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We derive generalisations of the Weingarten–Witten QCD mass inequalities for particular multi-hadron systems. For systems of any number of identical pseudo-scalar mesons of maximal isospin, these inequalities prove that near threshold interactions between the constituent mesons must be repulsive and that no bound states can form in these channels. Similar constraints in less symmetric systems are also extracted. These results are compatible with experimental results (where known) and recent lattice QCD calculations, and also lead to a more stringent bound on the nucleon mass than previously derived, $m_N \geq \frac{3}{2} m_\pi$.

Analytic relationships between low-energy hadronic quantities are difficult to obtain in Quantum Chromodynamics (QCD) because it is a strongly interacting field theory, and only a few such relationships are known. Consequently, the various inequalities between hadron masses that have been derived by Weingarten [1], Witten [2], and (under some assumptions) by Nussinov [3] have an important place in our understanding of QCD. The rigorous relations can be summarised by stating that the pion is the lightest colourless state of non-zero isospin [1] ($m_X \geq m_\pi$ for X being any $I \geq 1$ isospin-charged meson), that the pion electromagnetic mass difference $m_{\pi^+} - m_{\pi^0}$ is positive [2] and that baryons are heavier than pions, $m_B \geq m_\pi$ [1, 4]. The status of QCD inequalities is reviewed in Ref. [5]. The known results concern a relatively small number of static quantities, and it is important to consider whether further relations exist. In this direction, Nussinov and Sathiapalan [6] found that in QCD motivated models there are relationships between scattering lengths in various two-particle channels, and Gupta *et al.* [7, 8] showed that an unphysical combination of $\pi\pi$ interactions is attractive. Finally, Alfaro *et al.* [9] showed that relationships existed between $K \to \pi$ matrix elements of various four-quark operators. In this letter, we demonstrate that there are additional rigorous QCD inequalities that pertain to the spectrum of particular physical, multi-hadron systems and thereby to the nature of the corresponding hadronic interactions. As simple examples, we prove that there are no bound states in the $I = 2$ $\pi^+\pi^+$ or $I = 3/2$ $\pi^+ K^+$ channels and also improve on a previous baryon-meson mass inequality, showing that $m_N \geq \frac{3}{2} m_\pi$. As with the original inequalities, an experimental demonstration that these inequalities are violated would strongly suggests that QCD does not describe the strong interaction (modulo possible effects of electroweak interactions).

A central observation of Vafa and Witten [4] is that the measure of the QCD functional integrals that define QCD correlation functions is positive definite in the absence of a θ-term or baryon chemical potential (we will ignore these cases throughout this work). After integrating over the quark degrees of freedom, the functional integration measure can be expressed as

$$d\mu = \prod_{x,\mu,a} dA^a_\mu(x) e^{-S_{YM}[A]} \prod_f \det\left[\not{D} + \tilde{m}_f\right] , \qquad (1)$$

where A_μ represents the gauge field, $\not{D} = \not{D}[A]$ is the fermion Dirac operator, $\tilde{m}_f$ is the bare quark mass of flavour f, and $S_{\rm YM} = \frac{1}{2}\int d^4x {\rm Tr}[F^{\mu\nu}F_{\mu\nu}]$ is the Yang-Mills action with $F^{\mu\nu} = [D^\mu, D^\nu]$. Throughout our discussion, we use a Euclidean metric; for the correlators that we consider, analytic continuation to Minkowski space is straightforward. Correlation functions involving field operators at n spacetime points are defined as

$$\langle \mathcal{O}(x_1, \ldots, x_n)\rangle = \frac{1}{\mathcal{Z}} \int d\mu \, \hat{\mathcal{O}}(x_1, \ldots, x_n) \, , \qquad (2)$$

where $\mathcal{Z} = \int d\mu$, and the operator $\hat{\mathcal{O}}$ results from the operator $\mathcal{O}$ after integration over quark fields. These functional integrals are only defined after the imposition of a regulator, and we assume the use of a regulator that does not spoil positivity [1, 4]. As a consequence of the positivity of the measure, field independent relations that are shown to hold for any particular gauge field configuration also hold for the integrated quantity, the corresponding correlation function. Vafa and Witten used measure positivity to derive the celebrated result that vector symmetries do not break spontaneously.

