Locality of Gravitational Systems from Entanglement of Conformal Field Theories

Jennifer Lin, Matilde Marcolli, Hirosi Ooguri, and Bogdan Stoica

Phys. Rev. Lett. **114**, 221601 — Published 2 June 2015

DOI: 10.1103/PhysRevLett.114.221601

CALT-TH 2014-162, IPMU14-0349

Locality of Gravitational Systems from Entanglement of Conformal Field Theories

Jennifer Lin,[1] Matilde Marcolli,[2] Hirosi Ooguri,[3,4] and Bogdan Stoica[3]

[1] *Enrico Fermi Institute and Department of Physics, University of Chicago, Chicago, IL 60637*
[2] *Department of Mathematics, California Institute of Technology, 253-37, Pasadena, CA 91125*
[3] *Walter Burke Institute for Theoretical Physics, California Institute of Technology, 452-48, Pasadena, CA 91125*
[4] *Kavli Institute for the Physics and Mathematics of the Universe (WPI), University of Tokyo, Kashiwa 277-8583, Japan*

(Dated: March 11, 2015)

The Ryu-Takayanagi formula relates the entanglement entropy in a conformal field theory to the area of a minimal surface in its holographic dual. We show that this relation can be inverted for any state in the conformal field theory to compute the bulk stress-energy tensor near the boundary of the bulk spacetime, reconstructing the local data in the bulk from the entanglement on the boundary. We also show that positivity, monotonicity, and convexity of the relative entropy for small spherical domains between the reduced density matrices of any state and of the ground state of the conformal field theory are guaranteed by positivity conditions on the bulk matter energy density. As positivity and monotonicity of the relative entropy are general properties of quantum systems, this can be interpreted as a derivation of bulk energy conditions in any holographic system with the Ryu-Takayanagi prescription applies. We discuss an information theoretical interpretation of the convexity in terms of the Fisher metric.

PACS numbers: 11.25.Tq

Introduction.— Gauge/gravity duality posits an exact equivalence between certain conformal field theories (CFT's) with many degrees of freedom and higher dimensional theories of gravity. It is of obvious interest to understand how bulk spacetime geometry and gravitational dynamics emerge from a non-gravitating theory. In recent years, there have appeared hints that quantum entanglement plays a key role. One important development in this direction was the Ryu–Takayanagi proposal [1, 2] that the entanglement entropy (EE) between a spatial domain D of a CFT and its complement equals the area of the bulk extremal surface Σ homologous to it,

$$S_{EE} = \min_{\partial D = \partial\Sigma} \frac{\text{area}(\Sigma)}{4G_N}\,. \tag{1}$$

Using (1), [3–9] studied the connection between linearized gravity and entanglement physics of the CFT. In this letter, we continue this program. We develop tomographic techniques to diagnose local energy density in the bulk by the Radon transform of quantum entanglement data on the boundary. Moreover, we show that properties of entanglement on the boundary such as the positivity, monotonicity, and convexity of the relative entropy are guaranteed by energy conditions in the bulk. As the positivity and monotonicity of the relative entropy are general properties of quantum systems, we can interpret this as a derivation of energy conditions in the bulk for any holographic system, assuming the Ryu-Takayanagi formula.

Relative entropy (see *e.g.* [4]) is a measure of distinguishability between two quantum states in the same Hilbert space, for two density matrices ρ_0 and ρ_1 being defined as

$$S(\rho_1|\rho_0) = \text{tr}(\rho_1 \log\rho_1) - \text{tr}(\rho_1 \log\rho_0)\,. \tag{2}$$

