Nonperturbative Leakage Elimination Operators and Control of a Three-Level System

Jun Jing, Lian-Ao Wu, Mark Byrd, J. Q. You, Ting Yu, and Zhao-Ming Wang

Phys. Rev. Lett. **114**, 190502 — Published 15 May 2015

DOI: 10.1103/PhysRevLett.114.190502

Nonperturbative Leakage Elimination Operators and Control of a Three-Level System

Jun Jing[1,2], Lian-Ao Wu[2,3*] and Mark Byrd[4], J. Q. You[5], Ting Yu[6], Zhao-Ming Wang[7]
[1] *Institute of Atomic and Molecular Physics and Jilin Provincial Key Laboratory of Applied Atomic and Molecular Spectroscopy, Jilin University, Changchun 130012, Jilin, China*
[2] *Department of Theoretical Physics and History of Science, The Basque Country University (EHU/UPV), PO Box 644, 48080 Bilbao, Spain*
[3] *Ikerbasque, Basque Foundation for Science, 48011 Bilbao, Spain*
[4] *Physics Department, Southern Illinois University, Carbondale, Illinois 62901-4401, USA*
[5] *Beijing Computational Science Research Center, Beijing 100084, China*
[6] *Center for Controlled Quantum Systems and Department of Physics and Engineering Physics, Stevens Institute of Technology, Hoboken, New Jersey 07030, USA*
[7] *Department of Physics, Ocean University of China, Qingdao 266100, China*
(Dated: March 30, 2015)

Dynamical decoupling operations have been shown to reduce errors in quantum information processing. Leakage from an encoded subspace to the rest of the system space is a particularly serious problem for which leakage elimination operators (LEO) were introduced. Here we provide an analysis of non-ideal pulses, rather than the well-understood idealization or bang-bang controls. Under realistic conditions, we show that these controls will provide the same protection from errors as idealized controls. Our work indicates that the effectiveness of LEOs depends on the *integral of the pulse sequence* in the time domain, which has been missing because of the idealization of pulse sequences. Our results are applied to a three-level system for the nitrogen-vacancy centers under an external magnetic field and are illustrated by the fidelity dynamics of LEO sequences, ranging from regular rectangular pulses, random pulses and even disordered (noisy) pulses.

Introduction.—Quantum information is often stored in, and processed with, logical or encoded qubits which use physical d-state systems for the encoding. This is done to suppress or avoid noise (as in decoherence-free/noiseless subsystems), to enable errors to be detected and corrected, to enable universal quantum computing, or simply because the physical system being used is really not an ideal qubit (see [1] and references therein). In these cases, the encoded or logical subspace has a particular advantage which motivates the encoding.

When this encoded qubit is subjected to noise, it may lose some or all of its advantage due to its coupling with the other states of the system which are not used in the encoding. Such loss of information is called *leakage* since information leaks from the encoded states into the surrounding Hilbert space. This is particularly troublesome when the encoding is *required* to ensure the qubit retains one of the properties above.

Leakage elimination operators (LEOs) were proposed to counteract leakage in a two-level system which encodes one logical qubit in a multilevel Hilbert space [2–9] by employing unbounded fast and strong pulses called "bang-bang" (BB) control [10–13], which originates from the spin-echo effect [14] applied for the first order corrections of the evolution. In general, the total Hamiltonian including system and bath can be written as $H_{\rm SB} = H_P + H_Q + H_L$, where H_P acts within the qubit subspace, i.e., the subspace of interest; H_Q has no effect on the qubit subspace because it acts only within the remaining Hilbert space orthogonal to the subspace of P; and H_L (the leakage operator) represents the diffusion between the P- and Q-subspaces [9, 15, 16]. If an operator R_L satisfies $\{R_L, L\} = 0$ and $[R_L, P] = [R_L, Q] = 0$, then it follows that R_L serves as a leakage elimination operator: $\lim_{m\to\infty}(e^{-i\frac{H_{\rm SB}t}{m}}R_L^\dagger e^{-i\frac{H_{\rm SB}t}{m}}R_L)^m = e^{-iH_Pt}e^{-iH_Qt}$. This holds to the order of t^2 when $m = 1$.

