Correlation Lengths and Topological Entanglement Entropies of Unitary and Nonunitary Fractional Quantum Hall Wave Functions

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Correlation Lengths and Topological Entanglement Entropies of Unitary and Non-Unitary Fractional Quantum Hall Wavefunctions

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Using the newly developed Matrix Product State (MPS) formalism for non-abelian Fractional Quantum Hall (FQH) states, we address the question of whether a FQH trial wave function written as a correlation function in a non-unitary Conformal Field Theory (CFT) can describe the bulk of a gapped FQH phase. We show that the non-unitary Gaffnian state exhibits clear signatures of a pathological behavior. As a benchmark we compute the correlation length of Moore-Read state and find it to be finite in the thermodynamic limit. By contrast, the Gaffnian state has infinite correlation length in (at least) the non-Abelian sector, and is therefore gapless. We also compute the topological entanglement entropy of several non-abelian states with and without quasiholes. For the first time in FQH the results are in excellent agreement in all topological sectors with the CFT prediction for unitary states. For the non-unitary Gaffnian state in finite size systems, the topological entanglement entropy seems to behave like that of the Composite Fermion Jain state at equal filling.

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Our understanding of the Fractional Quantum Hall (FQH) effect has benefited substantially from the use of model wavefunctions [1–4]. These wavefunctions, although not ground-states of realistic hamiltonians, are nonetheless supposed to capture the universal behavior of the state such as quasiparticle charge, statistics, braiding in the gapped bulk, as well as electron and quasihole exponents on the gapless edge. In a seminal paper[4] Moore and Read proposed to use conformal blocks, *i.e.* correlation functions in a Conformal Field Theory (CFT), as a building block to write down bulk model wavefunctions for the ground state and its quasihole excitations. This construction relies on a number of conjectures, the most important being that such a model bulk wavefunction describes a gapped topological state. Another assumption is that the universality class of the fractional quantum Hall state - most notably the braiding and fusion properties of the excitations - can be read off directly from the bulk CFT. Finally the *bulk-edge correspondence* is usually assumed. It states that (most) properties of the physical gapless edge states should be described by the same CFT that was used to build the bulk wavefunctions. Despite the nontrivial nature of these conjectures, there is a large body of (mostly exact diagonalization) evidence that supports the Moore-Read construction.

However, this program has been observed recently to break down for non-unitary CFTs. While large sets of bulk trial wavefunctions can be written as correlation functions in a non-unitary CFT[5–9], the bulk and edge CFT can no longer match. Indeed the edge CFT is a low-energy effective theory describing the physical edge states, and as any proper quantum field theory it has to be unitary [10]. In that case one of the aforementioned hypothesis has to break down : either the edge CFT is different from the one used to write the bulk state - which was shown to be unlikely, at least for the non-unitary Gaffnian state [10]- or the bulk state itself has to be gapless[11]. The Gaffnian state is also the prototype of a two dimensional phase where one could study how gapless modes spoil the topological degrees of freedom[12].

Unfortunately, a direct numerical observation of the pathology of the non-unitary state as a FQH model wavefunction has been plagued by the relatively small system sizes[5, 13, 14] that can be reached within exact diagonalization and even with Jack polynomial techniques[15]. Thus the question of if and how the non-unitary states fail to be bona fide gapped bulk states has remained unsolved.

Recently, great progress[16–18] has been made in rewriting many model states using an exact matrix product state[19, 20] (MPS) description. This description allows for efficient encoding of the states, basically allowing - with excellent accuracy and controlled truncation parameter - the squaring of the sizes previously attained with exact diagonalization. A detailed description of the general method for obtaining the states and their entanglement spectra, approximation parameters, as well as examples of the MPS description of a large number of non-abelian unitary and non-unitary wavefunctions has been provided in Ref. 18.