It is positive and monotonic:

$$S(\rho_1|\rho_0) \geq 0\,, \quad S(\rho_1^W|\rho_0^W) \geq S(\rho_1^V|\rho_0^V), \tag{3}$$

where $W \supseteq V$. When ρ_0 and ρ_1 are reduced density matrices on a spatial domain D for two states of a quantum field theory (QFT), which is the case we specialize to from this point on, monotonicity implies that $S(\rho_1|\rho_0)$ increases with the size of D. That is, over a family of scalable domains with characteristic size R,

$$\partial_R S(\rho_1|\rho_0) \geq 0\,. \tag{4}$$

Defining the modular Hamiltonian H_{mod} of ρ_0 implicitly through

$$\rho_0 = \frac{e^{-H_{mod}}}{\text{tr}(e^{-H_{mod}})}\,, \tag{5}$$

Eq. (3) is equivalent to

$$S(\rho_1|\rho_0) = \Delta\langle H_{mod}\rangle - \Delta S_{EE} \geq 0 \tag{6}$$

with $\Delta\langle H_{mod}\rangle = \text{tr}(\rho_1 H_{mod}) - \text{tr}(\rho_0 H_{mod})$ the change in the expectation value of the operator H_{mod} (5) and $\Delta S_{EE} = -\text{tr}(\rho_1\log\rho_1) + \text{tr}(\rho_0\log\rho_0)$ the change in the entanglement entropy across D as one goes between the states.

When the states under comparison are close, the positivity (6) is saturated to leading order [4, 6, 8]:

$$S(\rho_1|\rho_0) = \Delta\langle H_{mod}\rangle - \Delta S_{EE} = 0\,. \tag{7}$$

This is the entanglement first law for its resemblance to the first law of thermodynamics. Indeed, when ρ_0 is a thermal density matrix $\rho_0 = e^{-\beta H}/\,\text{tr}(e^{-\beta H})$, (7) reduces to $\Delta\langle H\rangle = T\Delta S$, an exact quantum version of the thermal first law.

In general, the modular Hamiltonian (5) associated to a density matrix is nonlocal. However, there are a few simple cases where it is explicitly known. When ρ_0 is the reduced density matrix of a CFT vacuum state on a disk of radius R which we take to be centered at $\vec{x}_0 = 0$ [10],

$$H_{mod} = \pi \int_D d^{d-1}x \frac{R^2 - |\vec{x}|^2}{R} T_{tt}(x)\,, \tag{8}$$

where T_{tt} is the energy density of the CFT.

Our goal in this letter is to use the entanglement in the CFT, in particular the relative entropy, to elucidate local physics in the bulk (for related recent work see also [3, 7, 11–16]).

Our starting point is a d-dimensional CFT whose vacuum state is dual to anti-de Sitter space (AdS_{d+1}). We consider an arbitrary excited state of the CFT which has a semiclassical holographic bulk dual, whose metric can be parametrized as

$$g_{AdS} = \frac{\ell_{AdS}^2}{z^2}\left[dz^2 + (\eta_{\mu\nu} + h_{\mu\nu})\, dx^\mu dx^\nu\right]\,. \tag{9}$$

Spacetime indices $a, b, \ldots$ run over (z, t, x^i) while $\mu, \nu, \ldots$ run over (t, x^i) and $i \in 1, \ldots, d-1$ are boundary spatial directions. We assume that the Ryu-Takayanagi formula holds in the excited state, and the relative entropy between the reduced density matrix ρ_1 of the excited state and ρ_0 of the ground state for the entangling disk D of radius R is computable using the formulae discussed previously.

To apply a perturbative analysis in the bulk, we assume that the radius R of the entangling domain is small compared to the typical energy scale $\mathcal{E} \approx \langle T_{\mu\nu}\rangle^{\frac{1}{d}}$ of the state measured by the boundary stress tensor $T_{\mu\nu}$ as in $\mathcal{E}R \ll 1$.