During BB control, R_L or $R_L^\dagger$ is so strong and fast that the system-bath Hamiltonian can be neglected. This assumption is impractical or almost experimentally inaccessible for most existing setups. For the choice of free evolution time t, one should make $t \ll 1/\omega_c$, where ω_c is the upper bound of the frequencies of those bath-modes coupling to the system. It is reasonable for the situation where the characteristic frequency of the environment is much less than that of the system. When the frequencies of both system and environment are comparable to each other, it is difficult to meet these requirements.

Therefore, a *nonperturbative* version of LEO theory is desirable for the coherence-protection/diffusion-suppression protocol for open quantum systems. This would enable the use of sequences in a wider domain of the system's characteristic parameters, which would apply for non-ideal pulses in the presence of a non-Markovian environment. A practical scenario is the effective three-level Hamiltonian for the spin of electronic ground states of a nitrogen vacancy (NV) center [17–22] in a diamond in presence of an external magnetic field. An NV center has an $S = 1$ state with zero-field splitting $D = 2.88$ GHz between the $m_s = 0$ and $m_s = \pm 1$ states. An external magnetic field along the crystalline axis of the diamond will lift the degeneracy of the $m_s = \pm 1$ states. The lowest two levels with $m_s = 0$ and $m_s = -1$ have an energy gap $\omega_{\rm NV} = D - g_e\mu_B B_z \approx 2.88(1 - 0.01B_z/mT)$ GHz and it can be used as a spin-based quantum memory unit [23–25]. However, it neglects the influence of the $m_s = 1$ state. Fluctuations of the external magnetic field would violate the far off-resonance condition for the transition between $m_s = 0$ and $m_s = -1$ and also $m_s = 0$ and $m_s = 1$. Such leakage problems in the $m_s = 0, -1$ subspace can be serious, especially in a small magnetic field.

In this work, the theory of nonperturbative LEO is presented in the framework of non-Markovian quantum-state-diffusion (QSD) equation [26], by which an arbitrary sequence of LEO pulses as well as their fluctuations can be taken into account without any approximation. The nonperturbative LEO approach enables us to find out something missing due to the above-mentioned approximations and idealizations. Significantly, we find that the effectiveness of LEO, also for conventional dynamical decoupling (DD), is *not determined* by details of a fast pulse sequence, but the integral of the pulse sequence over time, which effectively increases the energy splitting between P- and Q-subspaces. In other words, it shows that any two fast pulse sequences, for example two sequences $c(t)$ and $c'(t) = c(t)$+fluctuation, will have the same effectiveness if they have the same integrals over time. Incidentally, our scheme automatically solves the main problem, the fluctuations in the driving fields, that the continuous decoupling scheme targets [27]. In particular, we focus on the leakage problem of three-level system that is ubiquitous in quantum optics and quantum solid-state devices. Our protocol allows a straightforward extension to multilevel systems. Distinguished from other work targeting the optimized pulse sequences [28], below we use the regular, random and noisy pulse sequences [29] to identify the key elements for attaining decoherence-suppression.

Construction of nonperturbative LEO.—Our LEO constitutes one part of system Hamiltonian in QSD equation. The total system space is separated into the (logical) P-subspace, and the remaining (orthogonal) Q-subspace as explained above. The LEO acts as I in P and $-I$ in Q, i.e., $R_L = \mathrm{diag}[c_1 I, -c_2 I]$, where the two identity operators have the same dimensions as P and Q, respectively; and c_k's ($k = 1, 2$) are non-negative real numbers. This LEO will be performed nonperturbatively by solving the QSD equation with both the pulse and H_{SB} present rather than using the BB pulse approximation as in Ref. [3].

Consider a general three-level atomic system [30]: $H_{\rm sys} = \sum_{j=1}^{3} \omega_j |j\rangle\langle j|$. The Lindblad operators for the V-type and λ-type atoms are denoted by $L_V = \mu_1|3\rangle\langle 1| + \mu_2|3\rangle\langle 2|$ and $L_\lambda = \nu_3|3\rangle\langle 1| + \nu_2|2\rangle\langle 1|$, respectively. Then by the LEO protocol, the nonperturbative operators are

$$R_L^V = c(t)\mathrm{diag}[1,1,0], \quad R_L^\lambda = c(t)\mathrm{diag}[1,0,0], \tag{1}$$