In related work, Weingarten [1] considered correlation functions from which meson and baryon masses can be determined, and made use of measure positivity and the Cauchy-Schwarz and Hölder inequalities to show that relationships exist between the corresponding functional integrals. The inequalities show that $m_\pi \leq m_X$, and $m_N \geq \frac{N_f-2}{N_f-3} m_\pi$ for a theory with $N_f \geq 6$ flavours. Using a further constraint on the spectrum of the inverse of the Dirac operator, shown to hold in Ref. [4], this latter constraint was extended to $m_N \geq m_\pi$, independent of the number of flavours.

Our analysis shares similarities with the approaches discussed above, but also makes use of a novel eigenvalue decomposition of correlation functions. We begin by considering an $I = I_z = n$ many-π^+ correlator of the form

$$\left\langle \Omega \left| \prod_i^n u\gamma_5 \bar{d}(x_i) \prod_j^n d\gamma_5 \bar{u}(y_j) \right| \Omega \right\rangle , \tag{3}$$

where $|\Omega\rangle$ is the vacuum state and the clusters of points $\{x_i\}$ (sources) and $\{y_j\}$ (sinks) are taken to be well separated in Euclidean space. The combination $d\gamma_5\bar{u}(y)$ is an interpolating operator that creates the quantum numbers of a π^+ meson. We specify to vanishing total momentum by separately summing over the spatial components of the y_i coordinates and for simplicity set the temporal components $x_i^4 = 0\ \forall i$ and $y_j^4 = t\ \forall j$ and allow for some of the source locations to be the same (nonzero correlators result provided that $4N_c$ or less quark fields are placed at the same spacetime point). This leads to

$$\begin{aligned} \mathcal{C}_n &\equiv C_n(\mathbf{x}_1, \dots \mathbf{x}_n; t; \mathbf{P}=0) \\ &= \left\langle \Omega \left| \prod_i^n u\gamma_5 \bar{d}(\mathbf{x}_i, 0) \left[\sum_{\mathbf{y}} d\gamma_5 \bar{u}(\mathbf{y}, t) \right]^n \right| \Omega \right\rangle . \end{aligned} \tag{4}$$

As shown in Refs. [10, 11], these correlation functions can be written in terms of products of traces of powers of the matrix

$$\mathbf{\Pi}_A = \begin{pmatrix} P_{1,1} & P_{1,2} & \cdots & P_{1,N_s} \\ P_{2,1} & \ddots & \ddots & P_{2,N_s} \\ \vdots & \ddots & \ddots & \vdots \\ P_{N_s,1} & \cdots & \cdots & P_{N_s,N_s} \end{pmatrix} , \tag{5}$$

where N_s is the number of source locations being considered, the $4N_c \times 4N_c$ blocks are given by

$$P_{i,j}(t) = \sum_{\mathbf{y}} S_u(\mathbf{x}_i, 0; \mathbf{y}, t)\gamma_5 S_d(\mathbf{y}, t; , \mathbf{x}_j, 0)\gamma_5 , \tag{6}$$

and S_u and S_d are propagators for the up and down quarks, respectively. The subscript A indicates that the matrix depends on the background gauge field and $\mathbf{\Pi}_A$ is a matrix of dimension $N = 4N_cN_s$ and by increasing N_s, this can be taken to infinity.

This can be further simplified in the isospin limit where the up and down quark propagators are the same, $S_u = S_d$, and by using the γ_5 hermiticity of the Dirac operator that implies that $\gamma_5 S_d(y,x)\gamma_5 = S_d^\dagger(x,y)$ so that the $P_{i,j}$ take the form

$$P_{i,j}(t) = \sum_{\mathbf{y}} S_u(\mathbf{x}_i, 0; \mathbf{y}, t) S_u^\dagger(\mathbf{x}_j, 0; \mathbf{y}, t) . \tag{7}$$

Consequently, we see that $\mathbf{\Pi}_A$ is a non-negative definite Hermitian matrix, as are all its diagonal sub-blocks. In Ref. [10], it was shown that the contributions to the correlation functions $\mathcal{C}_j$ for $j \leq N$ determined on a given gauge configuration arise as coefficients of the characteristic polynomial

$$\mathcal{P}_A(\alpha) \;=\; \det(1 + \alpha\, \mathbf{\Pi}_A) = \sum_{j=0}^N c_j[A]\alpha^j \tag{8}$$

of the matrix $\mathbf{\Pi}_A$.[1] Since the roots of the characteristic polynomial are determined by the eigenvalues π_i of $\mathbf{\Pi}_A$, it follows that

$$c_n[A] = \sum_{i_1 \neq i_2 \neq \dots \neq i_n = 1}^N \pi_{i_1}\pi_{i_2}\dots\pi_{i_n} . \tag{9}$$