In this article, we use the MPS machinery to show that model wavefunctions built from non-unitary CFTs exhibit clear signatures of their pathological behavior. We first analyze the topological entanglement entropy

[21, 22] of the unitary Moore-Read and $\mathbb{Z}_3$ Read-Rezayi and show that it accurately matches the CFT prediction both for the ground and for the quasihole states. This is the first time such an agreement is obtained for model FQH states and all topological sectors. Indeed, previous studies have been plagued by finite-size issues[23] or have not been able to access all the topological sectors[16]. We then perform the same analysis on the non-abelian non-unitary Gaffnian wavefunction, and find that its topological entanglement entropy is that of any abelian state at the same filling $\nu = 2/5$. By adding quasiholes to the Gaffnian wavefunction, we are able to accurately obtain their quantum dimension from topological entropy studies, and find it to be unity. This implies that the quasiholes would be abelian, despite the fact that their fusion rule is non-abelian. This suggests that the bulk of the Gaffnian state is gapless. While the fate of the entanglement entropy in critical one dimensional models based on non-unitary CFTs has been recently discussed[24], an analogous study for two dimensional systems was missing until now. We test this by computing the gap of the transfer matrix, which encodes the correlation length of local observables. For the unitary Laughlin, Moore-Read and $\mathbb{Z}_3$ Read-Rezayi state we find a finite correlation length in every abelian and non-abelian sector, while for the non-unitary Gaffnian we find that the correlation length in the quasihole sector extrapolates to infinity in the thermodynamic limit. In the electron sector the extrapolation is equivocal since the Gaffnian correlation length exhibits a level-crossing behavior.

One tool to extract topological information from the ground state wave function is the (topological) entanglement entropy[21, 22, 25]. We consider the simplest case of bipartite orbital entanglement on the cylinder between two semi-infinite parts A and B of the system in its ground state $|\Psi\rangle$. This partition is characterized by the reduced density matrix $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$ of subsystem A, obtained by tracing out all the B degrees of freedom, a procedure which uses, in the thermodynamic limit, only the highest eigenvalue eigenstate of the MPS transfer matrix, as detailed in Ref. 18. Among the various entropies that have been considered as an entanglement measurement, the entanglement entropy is the most popular one. It is defined as the Von Neumann entropy associated with ρ_A *i.e.* $S = -\mathrm{Tr}_A[\rho_A \ln \rho_A]$. For a system in d dimensions with a finite correlation length ζ, the entanglement entropy satisfies the area law[26]. For two dimensional topological phases, Refs. [21] and [22] showed that the first correction to the area law is a topological term:

$$S_A \simeq \alpha L - \gamma \tag{1}$$

where $L \gg \zeta$ is the length of the boundary of region A (the cylinder perimeter in our case) and α is a non-universal constant. The sub-leading term γ is called the topological entanglement entropy: it is a constant for a given topologically ordered phase and is related to the to-

