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**Wigner translations and the observer-dependence of the position
of massless spinning particles**

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Abstract

The Wigner little group for massless particles is isomorphic to the Euclidean group $SE(2)$. Applied to momentum eigenstates, or to infinite plane waves, the Euclidean “Wigner translations” act as the identity. We show that when applied to finite wavepackets the translation generators move the packet trajectory parallel to itself through a distance proportional to the particle’s helicity. We relate this effect to the spin Hall effect of light and to the Lorentz-frame dependence of the position of a massless spinning particle.

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I. INTRODUCTION

The Poincaré group provides the fundamental kinematic symmetry of a relativistic particle. As a non-compact group its unitary representations are infinite dimensional — but Wigner showed [1] that the physically interesting representations can be induced from finite-dimensional unitary representations of a *little group*, which is the subgroup of homogeneous Lorentz transformations that leaves some reference four-momentum p_0^μ invariant. The representation space of the little group is the Hilbert space for the particle’s spin. If the particle has positive mass m , we may take as reference the four-momentum in the particle’s rest frame where $p_0^\mu = (m, \mathbf{0})$. The little group then consists of the space rotations $\text{SO}(3)$. For a massless particle there is no rest frame and the reference momentum must be a null vector $p_0^\mu \rightarrow (|\mathbf{p}_0|, \mathbf{p}_0)$. The little group now consists of $\text{SO}(2)$ space rotations about the three-vector \mathbf{p}_0 , together with operations that are generated by infinitesimal Lorentz boosts in directions perpendicular to \mathbf{p}_0 combined with compensating infinitesimal rotations. Remarkably the combined operations mutually commute, possess all the algebraic properties of Euclidean translations, and the resulting little group is isomorphic to the symmetry group $\text{SE}(2)$ of the two-dimensional Euclidean plane. What are these Euclidean “Wigner translations” translating? The answer given by Wigner is nothing: if the translation generators had a physical effect, the little-group representation would be infinite dimensional and the particle being described would have “continuous spin” — a property possessed by no known particle. Indeed the Wigner translations have no effect when applied to plane-wave solutions of the massless Dirac equation, and act as gauge transformations when applied to the vector potentials of plane-wave solutions of Maxwell’s equations [2]. Consequently they act as the identity on the momentum eigenstates created by the operator-valued coefficients of the plane-wave modes, thus ensuring that the spin of a massless particle is entirely specified by a finite-dimensional representation of the $\text{SO}(2)$ helicity subgroup [3].

Here we show that, while they have no effect on *infinite* plane waves, when applied to *finite-size* wave packets of non-zero helicity — and in particular to circularly polarized Gaussian packets — the Wigner translations do have an effect: they shift the wave-packet trajectory parallel to itself. This shift is related to the optical spin Hall effect [4–8] and to the observer dependence of the location of massless particles [9]. It gives rise to the unusual Lorentz covariance properties found [10, 11] in the chiral kinetic theory approach

to anomalous conservation laws [12–14] and is also the source of the difficulty of obtaining a conventionally-covariant classical mechanics for a massless spinning particle in a gravitational field [15, 16].

In section II we will provide a suggestive algebraic argument for a sideways shift. In section III we will show that the shift actually occurs in finite-width beam solutions to Maxwell’s equations. In section IV we will discuss and resolve a potential paradox implied by the trajectory displacement.

II. POINCARÉ ALGEBRA AND MASSLESS PARTICLES

As an indication that Wigner translations can have a physical effect, we review a realization of the Poincaré algebra for massless particles of helicity λ in terms of quantum mechanical position and momentum operators [17–20]. We start from the familiar commutators

$$[\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad (1)$$

and use the fact that the spin of a massless particle is slaved to its direction of motion to motivate a definition of the angular momentum operator as

$$J_k = \epsilon_{klm} \hat{x}_l \hat{p}_m + \lambda \frac{\hat{p}_k}{|\hat{\mathbf{p}}|}. \quad (2)$$

This unconventional definition preserves the usual commutation relation

$$[J_k, \hat{p}_l] = i\hbar \epsilon_{klm} \hat{p}_m, \quad (3)$$

but in order to recover

$$[J_k, \hat{x}_l] = i\hbar \epsilon_{klm} \hat{x}_m \quad (4)$$

and

$$[J_k, J_l] = i\hbar \epsilon_{klm} J_m, \quad (5)$$

we need to modify the commutator of the position-operator components to

$$[\hat{x}_k, \hat{x}_l] = -i\hbar \lambda \epsilon_{klm} \frac{\hat{p}_m}{|\hat{\mathbf{p}}|^3}. \quad (6)$$