where $c_1 = c(t)$ is the implemented pulse sequence and c_2 is taken to be zero without loss of generality since it is equivalent to adding an overall constant term to the Hamiltonian. The exact stochastic wave-function for the system including the LEO is governed by the following QSD equation [26] (setting $\hbar = 1$):

$$\partial_t \psi_t(z^*) = [-iH_{\rm sys} - iR_L^x + L_x z_t^* - L_x^\dagger \bar{O}_x(t)]\psi_t(z^*), \tag{2}$$

where $x = V$ or λ. For a V-type three-level system, $\bar{O}_V(t) = F_1(t)|3\rangle\langle 1| + F_2(t)|3\rangle\langle 2|$, where $F_k(t) \equiv \int_0^t ds\alpha(t,s)f_k(t,s)$, $k = 1, 2$. $\alpha(t,s)$ is the environmental correlation function and $f_k(t,s)$ satisfies $f_k(t,t) = \mu_k$ and $\partial_t f_k(t,s) = i[\omega_k - \omega_3 + c(t)]f_k + F_k(t)(\mu_1 f_1 + \mu_2 f_2)$. While for the λ-type system, $\bar{O}_\lambda(t) = P_2(t)|2\rangle\langle 1| + P_3(t)|3\rangle\langle 1|$, where $P_k(t) \equiv \int_0^t ds\alpha(t,s)p_k(t,s)$, $k = 2, 3$, and $p_k(t,s)$ satisfies $p_k(t,t) = \nu_k$ and $\partial_t p_k(t,s) = i[\omega_1 - \omega_k + c(t) + \nu_2 P_2 + \nu_3 P_3]p_k$. According to Eq. (2), the ansätz $\bar{O}_x$, and the Novikov theorem, the exact master equation in the rotating frame with respect to $H_{\rm sys} + R_L^x$ is

$$\partial_t \rho_{\rm sys} = [L_x, \rho_{sys}\bar{O}_x^\dagger] + [\bar{O}_x\rho_{sys}, L_x^\dagger]. \tag{3}$$

The fidelity describing the survival probability of the initial state ψ_0 is defined by $\mathcal{F}(t) \equiv \sqrt{\langle\psi_0|\rho_{\rm sys}|\psi_0\rangle} = \sqrt{M[\langle|\psi_0|\psi_t(z^*)\rangle\langle\psi_t(z^*)|\psi_0\rangle]}$.

Result of nonperturbative LEO.– A BB pulse is a limiting case of more practical rectangular pulse, which can be characterized by the period τ, the duration time Δ, and the strength Φ_0. Particularly, $c(t) = \Phi_0/\Delta$ for $n\tau - \Delta \le t \le n\tau$, where $n \ge 1$ is an integer; otherwise, $c(t) = 0$. In Fig. 1(a), we demonstrate a typical parameter diagram for the fidelity of a three-level atom coupled to a non-Markovian environment with an exponential decay correlation function, where $1/\gamma$ characterizes the environmental memory time. Here the LEO control is chosen to be a rectangular pulse sequence and the y-axis denotes the dark time in one period of pulse. This diagram, parameterized by Δ and τ, divides the whole space into regions, where the fidelity can be preserved at 0.9, 0.99, etc. and is reminiscent of a phase transition. The ideal pulse occupies only the lower left corner of the diagram and one can expect a tolerance from non-ideal pulses which achieve the same fidelity at any desired moment. It indicates a great deal of freedom in choosing an efficient combination of duration time and period. For instance, $\mathcal{F} \ge 0.99$ can be obtained at $\omega t = 10$ as long as the ratio of dark time and duration time is not larger than about 3/2 when $\Delta \le 0.04\omega t$.