In this limit, to order less than $\mathcal{E}^{2d}R^{2d}$, we will show that the relative entropy is expressed as the integral of the local bulk energy density ε,

$$S(\rho_1|\rho_0) = 8\pi^2 G_N \int_V \frac{R^2 - (z^2 + x^2)}{R} \varepsilon \sqrt{g_V}\,, \tag{10}$$

where G_N is Newton's constant, V is a d-dimensional region on a constant-time slice bounded by the domain D on the boundary and the Ryu-Takayanagi surface Σ in the bulk, and $\sqrt{g_V}$ is the volume form in the bulk (including the time direction). In [8], it was shown that the first law $S(\rho_1|\rho_0) = 0$ in the linear approximation is equivalent to the linearized Einstein equation. This holds to order $O(\mathcal{E}^d R^d)$. Our result (10) improves the approximation to the order less than $\mathcal{E}^{2d}R^{2d}$ by taking into account the backreaction from the bulk stress-tensor.

Taking one derivative with respect to R, we find

$$\partial_R S(\rho_1|\rho_0) = 8\pi^2 G_N \int_V \left(1 + \frac{z^2 + x^2}{R^2}\right) \varepsilon \sqrt{g_V}\,. \tag{11}$$

Both positivity $S \geq 0$ and monotonicity $\partial_R S \geq 0$ are universal properties of the relative entropy. We find that, in the gravitational dual, they are translated to positivity of the integrals of the bulk energy density ε weighted by $(R^2 \pm (z^2 + x^2))\sqrt{g_V}$ (≥ 0 in V). In particular, the weak energy condition $\varepsilon \geq 0$ guarantees these properties. Though the weak energy condition is not necessarily satisfied in AdS, it holds near the boundary of AdS, where we are evaluating (10) and (11).

One more derivative relates the relative entropy to the integral of the energy density on the boundary Σ of V,

$$\left(\partial_R^2 + R^{-1}\partial_R - R^{-2}\right) S(\rho_1|\rho_0) = 16\pi^2 G_N \int_\Sigma \varepsilon \sqrt{g_\Sigma}\,, \tag{12}$$

where $\sqrt{g_\Sigma}$ is the volume form on the Ryu-Takayanagi surface. We will show that (12) can be inverted using the Radon transform to express the bulk stress tensor point-by-point in the near-AdS region using the entanglement information of the CFT.

In the holographic setup, it is generally believed that bulk locality emerges from the entanglement information in the boundary CFT and that the relation between bulk and boundary observables is non-local. In this paper, we give an explicit example in which the bulk stress tensor is expressed in terms of the boundary relative entropy, showing how these general expectations are realized in a specific setup.

ΔH_{mod} and ΔS_{EE} in holography.— We first review how each quantity appearing in the relative entropy definition (6) is mapped holographically. The modular Hamiltonian H_{mod} for the reduced density matrix of a CFT vacuum state on the entangling disk D of radius R, centered at a point on the boundary, is expressed in terms of the CFT stress tensor T_{tt} as in (8). It vanishes in the CFT vacuum. We can also use (8) to evaluate the expectation value of H_{mod} for any excited state in the same Hilbert space, by computing the expectation value $\langle T_{tt}\rangle$ of the CFT stress tensor using holographic renormalization (see *e.g.* [17–19]) or the shortcut of [8] to exploit the fact that the relative entropy in the CFT vanishes in the limit that the entangling domain shrinks to zero. As long as the bulk matter fields contributing to $\langle T_{\mu\nu}\rangle$ are dual to operators with scaling dimension $\delta > d/2$, both methods give

$$\Delta\langle H_{mod}\rangle = \lim_{z\to 0} \frac{d\ell_{AdS}^{d-1}}{16 G_N} \int_D d^{d-1}x \frac{R^2 - |\vec{x}|^2}{R} z^{-d} \eta^{ij} h_{ij}\,. \tag{13}$$

In general, the right-hand side is modified by boundary counter terms if it involves operators with $\delta \leq d/2$.