Accepting that the position-operator components no longer commute, we can still use $p^0 \equiv |\mathbf{p}|$ to define a generator of Lorentz boosts in direction k as

$$K_k = \frac{1}{2}(\hat{x}_k |\hat{\mathbf{p}}| + |\hat{\mathbf{p}}| \hat{x}_k). \quad (7)$$

These generators satisfy the remaining relations of the Lorentz Lie algebra

$$\begin{aligned} [J_k, K_l] &= i\hbar \epsilon_{klm} K_m, \\ [K_k, K_l] &= -i\hbar \epsilon_{klm} J_m, \end{aligned} \quad (8)$$

and act as expected on the momentum components:

$$\begin{aligned} [K_k, |\hat{\mathbf{p}}|] &= i\hbar \hat{p}_k, \\ [K_k, \hat{p}_l] &= i\hbar \delta_{kl} |\hat{\mathbf{p}}|. \end{aligned} \quad (9)$$

We have therefore constructed a representation of the Poincaré algebra on a quantum-mechanical Hilbert space.

When we extend the algebra to include the position operators we find (at $t = 0$)

$$[K_k, \hat{x}_l] = -i\hbar \left\{ \frac{1}{2} \left(\hat{x}_k \frac{\hat{p}_l}{|\hat{\mathbf{p}}|} + \frac{\hat{p}_l}{|\hat{\mathbf{p}}|} \hat{x}_k \right) + \lambda \epsilon_{klm} \frac{\hat{p}_m}{|\hat{\mathbf{p}}|^2} \right\}. \quad (10)$$

Neither term is immediately familiar. The expression in parenthesis arises because the underlying Hamiltonian formalism automatically maintains the non-Lorentz invariant condition $x_0 = t$ [21]. The term containing the helicity λ will be more interesting.

We select a reference four-momentum $p_0^\mu = (|\mathbf{p}_0|, \mathbf{p}_0)$ where $\mathbf{p}_0 = (0, 0, p)$ and obtain the corresponding Wigner translation generators as the boosts and compensating rotations given by

$$\begin{aligned} \Pi_1 &= K_1 + J_2, \\ \Pi_2 &= K_2 - J_1. \end{aligned} \quad (11)$$

From (8) we see that these generators obey the SE(2) Lie algebra

$$[\Pi_1, \Pi_2] = 0, \quad [J_3, \Pi_1] = i\hbar \Pi_2, \quad [J_3, \Pi_2] = -i\hbar \Pi_1. \quad (12)$$

From (9) and (10) we also see that \hat{x}_1 , \hat{x}_2 , and the SE(2) generators collectively leave invariant the eigenspace with eigenvalues $\mathbf{p} = (0, 0, p)$ and any fixed x_3 . Acting within the particular invariant subspace with $x_3 = 0$, we find that

$$[\Pi_k, \hat{x}_l] = -i\hbar \epsilon_{kl3} \frac{\lambda}{p}, \quad (k, l = 1, 2). \quad (13)$$

In (13) the Wigner “translations” seemingly effect a genuine infinitesimal translation of the x_1 , x_2 coordinates in the $x_3 = 0$ plane, and hence a translation of the particle trajectory

$\mathbf{x}(t) = (x_1, x_2, t)$ parallel to itself. Is this apparent displacement merely an artifact of an unconventional representation of the Poincaré algebra, or is it physical?

At issue is the question: “what is \mathbf{x} ?”. There are several distinct candidates for the position operator of a relativistic particle that differ in both their commutation and covariance properties [19, 20, 22]. In the next section we will use solutions of Maxwell’s equations to demonstrate that the particular non-commuting position operator \mathbf{x} in eq. (13) represents the wavepacket energy-centroid in the observer’s $x_0 \equiv t$ Lorentz frame. This centroid is where a stationary detector would locate the photon, but is a frame-dependent quantity. As a consequence the trajectory of a circularly polarized photon is observer dependent and is translated parallel to itself under an infinitesimal Lorentz boost and aberration-compensating rotation.