Thus $c_1[A] = \sum_{i=1}^N \pi_i = \mathrm{tr}[\mathbf{\Pi}_A]$, $c_2[A] = \sum_{i=1}^N \sum_{j\neq i=1}^N \pi_i\pi_j$, $\dots$, $c_N[A] = \pi_1 \dots \pi_N = \det[\mathbf{\Pi}_A]$. Since these eigenvalues are non-negative, we can bound these expressions by products of the single pion expression by relaxing the restrictions on the summation above. That is,

$$c_n[A] \;\leq\; \sum_{i_1, i_2, \dots, i_n = 1}^N \pi_{i_1}\pi_{i_2}\dots\pi_{i_n} = \left[\sum_{i=1}^N \pi_i \right]^n = c_1^n[A] . \tag{10}$$

From this eigenvalue relation, valid on a fixed background gauge configuration, we can construct the field independent bound, $c_n[A] - c_1^n[A] \leq 0$, that holds for all A_μ^a. Measure positivity then implies that this relation holds at the level of QCD correlators.[2] The large separation behaviour of $\langle c_n \rangle$ is governed by the energy of the lowest energy eigenstates of the system, $\langle c_n \rangle \sim \exp(-E_n^{(0)} t)$. We also note that $\langle c_1^n \rangle \leq \sigma \langle c_1 \rangle^n$ for some source-sink separation independent σ. Together, this implies that $E_n^{(0)} \geq n\, E_1^{(0)} = n\, m_\pi$ and consequently that there are no bound states possible in these maximal isospin channels. Further, is also implies that the two-body interactions in these systems are repulsive or vanishing at threshold. This second result follows from the fact that the relations derived above are valid in a finite volume where the energy eigenvalues of two particle systems below inelastic thresholds are determined by the appropriate infinite volume scattering phase shift [16, 17]. Since the scattering phase shift near threshold is proportional to the negative of the non-negative definite energy shift, it must correspond to a repulsive or vanishing interaction.

The two-pion results are in accordance with expectations from chiral perturbation theory (χPT) [18, 19]

[1] There are normalisation differences between the c_j and $\mathcal{C}_j$, and for multiple source locations, the c_j are linear combinations of the $\mathcal{C}_j$ with different numbers of interpolators at each source. The spectrum is common to each term in this linear combination.

[2] We note that the results hold for lattice QCD discretisations that preserve measure positivity such as domain-wall [12] and overlap fermions [13, 14], or Wilson fermions [15] with even N_f.

which predicts at next-to-leading order (NLO) that

$$m_\pi a_{\pi\pi}^{(I=2)} \;=\; -2\pi\chi\left[1+\chi\left(3\log\chi - L_{\pi\pi}^{(I=2)}\right)\right], \quad (11)$$

where $\chi = [m_\pi/4\pi f_\pi]^2$, f_π is the pion decay constant, and $L_{\pi\pi}^{(I=2)}$ is a particular combination of low energy constants (LECs) renormalised at scale $\mu = 4\pi f_\pi$. At tree level, this expression is universally negative, and at NLO it remains negative given the phenomenological constraints on $L_{\pi\pi}^{(I=2)}$. However, the bounds derived above are statements directly about QCD and do not rely on a chiral expansion, and in fact provide a fundamental constraint on $L_{\pi\pi}^{(I=2)}$ (the use of single particle QCD inequalities to constrain χPT is discussed in Refs. [20, 21]). The $\pi\pi$ scattering phase shifts can be experimentally extracted from studies of kaon decays [22–24] and the lifetime of pionium [25], but the direct constraints of the $I = 2$ channel are relatively weak. A chiral and dispersive analysis of experimental data nevertheless allows for a precise extraction [26], giving $m_\pi a_{\pi\pi}^{(I=2)} = -0.0444(10)$ and lattice QCD calculations [27–33] are in agreement. The sign implies that these results are concordant with the QCD inequalities derived here.