This raises a question: *what are the important parameters or factors for attaining nearly the same control effect besides the parameters of environment?* Our research indicates that over all of the parameters of LEO pulse and within a fixed evolution time scale, the time integral over the pulse, i.e., the accumulation of the pulse strength in the control history, is a clear candidate. It is also clear that the pulse integral is linearly proportional to Φ_0. In Fig. 1(b), we compare the dynamics of a V-type three-level system under control with different Φ_0. The calculations are performed with $\omega_1 - \omega_3 = \omega$, $\omega_2 - \omega_3 = 0.8\omega$,

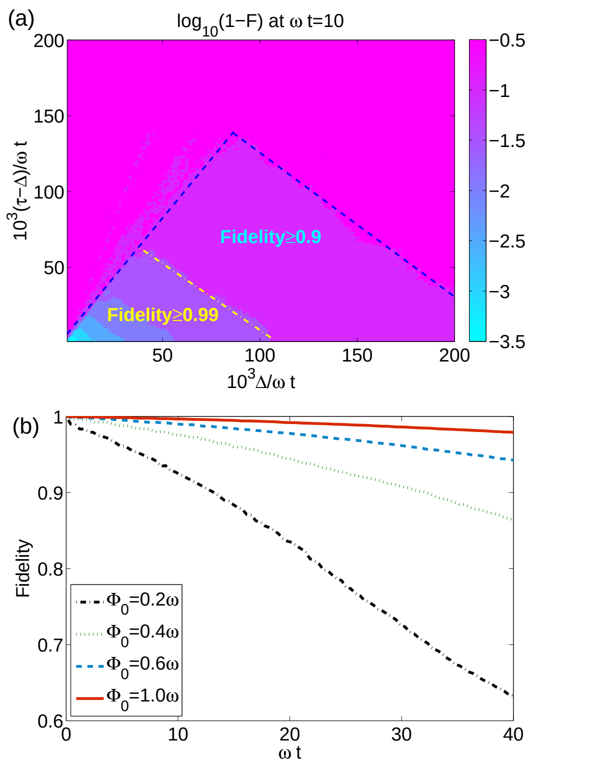

FIG. 1. (Color online) (a) Parameter diagram of fidelity at $\omega t = 10$ in cases of both $|\psi_0\rangle = |1\rangle$ for the λ-type system with $\omega_1 = \omega/2$, $\omega_2 = \omega_3 = -\omega/2$ and $|\psi_0\rangle = 1/\sqrt{2}(|1\rangle + |2\rangle)$ for the V-type system with $\omega_1 = \omega_2 = \omega/2$, $\omega_3 = -\omega/2$ (they share the same result), where $\Phi_0 = \omega$. (b) Dynamics of a V-type system under a regular LEO sequence with different Φ_0. $|\psi_0\rangle = 1/\sqrt{2}(|1\rangle + |2\rangle)$, $\Delta/\tau = 0.6$, $\tau = 0.02\omega t$. In the two frames and the following figures, the environmental correlation function is $\alpha(t,s) = \Gamma\gamma/2e^{-\gamma|t-s|-i\Omega(t-s)}$, where $\gamma = 1$, $\Gamma = \omega$, and $\Omega = 0.5\omega$.

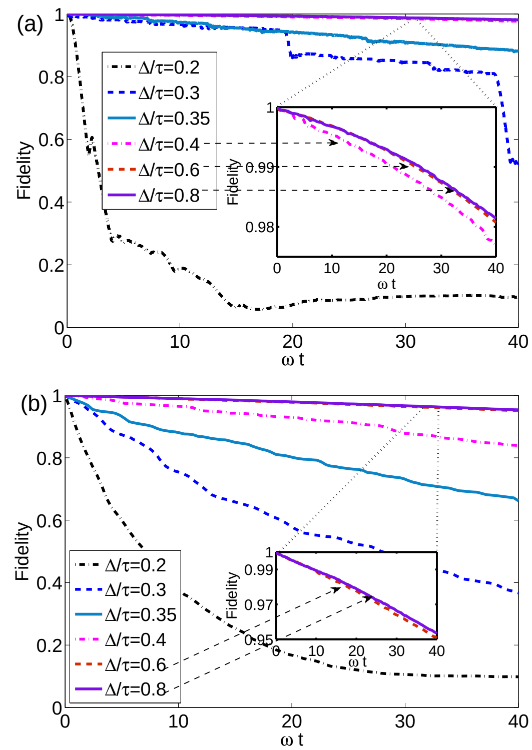

FIG. 2. (Color online) Dynamics of a V-type system under (a) regular LEO with different ratio $r = \Delta/\tau$ (b) random LEO with different *average ratio* r. $|\psi_0\rangle = 1/\sqrt{2}(|1\rangle + |2\rangle)$.