III. PARAXIAL MAXWELL BEAMS

We wish to consider the action of boosts and rotations on a finite-sized photon wavepacket. It will serve to consider their effect on a finite-width laser beam in the paraxial approximation. We will use units in which $\mu_0 = \epsilon_0 = c = 1$.

The scalar paraxial wave equation

$$\frac{\partial^2 \chi}{\partial x^2} + \frac{\partial^2 \chi}{\partial y^2} + 2ki \frac{\partial \chi}{\partial z} = 0 \quad (14)$$

is obtained from the full scalar wave equation by writing

$$\phi(\mathbf{r}, t) = \chi(\mathbf{r}) e^{ik(z-t)} \quad (15)$$

and assuming that $\chi(\mathbf{r})$ is sufficiently slowly varying that we can ignore its second derivative $\partial^2 \chi / \partial z^2$ in comparison to the remaining terms in (14).

The simplest solution of eq. (14) is the Gaussian-beam [23]

$$\chi(\mathbf{r}) = \frac{1}{(z - iz_0)} \exp \left\{ -\frac{x^2 + y^2}{2w^2(z)} + ik \frac{x^2 + y^2}{2R(z)} \right\}, \quad (16)$$

where

$$w^2(z) = \frac{z^2 + z_0^2}{kz_0}; \quad R(z) = \frac{z^2 + z_0^2}{z}. \quad (17)$$

In this solution the beam is propagating in the $+z$ direction, the quantity $w(z)$ is the width of the beam at a distance z from its waist, and $R(z)$ is the radius of curvature of the wavefront

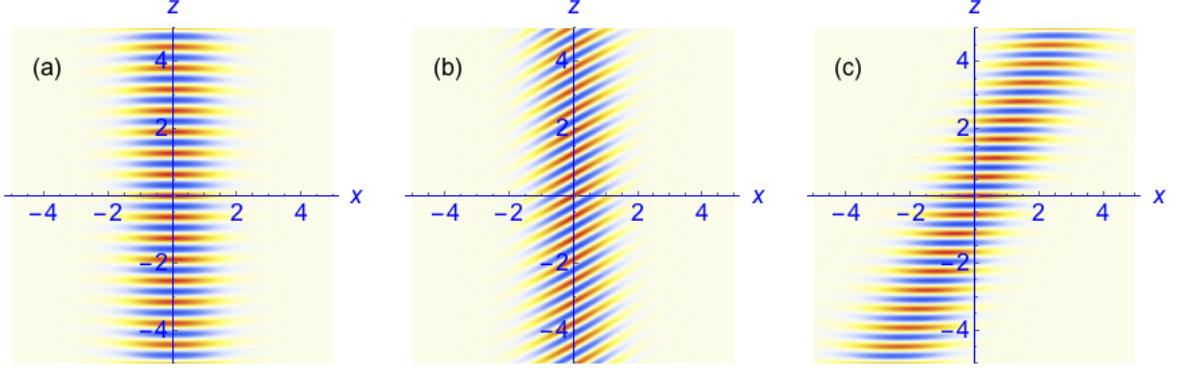


FIG. 1: Slice through a paraxial scalar beam with parameters $k = 10$, $z_0 = 10$. a) Density plot of original beam amplitude $\text{Re}\{\chi(x, 0, z, 0)e^{ikz}\}$; b) Beam amplitude after Lorentz transformation (eq. (25)) with rapidity $s = 0.5$; c) Beam amplitude after both Lorentz transformation and aberration-compensating rotation through $\theta = -\tan^{-1}(\sinh s) = -31.5^\circ$.

passing through the point $\mathbf{r} = (0, 0, z)$. The width grows linearly with z once $z \gg z_0$, and the angular half-width is $1/kw(0)$. The condition for the paraxial approximation to be accurate ($kz_0 \gg 1$) is equivalent to the beam having small asymptotic divergence. We will always be interested in the region $z < z_0$ where the beam is narrow and almost parallel sided.