The holographic EE in Einstein gravity is given by the Ryu-Takayanagi area formula (1). On a constant time slice of pure AdS, the codimension-2 bulk extremal surface Σ ending on a boundary sphere of radius R is the half-sphere

$$z_0(r) = \sqrt{R^2 - r^2}\,. \tag{14}$$

The EE of the entangling disk of radius R in the CFT vacuum is equal to the area functional of pure AdS evaluated on the surface (14). Suppose we perturb the bulk metric away from pure AdS by h_{ab} which is parametrically small. Because the original surface was extremal, the leading variation in the holographic EE comes from evaluating $z_0(r)$ (14) on the perturbed area functional. One finds [6]

$$\Delta S_{EE} = \frac{\ell_{AdS}^{d-1}}{8G_N R} \int_{\Sigma} d^{d-1}x(R^2\eta^{ij} - x^i x^j) z^{-d} h_{ij}. \quad (15)$$

At order h^2, one must account for corrections to the shape of the Ryu-Takayanagi surface, see *e.g.* [4].

Linearized Einstein equations.— We now summarize the derivation of the linearized gravitational equations of motion from the entanglement first law (7), as presented in [8]. The idea of [8] was to apply the Stokes theorem to the bulk d-dimensional region V on a constant-time slice bounded by the entangling disk D on the boundary and the extremal surface Σ in the bulk. One can write ΔH_{mod} and ΔS_{EE} as integrals over D and Σ respectively of a local $(d-1)$-form $\boldsymbol{\chi}$ that is a functional of the metric fluctuation h_{ab}. Within Einstein gravity, [8, 20, 21] explicitly construct a $\boldsymbol{\chi}[h_{ab}]$ that gives (13) and (15) when integrated over D and Σ,

$$\int_D \boldsymbol{\chi} = \Delta\langle H_{mod}\rangle, \quad \int_\Sigma \boldsymbol{\chi} = \Delta S_{EE}\,. \quad (16)$$

Moreover, the exterior derivative of this $\boldsymbol{\chi}$ is given by

$$\mathrm{d}\boldsymbol{\chi} = 2\xi^t E^g_{tt}[h] g^{tt}\sqrt{g_V} dz \wedge dx^{i_1} \cdots \wedge dx^{i_{d-1}}, \quad (17)$$

with $\sqrt{g_V}$ the natural volume form on V induced from the bulk spacetime metric, and

$$\xi = \frac{\pi}{R}\left\{[R^2 - z^2 - (t-t_0)^2 - x^2]\partial_t - 2(t-t_0)[z\partial_z + x^i\partial_i]\right\}$$

the Killing vector associated with Σ (14), which is a bifurcate Killing horizon in pure AdS. The linear gravitational equations of motion in vacuum are expressed as $E^g_{ab}[h] = 0$.

By the Stokes theorem, the relative entropy is

$$S(\rho_1|\rho_0) = \Delta\langle H_{mod}\rangle - \Delta S_{EE} = \int_V \mathrm{d}\boldsymbol{\chi}\,. \quad (18)$$

Considering (18) for every disk on a spatial slice at fixed time $t = 0$, the entanglement first law $S(\rho_1|\rho_0) = 0$ can be shown to be equivalent to $E^g_{tt}[h] = 0$. Considering it for Lorentz-boosted frames gives vanishing of the other boundary components, $E^g_{\mu\nu}[h] = 0$. Finally, an argument appealing to the initial-value formulation gives vanishing of the remaining components of the linearized Einstein tensor that carry z indices.

To summarize, [8] proved the existence of a $(d-1)$-form $\boldsymbol{\chi}$ as a functional of a metric fluctuation h_{ab}, for which (16) holds off-shell and (17) holds with $E^g_{ab}[h]$ the linearized gravity equations of motion in vacuum.

By accounting for the $1/N$ correction to the Ryu-Takayanagi formula [22], [9] showed that the entanglement first law (7) implies the bulk linearized Einstein equations sourced by the difference in the quantum expectation value of the bulk stress-energy tensor in the quantum state of bulk fields relative to their vacuum state, $\delta\langle t_{ab}\rangle$. Assuming that the source of the linearized Einstein equation is a local QFT operator, one can then argue that $\delta\langle t_{ab}\rangle$ in the derivation of [9] can be uplifted to the bulk operator t_{ab}. In contrast, in this note, we remain in the large N classical gravity limit, but assume the linearized Einstein equations sourced by the *classical* value of the bulk stress tensor.