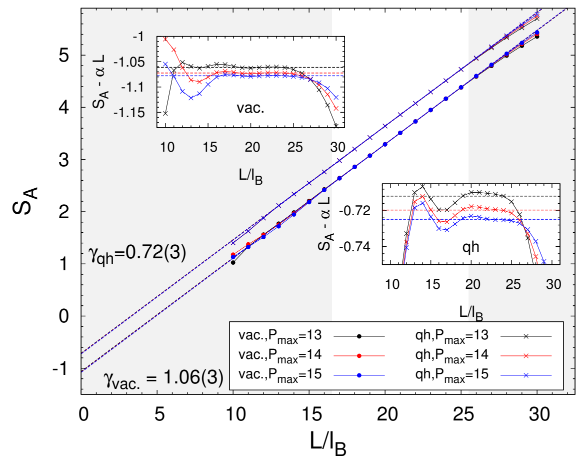

FIG. 1. Entanglement entropy S_A for the Moore-Read state in the vacuum and sigma sectors (*i.e.* with a non-abelian σ quasihole at positions $\pm\infty$) as a function of the cylinder perimeter L and the three largest truncation parameters $P_{\max}$ that can be reached. For large L, we observe the saturation due to the finite CFT truncation (the saturation increasing with $P_{\max}$. γ is extracted through a linear fit where we exclude the shaded regions. We consider only perimeters where a convergence of S_A as a function of $P_{\max}$ to be lower than 10^{-2} has been reached. The extrapolated $\gamma_{vac.}$ (resp. $\gamma_{qh.}$) for the vacuum (resp. sigma) sector is in agreement with the predicted value $\ln\sqrt{8}$ (resp. $\ln 2$). *Insets*: S_A minus the area law contribution αL for the vacuum (upper left corner) and the sigma (lower right corner) sectors. The value of α which is very close for both topological sectors ($\alpha \simeq 0.22/l_B$), is extracted from the linear fit.

tal quantum dimension of the phase. Additional changes in this term appear if regions A, B contain topological (quasihole) excitations, allowing for the determination of the specific quantum dimensions of each topological particle. For a given type of excitations a, the quantum dimension d_a defines how the Hilbert space dimension exponentially increases with the number of such excitations. Each type of excitations corresponds to a topological sector. Abelian excitations have a quantum dimension equal to 1 while non-abelian ones have $d_a > 1$. The topological entanglement entropy for a system with topological charge a in region A is given by

$$\gamma = \ln\left(\frac{\mathcal{D}}{d_a}\right) \tag{2}$$

where $\mathcal{D} = \sqrt{\sum_a d_a^2}$ is the total quantum dimension characterizing the topological field theory describing the phase. A state with abelian excitations, at filling factor $\nu = p/q$, such as the Laughlin states or more generally the Jain's composite fermions[3], has q abelian topological sectors. Thus its topological entanglement entropy is $\gamma = \log\left(\sqrt{q}\right)$. We numerically obtain the topological entanglement entropy for several states on an infinite cylinder of perimeter L. The bi-partition is performed

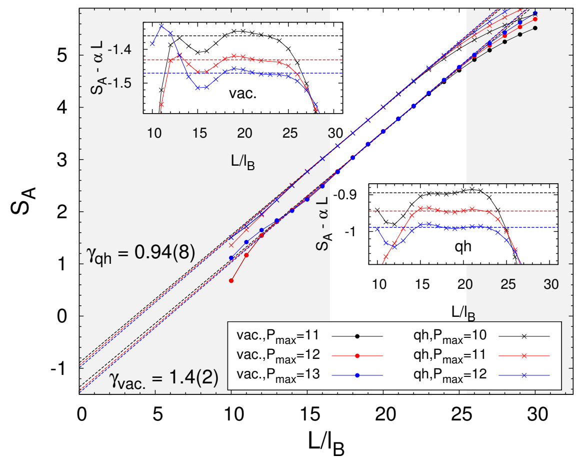

FIG. 2. Entanglement entropy S_A for the $\mathbb{Z}_3$ Read-Rezayi state in the vacuum sector and with a non-abelian quasihole at each end of the infinite cylinder as a function of the cylinder perimeter L and the three largest truncation parameters $P_{\max}$ that can be reached. We use the same conventions than those of Fig. 1, including for the insets. Note that the lower accuracy on γ is due to the data at $P_{\max} = 11$. Considering only the two largest truncation parameters would give $\gamma = 1.40(3)$. The fitted value of $\alpha \simeq 0.25/l_B$ is also very close for both topological sectors.