From any two independent solutions f, g of the scalar paraxial equation we can find [24] vector \mathbf{E} and \mathbf{B} fields that are internally consistent solutions of Maxwell's equations up to accuracy of order $1/(kl)^2$, where l is some characteristic length such as z_0

$$\begin{aligned}
 E_x(\mathbf{r}, t) &= f(\mathbf{r}, t) + \frac{1}{4k^2} \left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} \right) + \frac{1}{2k^2} \frac{\partial^2 g}{\partial x \partial y}, \\
 E_y(\mathbf{r}, t) &= g(\mathbf{r}, t) - \frac{1}{4k^2} \left(\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial y^2} \right) + \frac{1}{2k^2} \frac{\partial^2 f}{\partial x \partial y}, \\
 E_z(\mathbf{r}, t) &= \frac{i}{k} \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} \right),
 \end{aligned} \tag{18}$$

and

$$\begin{aligned}
 B_x(\mathbf{r}, t) &= -g(\mathbf{r}, t) + \frac{1}{4k^2} \left(\frac{\partial^2 g}{\partial x^2} - \frac{\partial^2 g}{\partial y^2} \right) + \frac{1}{2k^2} \frac{\partial^2 f}{\partial x \partial y}, \\
 B_y(\mathbf{r}, t) &= f(\mathbf{r}, t) - \frac{1}{4k^2} \left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} \right) - \frac{1}{2k^2} \frac{\partial^2 g}{\partial x \partial y}, \\
 B_z(\mathbf{r}, t) &= -\frac{i}{k} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right).
 \end{aligned} \tag{19}$$

To obtain a Gaussian beam that is circularly polarized with positive helicity we take $f(\mathbf{r}, t) =$

$\chi(\mathbf{r})e^{ik(z-t)}$ and $g(\mathbf{r}, t) = i\chi(\mathbf{r})e^{ik(z-t)}$, with $\chi(\mathbf{r})$ given by eq. (16).

Using MathematicaTM to manipulate the resulting rather lengthy expressions we find, for example, that the time-averaged energy density in the beam is

$$T^{00} \equiv \frac{1}{2}\langle |\mathbf{E}|^2 + |\mathbf{B}|^2 \rangle = \frac{(x^2 + y^2 + 4(z^2 + z_0^2))^2}{8(z^2 + z_0^2)^3} e^{-kz_0(x^2+y^2)/(z^2+z_0^2)}, \quad (20)$$

and the three components of the time-averaged Poynting vector $\mathbf{S} = \langle \mathbf{E} \times \mathbf{B} \rangle$ are

$$\begin{aligned} S_x &= T^{10} = \frac{(x^3z - x^2yz_0 + xy^2z - y^3z_0 + 4(xz - yz_0)(z^2 + z_0^2))}{2(z^2 + z_0^2)^3} e^{-kz_0(x^2+y^2)/(z^2+z_0^2)}, \\ S_y &= T^{20} = \frac{(y^3z + y^2xz_0 + x^2yz + x^3z_0 + 4(yz + xz_0)(z^2 + z_0^2))}{2(z^2 + z_0^2)^3} e^{-kz_0(x^2+y^2)/(z^2+z_0^2)}, \\ S_z &= T^{30} = \frac{(-x^4 - 2x^2y^2 - y^4 + 16(z^2 + z_0^2)^2)}{8(z^2 + z_0^2)^3} e^{-kz_0(x^2+y^2)/(z^2+z_0^2)}. \end{aligned} \quad (21)$$

The energy-flux streamlines twist in the direction of the beam helicity [25], consequently the z component of the angular momentum density

$$\Sigma_z = xS_y - yS_x \quad (22)$$

is non-zero. If we integrate over the plane $z = 0$ we find that

$$P_z \stackrel{\text{def}}{=} \iint_{z=0} S_z dx dy = \frac{\pi(-1 + 8k^2z_0^2)}{4k^3z_0^3} = \frac{2\pi}{kz_0} \left\{ 1 + O\left(\frac{1}{(kz_0)^2}\right) \right\}, \quad (23)$$

and

$$J_z \stackrel{\text{def}}{=} \iint_{z=0} \Sigma_z dx dy = \frac{\pi(1 + 2kz_0)}{k^3z_0^2} = \frac{1}{k} \frac{2\pi}{kz_0} \left\{ 1 + O\left(\frac{1}{(kz_0)^2}\right) \right\}. \quad (24)$$

The ratio P_z/J_z is equal to k in the region $kz_0 \gg 1$, where the paraxial approximation is accurate. This is what is to be expected: P_z gives the linear momentum per unit length, which should be $\hbar k$ per photon; J_z gives the angular momentum per unit length of the beam, which should be \hbar per photon.

