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Entanglement entropy of the $\nu = 1/2$ composite fermion non-Fermi liquid state.

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The so-called "non-Fermi liquid" behavior is very common in strongly correlated systems. However, its operational definition in terms of "what it is not" is a major obstacle for theoretical understanding of this fascinating correlated state. Recently there has been much interest in entanglement entropy as a theoretical tool to study non-Fermi liquids. So far explicit calculations have been limited to models without direct experimental realizations. Here we focus on a two dimensional electron fluid under magnetic field and filling fraction $\nu = 1/2$, which is believed to be a non-Fermi liquid state. Using a composite fermion wave-function which captures the $\nu = 1/2$ state very accurately, we compute the second Rényi entropy using variational Monte-Carlo technique. We find the entanglement entropy scales as $L \log L$ with the length of the boundary L as it does for free fermions, but has a pre-factor twice that of free fermions.

Despite its ubiquity in strongly correlated materials, the metallic "non-Fermi liquid" (nFL) behavior has been challenging to characterize theoretically. At the phenomenological level, non-Fermi liquid behavior is defined by a metallic system exhibiting physical properties that are qualitatively inconsistent with Landau's Fermi-liquid theory. Examples of non-Fermi liquid metals include the strange-metal phase of the high T_c cuprates[1], systems near a metallic quantum critical point[2–4] and two-dimensional electron system subject to a magnetic field at filling $\nu = 1/2$ (often referred to as Fermi-liquid-like state) [5–7]. However, there are many ways in which a system can deviate from a normal Fermi-liquid, such as diverging effective mass, vanishing quasiparticle weight, and anomalous transport[3, 8–12] and little is known about how different forms of deviation can be related. Hence the theoretical challenge of addressing a problem without a weakly interacting quasiparticle description has been compounded by the lack of a measure that can be used to define and classify non-Fermi liquids.

Here we turn to a quantum information measure that is sensitive to entangled nature of many-body wave-functions: the bi-partite entanglement entropy. For gapped systems, the entanglement entropy of the reduced density matrix $\rho_A \equiv \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$ of a subsystem A with respect to its complement B for a given ground state wave-function $|\Psi\rangle$ is widely believed to follow the area law, i.e., asymptotically proportional to the contact area of two subsystems, with rigorous arguments for lattice systems [13–15]. On the other hand, an explicit formula for a multiplicative logarithmic correction to the area law was suggested by Gioev and Klich [16] based on the Widom conjecture[17] and numerically confirmed in Ref. [18] for free fermions at dimensions $d > 1$ and rigorously proved in Ref. [19]. This dramatic violation of the area law for free fermions with a Fermi surface is in stark contrast to the area law found for critical bosons[15] up to subleading corrections[20, 21].

A key question is whether non-perturbative strong correlation effects can further enhance bi-partite entanglement entropy. Since the explicit form of bi-partite entanglement entropy found in Refs. [16, 18] follows from exact results on non-interacting one-dimensional Fermion systems associated each points in Fermi surface[22], strong interactions are likely to cause corrections to this explicit form. So far the only explicit result available for strongly interacting fermions at $d > 1$ is by Zhang *et al.* [23] for Gutzwiller projected two-dimensional (2D) Fermi-surface which is a model wave-function for a critical spin-liquid with spinon Fermi surface. Their variational Monte Carlo calculation of second Rényi entropy S_2 showed little change in both the functional dependence on L_A the linear dimension of the subsystem A (i.e., $S_2 \propto L_A \log L_A$) and the coefficient upon projection. Following this numerical work, Swingle and Senthil [24] argued that the entanglement entropy of certain nFL would follow the same scaling form. Alternatively AdS/CFT correspondence approach yielded a wealth of results on non-Fermi liquids and their entanglement entropy (see for instance [25]). However there is no established connection between these results and experimental systems.

In this letter we focus on a composite fermion[26] (CF) wave-function for the Halperin-Lee-Read[8] (HLR) half-filled Landau level $\nu = 1/2$ nFL state as a test case. There is strong numerical, theoretical and experimental support for the HLR description of the observed $\nu = 1/2$ lowest-Landau-level (LLL) compressible state. First, the pair correlation function calculated with a spherical-geometry wave-function[27][28] shows a good agreement with the structure factor obtained from the exact ground state wave-function [27]. Further, the wave-function is supported by field theoretical studies of fermions coupled to a Chern-Simons gauge field [8, 29] as it describes a state with diverging effective mass for fermions with flux attachment. Finally, the $\nu = 1/2$ state is experimentally established to be a non-Fermi liquid state with a Fermi

surface supporting anomalous sound propagation [5–7], in agreement with expectations of Refs. [8, 29]. We calculate second Rényi entropy S_2 using variational Monte Carlo techniques implementing a "particle number trick". We will focus on the comparison between subsystem linear dimension L_A dependence of S_2 for free fermions and for the CF wave-function.