Effects due to bulk stress tensor.— We now evaluate the $(d-1)$-form $\boldsymbol{\chi}$ of [8] on the bulk metric fluctuation h_{ab} of the dual to an arbitrary excited state of a CFT, but in the interior of the Ryu-Takayanagi surface for the entangling disk (14), whose radius satisfies $\mathcal{E}R \ll 1$. As the deviation of the bulk metric in the enclosed volume V is parametrically small, all results of the above discussion carry over and we can still use (18). However, $E^g_{ab}[h]$ in (17) should be evaluated on the h_{ab} which is reconstructed from CFT data at non-linear level and is not identically zero. Rather, the linearized Einstein tensor couples to bulk matter in the form of the bulk stress tensor t_{ab}. Our main result (10) can now be derived by using

$$E^g_{ab}[h_{ab}] = 8\pi G_N t_{ab}\,. \quad (19)$$

The energy density ϵ appearing on the right-hand side of (10) corresponds to the tt-component of the stress-energy tensor, $\varepsilon = -t^t{}_t$.

For example, a massive scalar field in the bulk can contribute to the metric perturbation h_{ab} as $\langle\mathcal{O}\rangle^2 z^{2\Delta}$, with Δ the scaling dimension of the corresponding operator on the boundary and $\langle\mathcal{O}\rangle$ its expectation value, leading to an $O(\langle\mathcal{O}\rangle^2 R^{2\Delta})$ effect in (10). On the other hand, corrections to the relative entropy by non-linear gravity effects are of the order $O(\mathcal{E}^{2d}R^{2d})$ or higher, which we ignore. Thus, effects due to relevant operators with $\Delta < d$ are visible in our approximation.

By taking a derivative of (10) with respect to the radius R of the entangling domain, we find (11). Though the derivative also generates an integral over the Ryu-Takayanagi surface Σ, it vanishes because ξ^t vanishes on the surface. Thus, we have shown that the positivity and the monotonicity of the relative entropy are translated to positivity of the integrals of the energy density ε weighted by $(R^2 \pm (z^2 + x^2))\sqrt{g_V}$. In other words, we derive the bulk energy conditions (10), (11) ≥ 0 from entropy inequalities on the boundary that hold for any CFT.

Inverting the bulk integral.— We found that $\partial_R S(\rho_1|\rho_0)$ is given by the integral of the energy density ε over the region V inside the Ryu-Takayanagi surface. We can invert this relation to compute ε point-by-point in the bulk by using the relative entropy $S(\rho_1|\rho_0)$.

To show this, note that

$$\left(\partial_R + R^{-1}\right) S(\rho_1|\rho_0) = 16\pi^2 G_N \int_V \varepsilon\sqrt{g_V} \tag{20}$$

so differentiating again,

$$\left(\partial_R^2 + R^{-1}\partial_R - R^{-2}\right) S(\rho_1|\rho_0) = 16\pi^2 G_N \int_\Sigma \varepsilon\sqrt{g_\Sigma} \tag{21}$$

where $\sqrt{g_\Sigma}$ is the natural volume form on the Ryu-Takayanagi surface Σ induced from the bulk spacetime metric. The right-hand side is still non-negative if we assume the positivity of the bulk energy density. Thus,

$$\left(\partial_R^2 + R^{-1}\partial_R - R^{-2}\right) S(\rho_1|\rho_0) \geq 0. \tag{22}$$