We now compute the \mathbf{E} and \mathbf{B} fields as seen from a reference frame moving along the $+x$ axis at rapidity s . The corresponding Lorentz transformation takes

$$\begin{aligned} E_x(x, y, z, t) &\mapsto E_x(x', y, z, t'), \\ E_y(x, y, z, t) &\mapsto E_y(x', y, z, t') \cosh s - B_z(x', y, z, t') \sinh s \\ E_z(x, y, z, t) &\mapsto E_z(x', y, z, t') \cosh s + B_y(x', y, z, t') \sinh s, \\ B_x(x, y, z, t) &\mapsto B_x(x', y, z, t') \\ B_y(x, y, z, t) &\mapsto B_y(x', y, z, t') \cosh s + E_z(x', y, z, t') \sinh s, \\ B_z(x, y, z, t) &\mapsto B_z(x', y, z, t') \cosh s - E_y(x', y, z, t') \sinh s, \end{aligned} \quad (25)$$

where

$$\begin{aligned}x' &= x \cosh s + t \sinh s \\t' &= t \cosh s - x \sinh s.\end{aligned}\tag{26}$$

The Lorentz transformation changes the wave vector from $\mathbf{k} = (0, 0, k)$ to $\mathbf{k}' = (k \sinh s, 0, k)$, so the direction of propagation has been rotated through an aberration angle of $|\theta| = \tan^{-1}(\sinh s)$. The wavefronts are therefore tilted. The beam envelope, however, still lies parallel to the z -axis, and is moving towards the observer at speed $\beta = \tanh s$ (see figure 1-b).

The Lorentz transformation also affects the energy density distribution and the Poynting-vector flux through the $z = 0$ plane. In addition to a Lorentz contraction it noticeably shifts the positions of their maxima (see figure 2). To quantify these shifts we can compute the location of the Lorentz transformed energy density and energy flux centroids. The required integrals are still Gaussian and can be done analytically. With the definition

$$\mathcal{E} = \iint_{z=0} T^{00} dx dy,\tag{27}$$

we have

$$\begin{aligned}[\Delta y]_{\text{density}} &= \frac{1}{\mathcal{E}} \iint_{z=0} y T^{00} dx dy, \\ &= \frac{z_0(4 + 8kz_0 \sinh s)}{(1 + 8kz_0 + 8k^2z_0^2) \cosh s - 4kz_0 \operatorname{sech} s}, \\ &= \frac{1}{k} \tanh s \left\{ 1 + O\left(\frac{1}{(kz_0)^2}\right) \right\},\end{aligned}\tag{28}$$

and

$$\begin{aligned}[\Delta y]_{\text{flux}} &= \frac{1}{P_z} \iint_{z=0} y S_z dx dy, \\ &= \frac{2z_0(1 - 2kz_0) \tanh s}{1 - 8k^2z_0^2}, \\ &= \frac{1}{2k} \tanh s \left\{ 1 + O\left(\frac{1}{(kz_0)^2}\right) \right\}.\end{aligned}\tag{29}$$

For positive helicity, both centroids are displaced to the left when seen by an observer moving towards the upward-propagating beam. The centroids do not coincide, the energy-flux centroid moving only half as far as the energy-density centroid. Such displacements are not restricted to Gaussian beams. A similar boost-induced sideways shift and centroid

separation was exhibited in [9] for Bessel beams possessing orbital angular momentum. The centroid separation arises solely from the geometrical effect pointed out in [26]: because of their corkscrew trajectories, energy-flux streamlines passing through a surface rotated away from perpendicular to the direction of propagation find themselves inclined at different angles to the surface to the right and left of the plane of rotation. Consequently, even in the absence of a Lorentz boost, the energy-flux centroid of a tilted beam is displaced with respect to its energy-density centroid [26].

We wish to obtain a finite-displacement version of the Wigner translations, so, after performing the boost, we rotate the Lorentz transformed beam about $\mathbf{r} = 0$ through an aberration-compensating angle of $\tan^{-1}(\sinh s)$. After the rotation the wavevector becomes $\mathbf{k} = (0, 0, k \cosh s)$ and the wavefronts again lie parallel to the x - y plane. Consequently the energy-flux streamlines no longer possess a left-right asymmetry. We find numerically that the position of the energy centroid in the $z = 0$ plane is unchanged by the rotation (T^{00} is a scalar under space rotations) while the energy-flux centroid moves into coincidence with the energy-density centroid. As a result of the combined boost and compensating rotation, *both* centroids have been shifted by $\Delta y = (1/k) \tanh s = \beta/k$, where $\beta = v/c$. The beam spot is restored to its pre-boost appearance, and we could repeat the operation and translate the beam spot through a further distance. If we reverse the helicity, we change the sign of this shift.