Rényi entanglement entropy and Widom formula.– The second Rényi entropy is defined as

$$S_2 \equiv -\ln\left[\mathrm{Tr}_A\left\{\rho_A^2\right\}\right], \tag{1}$$

where $\rho_A \equiv \mathrm{Tr}_B|\Psi\rangle\langle\Psi|$ is the reduced density matrix of region A. S_2 has become a quantity of growing interest as a measure of bi-partite entanglement since a convenient scheme for calculating S_2 using variational Monte Carlo technique was shown in Ref. [30]. For free fermions the leading L_A dependence of second Rényi entropy is given by[16, 18, 19]

$$\begin{aligned} S_2 &= \frac{3}{48}c(\mu)L_A\log L_A + o(L_A\log L_A), \\ c(\mu) &= (2\pi)^{1-d}\int_{\partial\Omega} dS_x \int_{\partial\Gamma} dS_k |\mathbf{n}_k\cdot\mathbf{n}_x|, \end{aligned} \tag{2}$$

where μ is the chemical potential, Ω is the real space region A and $\partial\Gamma$ is the Fermi surface. $\mathbf{n}_x$ and $\mathbf{n}_k$ denote the normal vectors on the spatial boundary $\partial\Omega$ and the Fermi surface respectively. For Eq. (2), the linear dimension of the system is scaled to unity. In this work, we will consider 37 fermions in 2D occupying momenta shown in Fig. 1 for both free fermions and for a $\nu = 1/2$ composite fermion non-Fermi liquid. We maintain the same density for both cases by setting $k_F^2 = 10\pi/37$ in units of ℓ, the magnetic length. A straight forward evaluation of $c(\mu)$ for the Fermi surface shown in Fig. 1 and a square-shaped $L_A \times L_A$ region A results in an asymptotic form for the second Rényi entropy

$$S_{2,\mathrm{Widom}} \sim (0.159)\lambda\log\lambda \tag{3}$$

as a prediction based on "Widom formula" Eq. (2). From here on we use the dimensionless quantity $\lambda \equiv k_F L_A$, where k_F is the radius of the Fermi surface.

Monte Carlo evaluation of S_2.– In order to calculate the Rényi entropy S_2 for the $\nu = 1/2$ CF wave-function, we use the scheme of Ref. [30] and consider two copies of the system to evaluate the expectation value of the SWAP operator which is related to S_2 as follows:

$$\begin{aligned} & e^{-S_2} \\ &= \sum_{\beta_1,\beta_2}\sum_{\alpha_1,\alpha_2} \langle\beta_2|\langle\alpha_1|\Psi\rangle\langle\Psi|\alpha_1\rangle|\beta_1\rangle\langle\beta_1|\langle\alpha_2|\Psi\rangle\langle\Psi|\alpha_2\rangle|\beta_2\rangle \\ &\equiv \langle\mathrm{SWAP}_A\rangle\,. \end{aligned} \tag{4}$$

Here α_i and β_i, with $i = 1, 2$ for the two copies, are real space coordinates within each copy of subregions, i.e., $\alpha_1 \in A_1, \beta_1 \in B_1$ and $\alpha_2 \in A_2, \beta_2 \in B_2$. Below we calculate the expectation value for the model wave-function by sampling the wave-function over the two copies, introducing a "particle number trick" which improves the computing time and allows for parallelization compared to the previous calculation of S_2 for itinerant fermions[31] .

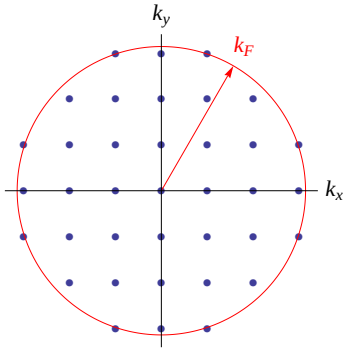

FIG. 1. Fermi surface for $N = 37$ particles in 2D. The set of momenta are shown as blue points. Red circle denotes the Fermi surface $\partial\Gamma$ of radius $k_F \approx \sqrt{10\pi/37}$.