Here the bulk geometry is the unperturbed AdS, and its space-like section is the d-dimensional hyperbolic space. The surface Σ is then totally geodesic. In this case, the integral (21) is the Radon transform and its inverse is known. For a smooth function f on d-dimensional hyperbolic space, the Radon transform $\mathcal{R}f$ is an integral of f over an n-dimensional geodesically complete submanifold with $n < d$. This gives a function on the space of geodesically complete submanifolds. The *dual* Radon transform $\mathcal{R}^*\mathcal{R}f$ gives back a function on the original hyperbolic space in the following way: pick a point in the hyperbolic space, consider all geodesically complete submanifolds passing through the point, and integrate $\mathcal{R}f$ over such submanifolds. It was shown by Helgason [23] that if d is odd, f is obtained by applying an appropriate differential operator on $\mathcal{R}^*\mathcal{R}f$. We are interested in the case $n = d-1$ for which

$$f = \left[(-4)^{(d-1)/2}\pi^{d/2-1}\Gamma(d/2)\right]^{-1} Q(\mathbf{\Delta})\mathcal{R}^*\mathcal{R}f\,, \tag{23}$$

where $Q(\mathbf{\Delta})$ is constructed from the Laplace-Beltrami operator $\mathbf{\Delta}$ on the hyperbolic space as

$$\begin{aligned} Q(\mathbf{\Delta}) = &\left[\mathbf{\Delta} + 1\cdot(d-2)\right]\left[\mathbf{\Delta} + 2\cdot(d-3)\right] \\ &\times \cdots \times \left[\mathbf{\Delta} + (d-2)\cdot 1\right]. \end{aligned} \tag{24}$$

Applying this to (21), we find

$$\begin{aligned} \varepsilon = &\left[(-4)^{(d+3)/2}\pi^{d/2+1}\Gamma(d/2)G_N\right]^{-1} \times \\ &\times Q(\mathbf{\Delta})\mathcal{R}^*\left(\partial_R^2 + R^{-1}\partial_R - R^{-2}\right) S(\rho_1|\rho_0)\,, \end{aligned} \tag{25}$$

when d is odd. There exists a similar formula for d even [24].

Note that even if we are evaluating ε at (z,t,x) in the near-AdS region, there are totally geodesic surfaces that pass through this point and go deep into the bulk, where the geometry can depart significantly from AdS. However, contributions from these surfaces are negligible when $\mathcal{E}z \ll 1$, where $\mathcal{E}$ is the typical energy scale of the CFT state. In this case, we can choose another z_0 so that $z \ll z_0$ and the geometry under z_0 is still approximately AdS. Since most totally geodesic surfaces passing through (z,t,x) stay under z_0, an integral over such surfaces is well-approximated by the inverse Radon transform in the hyperbolic space.

The energy density is the time-time component of the stress-energy tensor t_{ab}. By computing the relative entropy in other Lorentz frames, we can also derive components $t_{\mu\nu}$ along the boundary. Finally, we can use the conservation law, $\nabla^a t_{ab} = 0$, to obtain the remaining components, $t_{z\mu}, t_{\mu\nu}$. Thus, we can use the entanglement data on the boundary to reconstruct all components of the bulk stress tensor.

Since the Radon transform preserves positivity, the positivity of the energy density implies the positivity of $\left(\partial_R^2 + R^{-1}\partial_R - R^{-2}\right) S(\rho_1|\rho_0)$. Conversely, the positivity of the latter implies the positivity of its dual Radon transform. It is interesting to note that $Q(\mathbf{\Delta})$ in (25) is a positive definite operator when acting on normalizable functions on the hyperbolic space, though this does not quite imply the positivity of the energy density.

Comparison with information theoretic analysis.— We now discuss to what extent we can recover the monotonicity and convexity (12) of the relative entropy from the following general property of the relative entropy. Consider a density matrix ρ (with $\rho^* = \rho$, $\rho \geq 0$, $\mathrm{tr}(\rho) = 1$), and two increments h, ℓ, given by matrices with $h = h^*$, $\ell = \ell^*$, $\mathrm{tr}(h) = \mathrm{tr}(\ell) = 0$. If the matrices ρ, h, ℓ satisfy $[\rho, h] = [\rho, \ell] = 0$, the relative entropy satisfies

$$S(\rho + h|\rho + \ell) \sim \langle (h-\ell), \frac{1}{2}\rho^{-1}(h-\ell)\rangle, \tag{26}$$

where the right-hand-side is the Fisher metric, with the Hilbert–Schmidt inner product $\langle a, b\rangle = \mathrm{tr}(a^* b)$. Thus, the second order term is non-negative definite, and the quadratic form only vanishes for $h = \ell$.