In the absence of the lateral shift, the combination of boost and compensating rotation would leave the trajectory of a short wavepacket emitted from $\mathbf{r} = 0$ at $t = 0$ unchanged. The continuous beam, which can be thought of as arising from a stream of sequentially emitted wavepackets, is *not* invariant, however. How it changes is shown in fig. 1-c. The transformed beam can be thought of as a sequence of pulses each fired in the $+z$ direction by an emitter that is moving rapidly to the left. It is reminiscent of a diagonal stream of strictly upward-moving projectiles fired from a horizontally moving gun in the old AtariTM game “Space Invaders.” Any particular packet continues to move parallel to the z axis, but as a result of the lateral shift in the $z = 0$ plane, its entire trajectory is shifted sideways by $\Delta y = (1/k) \tanh s$. Figure 1-c also shows why the action of the Wigner translations take their simple form (13) only in the plane $x_3 \equiv z = 0$. In any other plane the translations get mixed up with the geometric effect of the rotation.

The finite- s boosts considered in this section have effects on the photon energy and

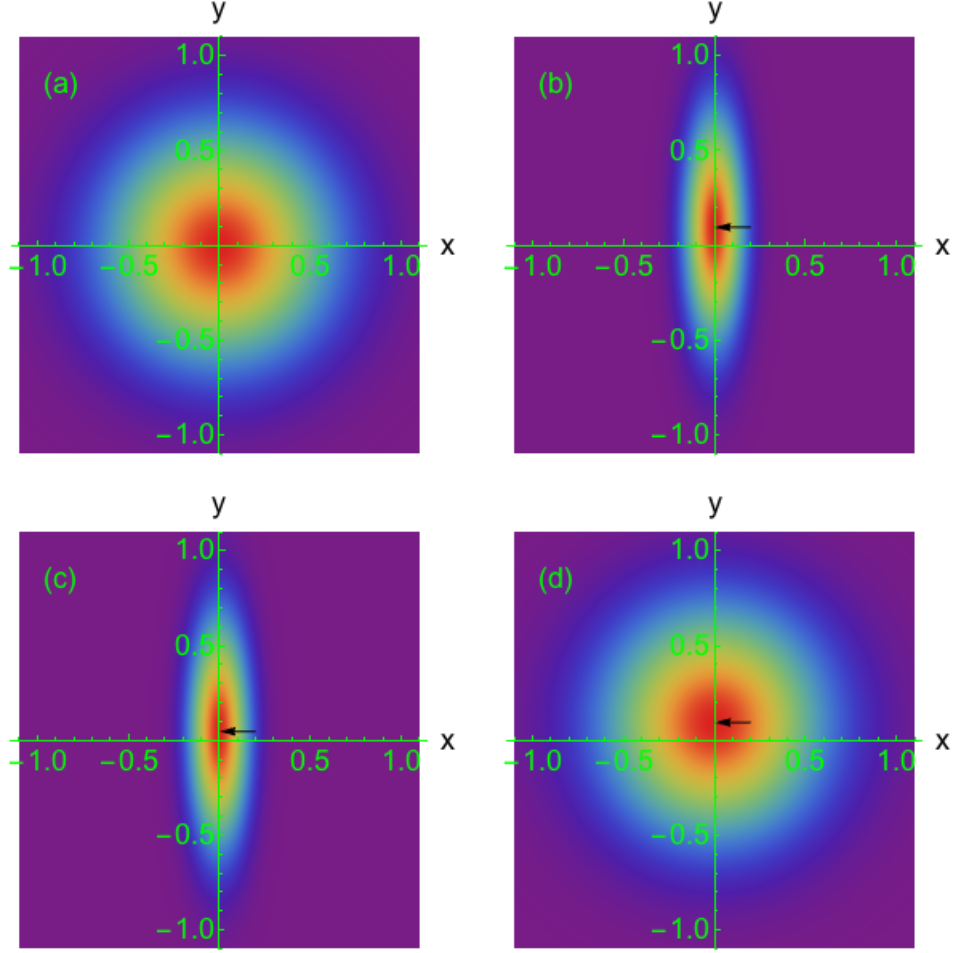


FIG. 2: Beam spot profiles in the $z = 0$ plane for $k = 10$, $z_0 = 3$. a) Original intensity $T^{00}(x, y)$; b) Lorentz transformation of $T^{00}(x, y)$ under eq. (25) with rapidity $s = 2.0$. The spot center is at $y = 0.095$ (black arrow); c) Poynting energy flux $S_z(x, y) = T^{30}$ after Lorentz transformation. The spot center is at $y = 0.0475$ (black arrow); d) Poynting energy flux $S_z(x, y) = T^{30}(x, y)$ after aberration-compensating rotation. The flux maximum is at $y = 0.095$ (black arrow). The rotated intensity distribution has similar appearance, and its maximum is also at $y = 0.095$.

intensity that appear at quadratic order in the rapidity s . If we alternate a sequence of infinitesimal boosts and compensating rotations, the quadratic terms can be neglected and only the sideways shift (now equal to λ/p times the total rapidity change) remains. We are in effect assembling a Trotter-product approximation that converges to exponentials of the Wigner translation generators (11).

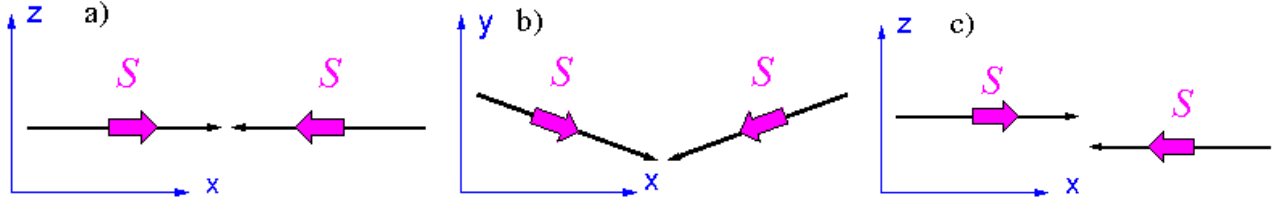