As a corollary, having shown that the $(\pi^+)^n$ systems do not bind, we can follow the discussion of Ref. [5] and strengthen the nucleon mass bound of Weingarten to $m_N \geq \frac{3}{2}m_\pi$. This improves on the bounds of Refs. [1, 6, 34] as it applies for arbitrary N_f and N_c and the inequality directly involves the pion mass. Furthermore, less complete modifications of the restricted sums in Eq. (9) show also that $E_n^{(0)} \geq E_{n-j}^{(0)} + E_j^{(0)}$ for all $j < n$. This then implies that the $I = 3$, $\pi^+\pi^+\pi^+$ interaction is separately repulsive at threshold, as are the $I = n$, $(\pi^+)^n$ interactions. In principle, the form of these interactions could be computed in the chiral expansion, and the constraints derived here would bound the LECs that enter. Lattice calculations show that the $\pi^+\pi^+\pi^+$ interaction is indeed repulsive [35, 36].

The inequalities above concern identical pseudoscalar mesons formed from quarks of equal mass, but they can be generalised in a number of ways. In particular, these inequalities can be extended to the case of unequal quark masses; thereby analogous results can be derived for multiple pion systems away from the isospin limit. Further, by defining

$$K_{i,j}(t) = \sum_{\mathbf{y}} S_u(\mathbf{x}_i, 0; \mathbf{y}, t) S_s^\dagger(\mathbf{y}, t; , \mathbf{x}_j, 0)\,, \quad (12)$$

where $S_s(x, y)$ is the strange quark propagator, in addition to $P_{i,j}$, correlators containing both π^+ and K^+ mesons can be studied. The matrix $\mathbf{K}_A$ can be constructed from the $K_{i,j}$ sub-blocks analogously to Eq. (5). To see how these generalisations arise, we need to examine the spectrum of the relevant matrices. If we denote the eigenvalues and eigenfunctions of the Dirac operator

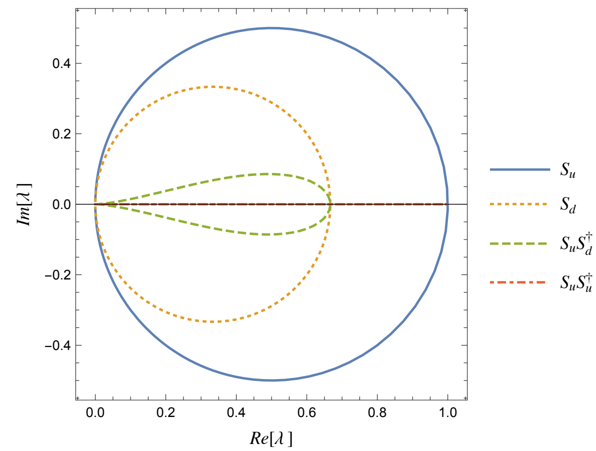

FIG. 1: Eigenvalues of S_u, S_d, $S_uS_u^\dagger$ and $S_uS_d^\dagger$ for $m_u = 1$ and $m_d = 1.5$.

as λ_i and v_i respectively, that is $\not{D}\, v_i = \lambda_i v_i$, we can decompose the quark propagators as

$$S_f = \sum_i \frac{v_i v_i^*}{\lambda_i + m_f} \equiv \sum_i \sigma_i^{(f)} v_i v_i^*\,, \quad (13)$$

where m_f is the renormalised quark mass[3] and the matrix $\mathbf{\Pi}_A$ as

$$\begin{aligned}\mathbf{\Pi}_A \;&\equiv\; \sum_i \pi_i v_i v_i^* = \sum_{i,j} \frac{v_i v_i^*}{\lambda_i + m_u}\left(\frac{v_j v_j^*}{\lambda_j + m_d}\right)^\dagger \\ &=\; \sum_i \frac{v_i v_i^*(-\lambda_i^2 + m_u m_d + \;\lambda_i(m_u - m_d))}{(m_u^2 - \lambda_i^2)(m_d^2 - \lambda_i^2)}\,, \quad (14)\end{aligned}$$