The entanglement density matrices $\rho(R)$ and $\rho_0(R)$ discussed in this paper have additional properties for small R. Since H_{mod} is given by the integral (8), the Taylor expansion of T_{tt} around $\vec{x} = \vec{x}_0$ gives $H_{mod} = h_0 R^d + \cdots$. Therefore, the density matrix for the vacuum state can be expanded as

$$\rho_0(R) = \frac{1}{\mathcal{N}} - h_0' R^d + \cdots\,, \tag{27}$$

where $\mathrm{tr}\ 1 = \mathcal{N}$ and $h_0' = h_0 - \frac{1}{\mathcal{N}}\mathrm{tr}\ h_0$ so that $\mathrm{tr}\ h_0' = 0$. For $\rho(R)$, we postulate

$$\rho(R) = \frac{1}{\mathcal{N}} + \sum_i \ell_i R^{\delta_i} + hR^d + \cdots\,, \tag{28}$$

so that the small R expansion of the relative entropy $S = \sum_i R^{2\delta_i} s_i + \cdots$ expected from the holographic computation above is reproduced. Here $\mathrm{tr}\ \ell_i = 0$ and δ_i's are scaling dimensions of relevant operators, $\delta_i < d$.

The right-hand-side of (26) becomes

$$\sum_{ij} \frac{\mathcal{N}}{2} \langle \ell_i, \ell_j \rangle R^{\delta_i+\delta_j}. \tag{29}$$

Thus, the leading order term of $S(\rho_1|\rho_0)$ is

$$S(\rho_1|\rho_0) \sim \frac{\mathcal{N}}{2} |\ell_1|^2 R^{2\delta_1}, \tag{30}$$

where $\delta_1 = \min_i\{\delta_i\}$. Its first and second derivatives in R have leading term

$$\partial_R S(\rho_1|\rho_0) \sim \mathcal{N}\delta_1 |\ell_1|^2 R^{2\delta_1-1}, \tag{31}$$
$$\left(\partial_R^2 + R^{-1}\partial_R - R^{-2}\right) S(\rho_1|\rho_0) \sim \frac{\mathcal{N}}{2} |\ell_1|^2 (4\delta_1^2 - 1) R^{2\delta_1-2}.$$

The first is manifestly positive, and the second is non-negative provided $\delta_1 \geq 1/2$, which is satisfied by our assumption $\delta_1 > d/2$ for $d \geq 2$.

Our holographic analysis shows that the positivity and convexity of the relative entropy hold for subleading terms up to $O(R^{2d})$. On the other hand, corrections to (30) may involve not only quadratic terms with $\delta_i + \delta_j < 2d$, but also cubic terms with $\delta_i + \delta_j + \delta_k < 2d$, *etc*. It appears that additional assumptions on the density matrices are required to explain the convexity from this point of view.

Acknowledgments.— We thank N. Bao, D. Gaiotto, D. Harlow, T. Hartman, P. Hayden, N. Hunter-Jones, C. Keller, D. Kutasov, H. Liu, Y. Nakayama, S. Pufu, P. Sulkowski, T. Takayanagi, M. Van Raamsdonk and E. Witten for useful discussion. JL acknowledges support from the Sidney Bloomenthal fellowship at the University of Chicago. MM is currently supported by NSF grants PHY-1205440, DMS-1201512, and DMS-1007207. HO and BS are supported in part by the Walter Burke Institute for Theoretical Physics at Caltech, by U.S. DOE grant DE-SC0011632, and by a Simons Investigator award. The work of HO is also supported in part by the WPI Initiative of MEXT of Japan, and JSPS Grant-in-Aid for Scientific Research C-26400240. He also thanks the hospitality of the Aspen Center for Physics and the National Science Foundation, which supports the Center under Grant No. PHY-1066293. BS is supported in part by a Dominic Orr Graduate Fellowship. JL, HO and BS would like to thank the Institute for Advanced Study, Princeton University, and the Simons Center for Geometry and Physics for hospitality. JL also thanks Caltech for hospitality.