FIG. 3: a) A pair of massless particles with spin \mathbf{S} collide head-on; b) The particles viewed from above in a frame moving towards the collision with velocity $\mathbf{v} = v|\hat{\mathbf{y}}$; c) A front view from the moving frame shows the particles miss one another.

IV. DISCUSSION

The direction and magnitude of the boost-induced lateral shift can be understood from a geometric picture (See [10] for a related argument). Consider two massless particles, both possessing helicity $\mathbf{p} \cdot \mathbf{S}_{\text{spin}}/|\mathbf{p}| = \lambda$ and heading directly towards one another parallel to the x axis. Because they will collide head-on, they have no relative orbital angular momentum and the two spin angular momenta $\mathbf{S}_{\text{spin}} = (\pm\lambda, 0, 0)$ also sum to zero. Seen from a frame moving along the y axis towards the collision point, however, the unit vectors in the direction of the particles' motion have components $(\pm \text{sech } s, -\tanh s, 0)$. Because the spin of a massless particle is slaved to its direction of motion, there is now a net spin component of $2\lambda \tanh s$ directed towards the observer. Nonetheless, in the new frame, the total angular momentum is still zero, so the spin contribution must be offset by an orbital angular momentum of $-2\lambda \tanh s$. This orbital angular momentum can only come from a lateral shift of each particle's trajectory by $\Delta z = (\pm\lambda/|\mathbf{p}|) \tanh s$ (see fig. 3). For a photon $\mathbf{p} = \hbar \mathbf{k}$ and $\lambda = \hbar$, so we recover the shift seen in our Gaussian beam. Of course, if two particles collide and produce two pions in one frame, they must produce two pions when seen from another frame. That the particles apparently miss each other because of the sideways shift cannot affect the pion production. The incipient paradox is resolved by the fact that partial-wave scattering amplitudes depend only on the total relative angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}_{\text{spin}}$ of the particles, and this quantity is not affected by the shift. The shift still has physical consequences, though. If we move a detector such as a photographic emulsion through the beam, it will be sensitive to either the energy density or the energy flux in its own rest frame, and these quantities have been displaced by the motion.

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