$\mu_1 = \omega$, and $\mu_2 = 0.5\omega$. It is shown that the larger Φ_0 (essentially, the larger splitting between the interested and the remaining subspaces), the more resistant the system is to leakage. At the fixed moment $\omega t = 40$, when $\Phi_0 = 0.4\omega$, the fidelity decays to about 0.85; while when $\Phi_0 = \omega$, it is still above 0.98.

However, the strength Φ_0 cannot completely determine the most effective LEO control. Figure 2(a) shows a simulation with an unchanged pulse strength Φ_0 and various ratios of duration time and period. It exhibits a phenomenon analogous to a phase transition, where $r \equiv \Delta/\tau = 0.35$ is a threshold value r_c. If $r < r_c$, then an accelerated decay emerges in the fidelity dynamics. The line of $r = 0.2$ shows a rapid decay in a very short time. Although when $\omega t < 19$, the effect of $r = 0.3$ is nearly the same as $r = 0.35$, it deviates from the asymptotic curve by several sudden jumps. If $r = r_c = 0.35$, then the fidelity decays to sightly less than 0.9 at $\omega t = 40$, and presents a significant gap from the results with even larger r. While if $r > r_c$, almost the same degrees of decoherence-suppression are achieved. The inset in Fig. 2(a) shows the curves for $r = 0.4, 0.6, 0.8$, which have an approximate fidelity of 0.98 at $\omega t = 40$, and the maximal relative error is less than 0.5%. So here the fidelity will be saturated with $0.4 < r < 1$, where the LEO control effect is nearly completely determined by the pulse integral over time rather its configuration.

Stochastic quantum fluctuations and environmental noise inevitably yield a random rather than regular rectangular pulse. Such a sequence can be “simulated” in the following way: based on a regular sequence with fixed τ, Δ and Φ_0, the time-dependent parameter $X = \tau$ or Φ_0 are obtained by $X' = X[1 + A_X R_X(t)]$, where R_X can be uniformly distributed between -1 and 1, and R_τ and R_{Φ_0} are uncorrelated. After a sufficiently long time of evolution and ensemble averaging, $M[X'] = X$. Therefore, the integral of the pulse strength over time is the same as that of the original regular sequence. Figure 2(b) shows the result where random amplitudes are $A_\tau = 40\%$ and $A_{\Phi_0} = 90\%$. Compared to Fig. 2(a), accelerated-decoherence phenomenon has been alleviated. Under such a large fluctuation in parameters, the controls with the same ratio $r \le 0.4$ seem to be only a little less effective than that of the regular LEO pulse. The fidelity also asymptotically saturates in the regime $r \ge 0.6$ [see the inset in Fig. 2(b)], where it decays to around 0.96 at $\omega t = 40$.

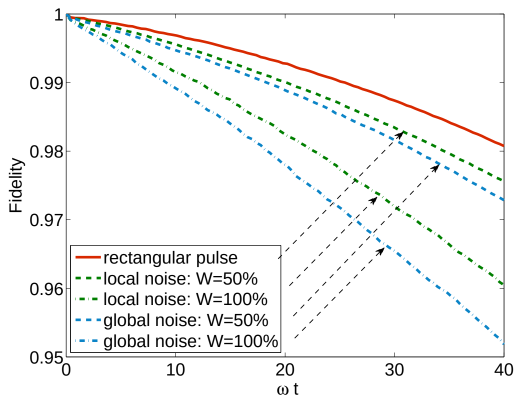

FIG. 3. (Color online) Dynamics of a V-type system under noisy LEOs with different amplitudes of Gaussian noise. $|\psi_0\rangle = 1/\sqrt{2}(|1\rangle + |2\rangle)$, $r = \Delta/\tau = 0.6$, $\tau = 0.02\omega t$.

Influenced by some uncontrollable factors, a real nonperturbative LEO control could be realized by the noisy pulse. To keep unchanged the pulse integral during the same period of time, one can expect two types of noises. (a) Global noise: $c(t) \to c(t) + \Phi_0/\tau W n(t)$, where W is a percentage measuring the dimensionless noise strength and $n(t)$ is a white noise. (b) Local noise: only in each duration time Δ, the strength of pulse is equivalent to the fixed value plus the noise $\Phi_0/\tau W n(t)$, while it remains dark during the original dark intervals. The results from Gaussian noise are presented in Fig. 3. To exhibit the effect of the pulse integral, we use a greatly exaggerated amplitude of the noise. Particularly, even when W is as large as 100%, for local and global noises, the deviations from the regular pulse are still less than 0.02 and 0.03, respectively. Therefore, the accumulation of pulse strength remains the key element even for a noisy LEO.

