(\lambda) \rightarrow 2\theta/\lambda, \quad (12)$$

where we assumed that ρ_{ss} follows Eq. (4). This result is tested in Fig. 1(d). A similar result was obtained in Ref. [12] following a quench.

Emergence of correlations: Eq. (12) implies that $\gamma = 1$ in Eq. (7). Such singular correlations are unexplained [40]. We now argue that they naturally build up in the dynamics through the spin-flip induced motion of stabilities of frustrated and unfrustrated spins, as illustrated in Fig. 3. To quantify this effect we define respectively $C_f(\lambda)$ and $C'_f(\lambda)$ as the correlation between the flipping spin 0 and the spins at λ *before* and *after* a flip event. As s_0 flips, the stability of spin i increases by $x_i \equiv -2J_{0i}s_0s_i$, $\lambda'_i = \lambda_i + x_i$. The correlation $C'_f(\lambda)$ is an average over
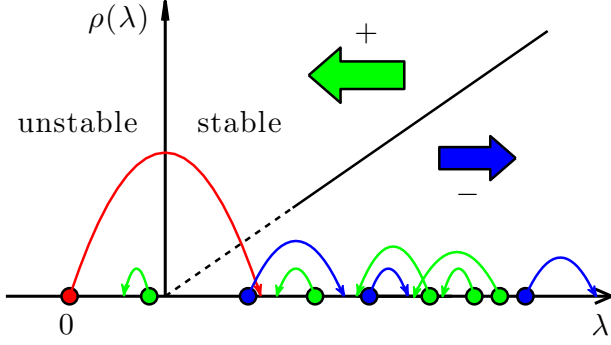
FIG. 3. Illustration of a step in the dynamics, in the SK model and the random walker model. Circles on the λ-axis represent the spins or walkers. At each step, the most unstable spin (in red) is reflected to the stable side, while all others (in green or blue) receive a kick and move. The dashed and solid line outlines the density profile $\rho(\lambda) \sim \lambda$ for $\lambda > 1/\sqrt{N}$. The blue spins were initially frustrated with the flipping spin 0. They are stabilized and are now unfrustrated with 0. In contrast, green spins become frustrated with spin 0 and are softer now. Because of the motion of spins depends on their frustration with spin 0, a correlation builds up at small λ, leading to an overall frustration of "soft" spins among each other.

all spins which migrated to λ due to the flip:

$$C_f'(\lambda) = \frac{1}{\rho'(\lambda)} \int \rho(\lambda - x)(-x)f_{\lambda-x}(x)\mathrm{d}x,$$

$$\rho'(\lambda) = \int \rho(\lambda - x)f_{\lambda-x}(x)\mathrm{d}x.$$

$f_\lambda(x)$ is the Gaussian distribution of kicks x given to spins of stability λ: $f_\lambda(x) = \exp\left[-\frac{(x-C_f(\lambda))^2}{4D/N}\right]/\sqrt{4\pi D/N}$. In the integrands we expand $\rho(\lambda - x)$ and $C_f(\lambda - x)$ for small x and keep terms of order $1/N$, which yields

$$C_f'(\lambda) = -C_f(\lambda) + 2\frac{D}{N}\frac{\partial_\lambda \rho(\lambda)}{\rho(\lambda)}, \tag{13a}$$

$$\rho'(\lambda) = \rho(\lambda) - \partial_\lambda\left[C_f(\lambda)\rho(\lambda) - \frac{D}{N}\partial_\lambda\rho(\lambda)\right]. \tag{13b}$$

Thus, even if correlations are initially absent, $C_f(\lambda) = 0$, they arise spontaneously, $C_f'(\lambda) = 2D\partial_\lambda\rho(\lambda)/N\rho(\lambda)$.

In the steady state, $\rho_{\mathrm{ss}}' = \rho_{\mathrm{ss}}$, and Eq. (13b) implies the vanishing of the spin flux, that is, Eq. (12) with $v = NC_f$. Plugged into Eq. (13a), we obtain that the correlations are steady, too,

$$C_f'(\lambda) = C_f(\lambda) = \frac{v_{\mathrm{ss}}(\lambda)}{N} = \frac{2\theta}{N\lambda}. \tag{14}$$

These correlations are expected once the quasi-statically driven dynamics reaches a statistically steady regime, and thus should be present both during avalanches and in the locally stable states reached at their end.

Interestingly, Eq. (14) implies that all the bounds of Eq. (9) are saturated if the first one is, i.e., if $\theta = 1$.

The latter value was previously derived from dynamical considerations in Ref. [1]. It is intriguing that the present Fokker-Planck description of the dynamics does not pin θ, as according to Eqs. (12, 14) any value of θ is acceptable for stationary states. However, additional considerations on the applicability of the Fokker-Planck description discard the cases $\theta > 1$ and $\theta < 1$, as discussed in the Supplemental Material, Sec. C.

Those are related to the interesting fact that that Eqs. (10, 11, 12) with $\theta = 1$ are equivalent to the Fokker-Planck equation for the radial component of unbiased diffusion in $d = 2$ (as derived in Supplemental Material, Sec. D), whose statistics is well known [41, 42]. We can use this analogy to predict $F(n)$, the number of times an initially soft spin flips in an avalanche of size n. Indeed, a discrete random walker starting at the origin will visit that point $\ln(t)$ times after t steps in two dimensions, and thus $F(n) \sim \ln(n)$, as supported by Fig. 2(b). Similarly we expect times between successive flips of a given spin to be distributed as $P(\delta t) \sim 1/(\delta t[\ln(\delta t)]^2)$.

Short range systems and experiments: In short range spin glasses we expect and have numerically checked analogous frustrated correlations between pairs of directly interacting soft spins as in the SK model, except that the growth of correlations at small λ is cut off at the typical coupling between spins. This prediction can be tested in experiments akin to NMR protocols: First flip the spins of stability λ by a π-pulse of appropriate frequency. Then flip those of stability λ' and observe the resulting shift in the fluorescence spectrum around λ. From our findings we predict a systematic shift to higher frequencies.

Conclusions: We have studied the quasi-static dynamics in a marginally stable glass at zero temperature, focusing on a fully-connected spin glass as a model system. Our central result is that the pseudo-gap appears dynamically due to a strong frustration among the softest spins, characterized by a correlation function $C(\lambda)$ which scales inversely with the stability λ. We provided a Fokker-Planck description of the dynamics that explains the appearance of both the pseudo-gap and the singular correlation, and suggests a fruitful analogy between spin glass dynamics and random walks in two dimensions.

We expect our findings to apply to other marginally stable systems, in particular hard sphere packings that display a pseudo-gap with a non-trivial exponent: $P(f) \sim f^{\theta_e}$ [16, 17, 20, 23] where f is the contact force. Our analysis above suggests that a singular correlation function $C(f) \sim 1/f$ characterizes how contacts are affected by the opening of a contact of very weak force, the relevant excitations in packings [16, 17]. Contacts with small forces should on average be stabilized by $C(f)$ - a testable prediction. Our analysis also suggests a connection between sphere dynamics and random walks in dimension $1 + \theta_e$, which is interesting to explore further.

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