Particle number trick.– Compared to the case of positive definite spin wave-functions studied in Ref. [30], itinerant fermion systems come with two major challenges against evaluation of $\langle\mathrm{SWAP}_A\rangle$: (1) the wave-function is not positive definite, (2) the number of fermions in the region A fluctuates. The first issue had been partially mitigated in Ref. [23] using the so-called "sign trick" exactly factorizing $\langle\mathrm{SWAP}_A\rangle$ into a product of two terms each concerning only magnitude or only sign. On the other hand, the fermion number fluctuation was not an issue in Ref. [23] as the Gutzwiller projector ensured one fermion per site. For the case of composite fermions however the statistical error increased dramatically at large L_A, and was only partially mitigated by the sign trick. This motivated us to implement a "particle number trick" that can further reduce error-bars.

The particle number trick uses the fact that $\langle\mathrm{SWAP}_A\rangle$ can be further exactly factorized into contributions from sectors of fixed particle numbers in each subregion[32]. By delegating more processor time to those number sectors that contribute most to the entropy, we could perform the computation more efficiently. For example, it took about 10^4 CPU hours per data point to achieve error bars comparable to the free fermion case with the composite fermion wave function, as supposed to 10^5 CPU hours per data point using the sign trick alone. Moreover, this "number trick" is particularly effective in controlling error-bars at large L_A enabling efficient calculation of the Rényi entropy in the large L_A limit. We note that analogous approaches that resolve particle numbers have been mentioned for studying entanglement in lattice systems [30] [33] and a subtly different number trick was used in

Ref.[34].

Wave-function.– Heuristically, a wavefunction that models a spin polarized $\nu = 1/2$ LLL nFL state combines features of a Slater determinant Fermi sea and the $\nu = 1/2$ boson Laughlin state. Most previous numerical studies use spherical geometry in which the Fermi sea is modeled as closed filled shells of spherical harmonic orbitals $Y_{\ell m}$ with $\ell = 0, 1, \ldots \ell_F$, so $(\ell_F + 1)^2$ particles fill $\ell_F + 1$ shells. This has a Fermi level, but no clear analog of a k-space Fermi surface. To reproduce a Fermi surface with a variable shape, we instead used the torus geometry with a periodic boundary condition (pbc). A straightforward model state [27, 35], is given by

$$\det_{ij} e^{i\boldsymbol{k}_i\cdot\boldsymbol{R}_j}|\Psi_L^{1/2}\rangle = \det_{ij} t_i(\boldsymbol{d}_j)|\Psi_L^{1/2}\rangle \tag{5}$$

where $|\Psi_L^{1/2}\rangle$ is the boson Laughlin state, and $\boldsymbol{R}_i$ are the non-commutive guiding-center coordinates that act within a Landau level. Note that the translation operator $\exp i\boldsymbol{k}\cdot\boldsymbol{R}_i = t_i(\boldsymbol{d})$ displaces particle i by $d^a = \epsilon^{ab}k_b\ell_B^2$, leading to the second form above. The pbc requires that each displacement has the form $N_\Phi \boldsymbol{d}_i \in \{m\boldsymbol{L}_1 + n\boldsymbol{L}_2\}$ where the primitive pbc translations $\boldsymbol{L}_1$, $\boldsymbol{L}_2$ define a unit cell with area $2\pi N_\Phi \ell_B^2$ with quantized flux N_Φ, and $\ell_B^2 = \hbar/|eB|$. With a pbc, the more familiar holomorphic polynomial factors $(z_i - z_j)$ in the Laughlin wavefuctions become holomorphic elliptic functions like $\vartheta_1(\kappa(z_i - z_j)|\tau)$. The formulation of Ref.[36] in terms of Jacobi elliptic functions ϑ_1 treats the translations $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ differently, and we found it convenient to use an equivalent form in terms of the Weierstrass σ function

$$\sigma(z) = \frac{\vartheta_1(\kappa z;\tau)}{\kappa\vartheta_1'(0;\tau)}\exp(i(\kappa z)^2/\pi(\tau - \tau^*)). \tag{6}$$

Here $\kappa = \pi/L_1$, $L = (L_x + iL_y)/\sqrt{2}\ell_B$ is the linear dimension of the system with $L_1^*L_2 - L_2^*L_1 = 2\pi i N_\Phi$, and $\tau = L_2/L_1$ is the modular parameter of the torus. Note that complex quantities z, L are made dimensionless by using a length unit $\sqrt{2}\ell_B$ (not ℓ_B) so the usual disk-geometry Gaussian factor $\exp -\frac{1}{4}r^2/\ell_B^2$ becomes $\exp -(\frac{1}{2}z^*z)$.