Discussion and conclusion.—A nonperturbative LEO sequence suppresses the leakage by rapidly enhancing the energy difference between the interested subspace P and the remaining Hilbert space Q in terms of the control $c(t)$, which is in contrast with the traditional DD controls averaging or symmetrizing the system-environment coupling away by repeatedly and instantaneously rotating the system. Specifically, under an LEO, the modulus of coefficients F_k's, $k = 1, 2$, of the O-operator in the master equation (3) that determines the decoherence rate can be preserved close to 0. Using the correlation function indicated in Fig. 1(a), one can find

$$F_k(t) = e^{iC(t)} \int_0^t ds e^{-iC(s)} G(s), \tag{4}$$

where $G(t) = \frac{\Gamma\gamma\mu_k}{2} + [-\gamma + i(\omega_k - \omega_3 - \Omega)]F_k + (\mu_1 F_1 + \mu_2 F_2)F_k$. When $C(t) \equiv \int_0^t ds c(s)$ is sufficiently large, the kernel in the integral of Eq. (4) consists of a fast-oscillation function $e^{-iC(s)}$ and a slowly varying function $G(s)$, so that F_k vanishes [29]. This shows that the effectiveness of LEOs depends on $C(t)$, the *integral of the pulse sequence* in the time domain, but not details of $c(t)$ such as the shape or arrangement of these pulses [31]. The first term in $G(t)$ is a constant, the second is the linear term of F_k with the modulus of the coefficient being $\sqrt{\gamma^2 + (\omega_k - \omega_3 - \Omega)^2}$, and the last term is proportional to F_k^2, which can be ignored if the variation of F_k is a first order perturbation. Therefore, the requirement of $G(t)$ as a slow variable is consistent with a vanishing F_k if $C(t)$ is large. Furthermore, it is reasonable to expect that nonperturbative LEO works well with small γ corresponding to a strong non-Markovian environment. Note that although this correlation function does not apply to every environment, many correlation functions, e.g., the $1/f$ noise, could be decomposed into a finite summation of this one with different γ's and Ω's [32]. Therefore our scheme can be adapted to more general situations.

The LEO approach can be applied to protect the NV spin (a V-type three-level system) from fluctuations of the magnetic field. Compared to previous DD prescriptions [28], this LEO suppresses the diffusion of the $m_s = \pm 1$ states by performing a nonperturbative control field with compatible frequency in the order of $\omega_{\rm NV}$. The results shown in Fig. 1(b) apply even in the presence of a much larger dissipative coupling than is often met in practice. This is indicated by a value $\Gamma \sim \omega_{\rm NV}$, in the order of GHz, while in practical situations, it is usually less than 50 MHz. In this extreme case, the fidelity of the system remains as large as 0.98 after 40 ns, which is already much larger than a typical operation time ~ 10 ns.

In summary, we presented a nonperturbative LEO approach to dynamical decoupling of an arbitrary multilevel system, and found that the integral over the pulse sequence is the most important quantity in determining the effect of decoherence-suppression under proper average ratio of (pseudo-) duration time to (pseudo-) period. This result is insensitive to fluctuations of the pulse strength and period and robustly removes the system disturbances due to environmental noise under imperfect and noisy controls.

Acknowledgments.—We acknowledge grant support from the Basque Government (grant IT472-10), the Spanish MICINN (No. FIS2012-36673-C03-03), the NBRPC No. 2014CB921401, the NSAF No. U1330201, the NSFCs No. 11175110 and No. 11475160, Science and Technology Development Program of Jilin Province of China (20150519021JH), the NSF No. PHY-0925174 and the AFOSR No. FA9550-12-1-0001.

* Author to whom any correspondence should be addressed. Email address: lianao.wu@ehu.es

[1] D. A. Lidar and T. A. Brun (editors), *Quantum Error Correction*, Cambridge University Press, Cambridge (2013).