---

[1] S. Ryu and T. Takayanagi, Phys.Rev.Lett. **96**, 181602 (2006), arXiv:hep-th/0603001 [hep-th].
[2] S. Ryu and T. Takayanagi, JHEP **0608**, 045 (2006), arXiv:hep-th/0605073 [hep-th].
[3] M. Nozaki, T. Numasawa, A. Prudenziati, and T. Takayanagi, Phys.Rev. **D88**, 026012 (2013), arXiv:1304.7100 [hep-th].
[4] D. D. Blanco, H. Casini, L.-Y. Hung, and R. C. Myers, JHEP **1308**, 060 (2013), arXiv:1305.3182 [hep-th].
[5] G. Wong, I. Klich, L. A. P. Zayas, and D. Vaman, Journal of High Energy Physics **12**, 20 (2013), arXiv:1305.3291 [hep-th].
[6] N. Lashkari, M. B. McDermott, and M. Van Raamsdonk, JHEP **1404**, 195 (2014), arXiv:1308.3716 [hep-th].
[7] J. Bhattacharya and T. Takayanagi, JHEP **1310**, 219 (2013), arXiv:1308.3792 [hep-th].
[8] T. Faulkner, M. Guica, T. Hartman, R. C. Myers, and M. Van Raamsdonk, JHEP **1403**, 051 (2014), arXiv:1312.7856 [hep-th].
[9] B. Swingle and M. Van Raamsdonk, (2014), arXiv:1405.2933 [hep-th].
[10] H. Casini, M. Huerta, and R. C. Myers, JHEP **1105**, 036 (2011), arXiv:1102.0440 [hep-th].
[11] J. Bhattacharya, M. Nozaki, T. Takayanagi, and T. Ugajin, Phys.Rev.Lett. **110**, 091602 (2013), arXiv:1212.1164.
[12] S. Banerjee, A. Bhattacharyya, A. Kaviraj, K. Sen, and A. Sinha, JHEP **1405**, 029 (2014), arXiv:1401.5089 [hep-th].
[13] S. Banerjee, A. Kaviraj, and A. Sinha, (2014), arXiv:1405.3743 [hep-th].
[14] V. Balasubramanian, B. D. Chowdhury, B. Czech, J. de Boer, and M. P. Heller, Phys. Rev. D **89**, 086004 (2014), arXiv:1310.4204 [hep-th].
[15] V. E. Hubeny, Journal of High Energy Physics **9**, 156 (2014), arXiv:1406.4611 [hep-th].
[16] B. Czech and L. Lamprou, ArXiv e-prints (2014), arXiv:1409.4473 [hep-th].
[17] V. Balasubramanian and P. Kraus, Commun.Math.Phys. **208**, 413 (1999), arXiv:hep-th/9902121 [hep-th].
[18] S. de Haro, S. N. Solodukhin, and K. Skenderis, Commun.Math.Phys. **217**, 595 (2001).
[19] K. Skenderis, Class.Quant.Grav. **19**, 5849 (2002), arXiv:hep-th/0209067 [hep-th].
[20] R. M. Wald, Phys. Rev. D **48**, 3427 (1993), gr-qc/9307038.
[21] V. Iyer and R. M. Wald, Phys.Rev. **D50**, 846 (1994), arXiv:gr-qc/9403028 [gr-qc].
[22] T. Faulkner, A. Lewkowycz, and J. Maldacena, JHEP **1311**, 074 (2013), arXiv:1307.2892.
[23] S. Helgason, Acta Math. **102**, 239 (1959).
[24] B. Rubin, Adv. Math **170**, 206 (2002).