The holomorphic factor of the symmetric-gauge wavefunction for (5) is

$$\Big(\sum_{P=1}^{N!}(-1)^P F_P(z_1,\ldots,z_N)\Big)F_{CM}(\sum_i(z_i - \bar{d}))$$
$$F_P(\{z_i\}) = \prod_{i<j}\sigma(z_i - z_j + d_{P(i)} - d_{P(j)})^2\prod_i e^{d^*_{P(i)}z_i}, \tag{7}$$

where $F_{CM}(z)$ $(= \sigma(z)^2)$ is the center of mass wavefunction, $d = (d_x + id_y)/\sqrt{2}\ell_B$, and $z = (x + iy)/\sqrt{2}\ell_B$ are complex distances and coordinates.

The wave-function in Eq. (7) requires explicit antisymmetrization of $N!$ terms, which for large N quickly becomes prohibitive for Monte-Carlo calculations. Instead, we present a new expression that is inspired by a spherical-geometry construction due to Jain and Kamilla[37]. In analogy to Ref. [37], we move the translation operators inside the determinant, which can be computed in $O(N^3)$ operations. Doing so reduces the number of times they act on $\sigma(z_i - z_k)$ (the relative part) from $2(N-1)$ to $N-1$, and would break the pbc, as half of their actions are lost. To compensate, we *double* each $d_i(t_j)$:

$$F_{CF} = \det_{i,j}(e^{d_j^* z_i}\prod_{k(\neq i)}\sigma(z_i - z_k + 2(d_j - \bar{d}))\times$$
$$F_{CM}(\sum_i(z_i - \bar{d})). \tag{8}$$

To complete the wave-functions a non-holomorphic exponential factor $e^{-\sum_i z_i z_i^*/2}$ has to be included. For convenience, we have made a specific choice of the zeroes of the coherent-state center-of-mass wave-function[36]. This resolves the two-fold topological degeneracy of the state.

The choice of the set of distinct values $\{\boldsymbol{k}_i\}$, where $k_{ia} = \epsilon_{ba}d_i^a/\ell_B^2$ corresponds to the choice of occupied Bloch states in a free-fermion Slater determinant. For the $\nu = 1/2$ state, the choice of how to fill the "Fermi sea", is non-trivial, as the usual kinetic energy is completely quenched by the magnetic field, and a uniform boost $\boldsymbol{k}_i \mapsto \boldsymbol{k}_i + \boldsymbol{k}$, all i, corresponds to a translation that leaves the variational energy unchanged. It can be empirically argued that clustering the $\boldsymbol{k}_i$ as close to each other as possible, while keeping them distinct, is the minimal antisymmetric distortion of the $\nu = \frac{1}{2}$ Laughlin state. A simple model Hamiltonian[38], that encodes this idea is

$$H = \frac{\hbar^2}{2mN}\sum_{i<j}|\mathbf{k}_i - \mathbf{k}_j|^2 \tag{9}$$

where m is the "CF mass", and for a finite system we choose the "Fermi sea" $\{\boldsymbol{k}_i\}$ to minimize (9), which is manifestly invariant under boosts, guaranteeing $F_1 = -1$[39–41], which is the only nonzero "Fermi-liquid parameter" of the model. The many-body momentum quantum number is the average $\boldsymbol{K} = N^{-1}\sum_i \boldsymbol{k}_i$ modulo the pbc-allowed values of $\boldsymbol{k}$. The effective model (9) is the minimal modification of a free-Fermi gas to incorporate the required boost-invariance, and the ground-state values of $\boldsymbol{K}$ that it predicts for finite-N closely track the values found in exact diagonalization studies of the $\nu = \frac{1}{2}$ LLL system with Coulomb interactions and a pbc[38]. We chose a square pbc with $N = 37$, for which the Fermi-sea configuration predicted by (9) has K = 0, like a free-fermions system. We can use this set of "occupied" momenta, shown in Fig. 1, for both the composite and free fermion wave-functions.