with a similar expression for $\mathbf{K}_A$ (in the second equality for $\mathbf{\Pi}_A$, we have used (Coulomb gauge spatial) completeness as we are integrating over the spatial position of the sink in defining $P_{i,j}$). Because of the spectral properties of the Dirac operator ($\lambda_i \in \mathbb{I}$, and $\{\lambda_i, \lambda_i^*\}$ both eigenvalues), the eigenvalues of quark propagators, $\sigma_i^{(f)}$, fall on circles (centre $(1/(2m_f), 0)$, radius $1/(2m_f)$) in the complex plane. For the matrix $\mathbf{\Pi}_A$ in the isospin limit, we immediately see that the eigenvalues are real and non-negative as stated above, occupying the interval $[0, 1/m_f]$. Away from the isospin limit, $\mathbf{\Pi}_A$ and $\mathbf{K}_A$ have eigenvalues, denoted π_i and κ_i respectively, that occur in complex conjugate pairs with non-negative real parts and imaginary parts that are proportional to the mass splitting $|m_1 - m_2|$. The locii of these eigenvalues are shown in Fig. 1 for exemplary masses.

The spectral properties[4] discussed above are sufficient

[3] In what follows, we assume a mass-independent and multiplicatively renormalisable regularisation and renormalisation scheme
[4] The overlap Dirac operator [13, 14], which is γ_5 hermitian and has eigenvalues on the circle $(1 + \cos\theta, \sin\theta) \in \mathbb{C}$ for $0 \leq \theta < 2\pi$, is an explicit regulator for which the argument that follows holds.

to show that even in the less symmetric cases mentioned previously, the generalisations of the eigenvalue inequality used in Eq. (10) still hold, at least for certain quark mass ratios in systems containing up to $n = 8$ particles (for example $\pi^+\pi^+\pi^+K^+$) where we have explicitly checked.[5] To see this, we reconsider the eigenvalue sums that occur in the expressions for correlators[6] with the quantum numbers of $(\pi^+)^j(K^+)^{n-j}$, denoted $c_{j,n-j}$. As the simplest example we consider

$$c_{1,1} \sim \sum_i \sum_{j\neq i} \pi_i \kappa_j \;=\; \sum_{i,j} \pi_i \kappa_j - \sum_i \pi_i \kappa_i\,, \qquad (15)$$

and shall show that the last sum is positive. This is most easily approached in the $N \to \infty$ limit in which the eigenvalue sums become continuous integrals. To make our notation simpler, we replace $\lambda \to i\lambda_R$ with $\lambda_R \in \mathbb{R}$ and subsequently drop the subscript. In this case, defining

$$f_{a,b}(\lambda) = \frac{\lambda^2 + m_a m_b + i\,\lambda(m_a - m_b))}{(\lambda^2 + m_a^2)(\lambda^2 + m_b^2)}\,, \qquad (16)$$

and $\pi(\lambda) = f_{u,d}(\lambda)$ and $\kappa(\lambda) = f_{u,s}(\lambda)$, we can replace $\sum_i \pi_i \kappa_i$ by

$$\int_{-\infty}^{\infty} \mathcal{D}\lambda\, \pi(\lambda)\, \kappa(\lambda)\,, \qquad (17)$$

where the measure $\mathcal{D}\lambda \equiv d\lambda\, \rho(\lambda)$ is weighted by the spectral density of the Dirac operator, $\rho(\lambda)$. Since the spectral density is non-negative, a non-negative integrand results in a non-negative integral. However, the integrand above is only positive definite for some ranges of the ratios m_d/m_u and m_s/m_u (we specify to a mass-independent multiplicative renormalisation scheme for the quark masses in which these ratios are scale independent; schemes involving a chiral lattice regularisations such as domain-wall fermions and overlap fermions are examples) as is shown for this case in Fig. 2. If the mass ratios are in the allowed region, then $c_{1,1} \leq c_{1,0}c_{0,1}$ and through the same logic that we employed for $I = 2$ $\pi\pi$ systems, we see that $E_{\pi^+K^+} \geq m_{\pi^+} + m_{K^+}$, so $I = 3/2$ π^+K^+ scattering can not result in bound states. This result is in agreement with lattice calculations [37–40]. Outside these parameter ranges, the integral has negative contributions at intermediate λ but is positive at large λ; given the expectations of the behaviour of the spectral density, $\rho(\lambda) \sim V\lambda^3$ for large λ, this suggests that the integral is always positive in Eq, (17), but this cannot be proven rigorously. For the important cases of $\pi^+\pi^+$ at $m_d \neq m_u$ and π^+K^+, the physical mass ratios [41] are such that the proof is complete, but for example for $I = 1$ K^+K^+ or D^+D^+, the mass ratios are such that the proof fails.