[2] L.-A. Wu, et al., Phys. Rev. Lett. **89**, 127901 (2002).
[3] M. S. Byrd, et al., Phys. Rev. A **71**, 052301 (2005).
[4] L. Tian and S. Lloyd, Phys. Rev. A **62**, 050301 (2000).
[5] M. Bhattacharya, et al., Am. J. Phys. **80**, 810 (2012).
[6] X. Zhou and A. Mizel, Phys. Rev. Lett. **97**, 267201 (2006).
[7] A del Campo, et al., Phys. Rev. Lett. **110**, 050403 (2013).
[8] A Del Campo, Phys. Rev. A **84**, 031606 (2013).
[9] L.-A. Wu, et al., Phys. Rev. Lett. **102**, 080405 (2009).
[10] L. Viola and S. Lloyd, Phys. Rev. A **58**, 2733 (1998).
[11] L. Viola, et al., Phys. Rev. Lett. **82**, 2417 (1999).
[12] D. Vitali and P. Tombesi, Phys. Rev. A **59**, 4178 (1999).
[13] P. Zanardi, Phys. Rev. A **63**, 012301 (2001).
[14] H. Y. Carr, and E. M. Purcell, Phys. Rev. **94**, 630 (1954).
[15] P. Gaspard and M. Nagaoka, J. Chem. Phys. **111**, 5676 (1999). L. Viola and E. Knill, Phys. Rev. Lett. **90**, 037901 (2003).
[16] J. Jing, et al., Phys. Rev. A **85**, 032123 (2012).
[17] F. Jelezko, et al., Phys. Rev. Lett. **92**, 076401 (2004); ibid **93**, 130501 (2004).
[18] R. Hanson, et al., Science **320**, 352 (2008).
[19] G. D. Fuchs, et al., Science **326**, 1520 (2009).
[20] P. Rabl, et al., Phys. Rev. B **79**, 041302(R) (2009).
[21] M. Schaffry, et al., Phys. Rev. Lett. **107**, 207210 (2011).
[22] L. Chotorlishvili, et al., Phys. Rev. B **88**, 085201 (2013).
[23] D. Marcos, et al., Phys. Rev. Lett. **105**, 210501 (2010).
[24] X.-Y. Lv, et al., Phys. Rev. A **88**, 012329 (2013).
[25] Y. Qiu, et al., Phys. Rev. A **89**, 042321 (2014).
[26] L. Diósi and W. T. Strunz, Phys. Lett. A **235**, 569 (1997); L. Diósi, et al., Phys. Rev. A **58**, 1699 (1998); W. T. Strunz, et al., Phys. Rev. Lett. **82**, 1801 (1999); J. Jing and T. Yu, ibid. **105**, 240403 (2010).
[27] J.-M. Cai, et al., New. J. Phys. **14**, 113023 (2012); P. Chen, Phys. Rev. A **73**, 022343 (2006); S. K. Mishra, et al., Phys. Rev. A **90**, 033817 (2014).
[28] A. G. Kofman and G. Kurizki, Phys. Rev. Lett. **93**, 130406 (2004); K. Khodjasteh and D. A. Lidar, ibid. **95**, 180501 (2005); G. S. Uhrig, ibid. **98**, 100504 (2007); J. R. West, et al., ibid. **105**, 230503 (2010); G. A. Paz-Silva, et al., ibid. **108**, 080501 (2012).
[29] J. Jing, et al., Phys. Rev. A **88**, 022333 (2013); J. Jing, et al., ibid. **89**, 032110 (2014); J. Jing and L.-A. Wu, Sci. Rep. **3**, 2746 (2013); ibid. **4**, 6229 (2014).
[30] A. R. P. Rau and W. Zhao, Phys. Rev. A **71**, 063822 (2005); D. Uskov and A. R. P. Rau, ibid. **78**, 022331 (2008).
[31] J. Bylander, et al., Nat. Phys. **7**, 565 (2011).
[32] M. Möttönen, et al., Phys. Rev. A **73**, 022332 (2006). Y. M. Galperin, et al., Phys. Rev. Lett. **96**, 097009 (2006). A. De, et al., Phys. Rev. A **83**, 042331 (2011).