Results.- - We obtained the second Rényi entropy S_2 as a function of λ for both the free fermion gas and

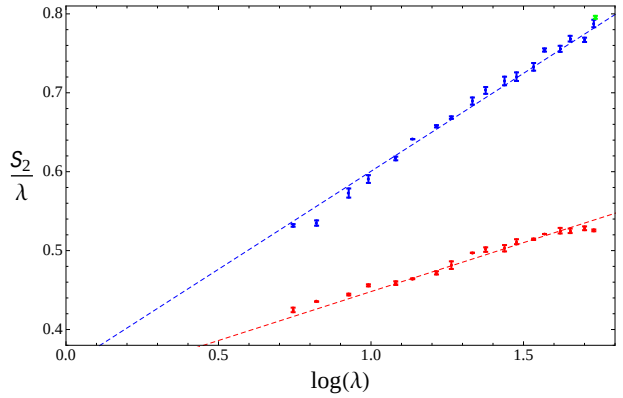

FIG. 2. Plot of S_2/λ as a function of λ. Red corresponds to free fermions, while blue ($N = 37$) and green ($N = 137$) correspond to composite fermions at half-filling in the LLL. Error bars indicate 95% confidence intervals. Dashed lines are best linear fits of S_2/λ to the form $a + c \log \lambda$, where a and c are fit parameters.

the composite fermion trial wave-function for $\nu = 1/2$ state. Here L_A is the linear dimension of the system and $k_F = \sqrt{10\pi/37}$ is the radius of the Fermi surface. The results and the error bar are shown in Fig. 2. Our results for the free fermion gas shown in red are in good agreement with previous numerical results[18, 31] and hence demonstrate reliability of our algorithm using the particle number trick. The ratio S_2/λ is linear in $\log \lambda$ and the linear fit results in 95% confidence level shown in the red dashed line in Fig. 2 follows

$$S_2[\Psi_0] = (0.135 \pm 0.01)\lambda \log \lambda \tag{10}$$

where Ψ_0 denotes the free Fermi gas wave-function for the set of momenta shown in Fig. 1. Comparing the results Eq. (10) to Eq. (3) we confirm that our numerical results for free fermions are consistent with the exact results Eq. (2).

For the $\nu = 1/2$ non-Fermi liquid state, the results shown in blue in Fig. 2 again exhibit linear dependence of S_2/λ on $\log \lambda$, i.e. multiplicative logarithmic violation of the area law. However, the linear fit again at 95% confidence level:

$$S_2[\Psi_{\nu=1/2}] = (0.27 \pm 0.02)\lambda \log \lambda \tag{11}$$

reveals that the coefficient is no longer given by the "Widom formula". The steeper slope for the non-Fermi liquid state is evident even from the raw data. [42] As a check for finite-size effects, we computed the entanglement entropy at large $\lambda = 1.73$ for a large system size of $N = 137$. The fact that this data points lands on top of the the data points for the system size of $N = 37$ makes it unlikely for the observed enhancement to be a finite size effect. Curiously, the coefficient of the multiplicative logarithmic correction term for $\nu = 1/2$ non-Fermi liquid is not only larger compared to that for free fermions, but it appears to be double the value expected from the "Widom formula" Eq. (2).

Discussion.– In summary, we calculated the second Rényi entropy S_2 of the $\nu = 1/2$ composite fermion non-Fermi liquid state captured by the trial wave-function. We found the multiplicative logarithmic violation of the area law with the same functional dependence on the linear dimension of the subregion, i.e., $S_2 \propto \lambda \log \lambda$, as is the case for free fermions but with a coefficient that is roughly double what is found for free fermions. Our results support the conjecture that $S_2 \propto \lambda \log \lambda$ might be the strongest form of area law violation in 2D made in Ref. [24]. However, our results reveal a violation of the Widom formula, in the form of a doubling of its coefficient for the $\nu = 1/2$ non-Fermi liquid state, in contrast to the free fermion result obtained for exactly solvable models of fermions coupled to discrete gauge fields in Ref. [43].

Our explicit calculation of entanglement entropy for an established non-Fermi liquid state raises a number of interesting questions. For example, what is the physical significance of the coefficient of the $L_A \log L_A$ term in the entanglement entropy of strongly correlated fermions forming a non-Fermi liquid state? There is little literature on this coefficient for interacting fermions. While a Gutzwiller-projected Fermi surface constrained to maintain one fermion per site showed little difference from free fermions[23], Slater-Jastrow wave-functions for the interacting Fermi gas without a magnetic field showed small changes in the coefficient[31]. However, while Slater-Jastrow wave-functions are often used to add correlation effects and model conventional Fermi liquids, it is not clear if the effect of the Jastrow factor is perturbative as the S_2 result of Ref. [31] does not extrapolate to free fermion result in the limit of fermion residue $Z \to 1$. It will be interesting to use the particle number trick on a d-wave metal wave-function which is proposed to be stabilized by a ring-exchange Hamiltonian[44] in this context. Another question is whether it is possible to gain analytic insight into the enhancement of entanglement due to strong correlation. Finally, the investigation of entanglement spectra or shape dependence of entanglement entropy[45] may reveal more insight into $\nu = 1/2$ non-Fermi liquid state.

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