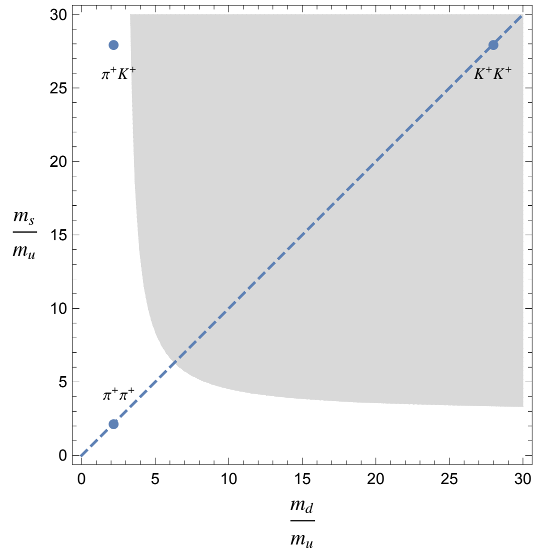

FIG. 2: Grey shading indicates the region of non-positivity of the integrand in Eq. (17). Also shown are relevant physical mass ratios for $\pi^+\pi^+$ at $m_d \neq m_u$ and π^+K^+ and K^+K^+.

In a more complicated case, such as $c_{3,1}$, the subtractions are more involved,

$$\begin{aligned} c_{3,1} \;\sim\; & \sum_i \sum_{j\neq i} \sum_{k\neq i,j} \sum_{l\neq i,j,k} \pi_i\pi_j\pi_k\kappa_l \qquad (18) \\ =& \sum_{i,j,k,l} \pi_i\pi_j\pi_k\kappa_l - 3\sum_i \sum_{j\neq i} \sum_{k\neq i,j} (\pi_i^2\pi_j\kappa_k + \pi_i\pi_j\pi_k\kappa_i) \\ & - \sum_i \sum_{j\neq i} (\pi_i^3\kappa_j + 3\pi_i^2\pi_j\kappa_i + 3\pi_i^2\pi_j\kappa_j) - \sum_i \pi_i^3\kappa_i \\ =& \sum_{i,j,k,l} \pi_i\pi_j\pi_k\kappa_l - \Big\{ 3\sum_{i,j,k} (\pi_i^2\pi_j\kappa_k + \pi_i\pi_j\pi_k\kappa_i) \\ & - \sum_{i,j} (2\pi_i^3\kappa_j + 6\pi_i^2\pi_j\kappa_i + 3\pi_i^2\pi_j\kappa_j) + 6\sum_i \pi_i^3\kappa_i \Big\}\,. \end{aligned}$$

However, by again taking the continuous limit and writing the eigenvalue sums as (multiple) integrals, the term in the braces can be proven to be positive for certain values of m_d/m_u and m_s/m_u, thereby showing $E_{\pi^+\pi^+\pi^+K^+} \geq 3m_{\pi^+} + m_{K^+}$. The region of guaranteed positivity varies with the number of pions and kaons in the system, but a region exists for all $c_{j,k}$.

As a further generalisation, we may consider modified correlators where we replace some of the γ_5 matrices in Eq. (4) by other Dirac structures. We can then use the Cauchy-Schwartz inequality to derive the related results that the energies of arbitrary J^P states with $I = I_z = n$ are bounded from below by $n\,m_\pi$ in the same manner in which Weingarten [1] showed that $m_X \geq m_\pi$. This

[5] We expect that these results hold for all n and all mass ratios, but have been unable to prove the necessary relations.

[6] The correlators for j π^+s and k K^+s can be constructed from the expansion of $\det(1 + \alpha\mathbf{\Pi}_A + \beta\mathbf{K}_A)$ as discussed in Ref. [37].

does not prohibit bound state formation if the quantum numbers prohibit an $n\,\pi^+$ state in the given channel (for example $\rho^+\rho^+\rho^+$ with $J^P = 3^-$), but limits the amount of binding that is possible.

In summary, we have shown that the hadron mass inequalities previously derived in QCD have an infinite set of analogues for certain multi-hadron systems that constrain the nature of the interactions between the constituent hadrons. These results provide important constraints on phenomenological, and lattice QCD studies of hadron interactions and serve as fundamental tests of QCD. The scope of the techniques used to derive the original hadron mass inequalities and the new techniques introduced here is more general than the two-point correlation functions considered so far, and there are a number of extensions that may be pursued productively.

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