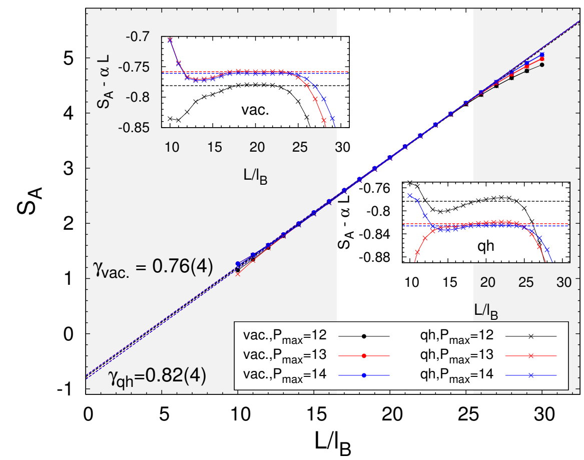

FIG. 3. Entanglement entropy S_A for the Gaffnian state in the vacuum and sigma sectors as a function of the cylinder perimeter L and the three largest truncation parameters $P_{\max}$ that can be reached. We use the same conventions than those of Fig. 1, including for the insets. The fitted value of $\alpha \simeq 0.20/l_B$ is also very close for both topological sectors. Note that the difference of the entanglement entropies between the two sectors at any perimeter (in the converged region) is always lower than 0.02.

perpendicular to the cylinder axis, such that the length of the boundary between two regions is L. Our numerical results contain a truncation parameter, $P_{\max}$, whose meaning is twofold : it is the maximum momentum we use for the edge CFT fields in the auxiliary bond of the MPS [18]; it is also the maximum descendant field level in the truncated CFT used to build the MPS[27].

For the Laughlin state, the topological entanglement entropy has been computed using a MPS representation in Ref. 16 and matches the theoretical prediction. Beyond abelian states, the simplest example of a non-abelian state is the Moore-Read state[4]. It has 6 topological sectors, 4 abelian ones and 2 non-abelian ones with a quantum dimension $d_\sigma = \sqrt{2}$. In Fig. 1, we show the entanglement entropy as a function of the cylinder perimeter for a for the Moore-Read state in the abelian (or vacuum) sector and the sigma sector. We first focus on the vacuum sector. Two size effects are present in the raw data: for very small perimeters, we are in a regime where the Moore-Read state is not fully developed (the "thin torus" regime). For very large perimeters, the entanglement entropy saturates due to the finite truncation in the CFT. We hence present only the regime where a clear area law of the entanglement entropy is observed. We extrapolate to obtain the topological entanglement entropy $\gamma_{\mathrm{vac.}}$ and find it in excellent agreement with the conjectured value $\ln\sqrt{8} \simeq 1.039$ for the groundstate. Adding a quasihole at each end of the infinite cylinder to access the sigma (i.e. quasihole) sector

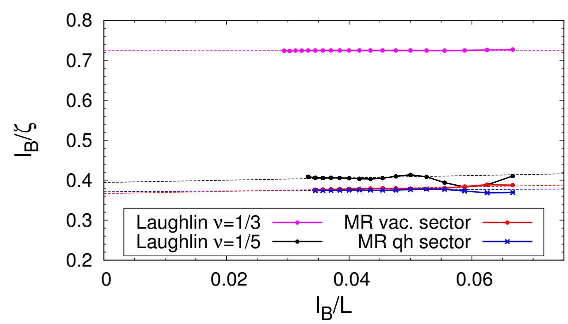

FIG. 4. Correlation lengths ζ of the Laughlin $\nu = 1/3$, $\nu = 1/5$, and the Moore-Read state for both the vacuum and the quasihole sectors as a function of cylinder perimeter L. A linear fit as a function of $1/L$ gives the thermodynamical values $\zeta_3/l_B = 1.381(1)$ for $\nu = 1/3$, $\zeta_5/l_B = 2.53(7)$ for $\nu = 1/5$, $\zeta_{\mathrm{vac}}/l_B = 2.73(1)$ for the Moore-Read vacuum sector and $\zeta_{\mathrm{qh}}/l_B = 2.69(1)$ for the Moore-Read quasihole sector.

and extracting the corresponding γ_{qh} from Fig. 1 give a non-abelian quasihole quantum dimension of $d_\sigma = 1.4$, again in excellent agreement with the CFT conjecture. Note that as expected, the area law linear factor α given by Eq. 1 is identical (within numerical accuracy) in both topological sectors. This result holds true for each state we have considered. Fig. 2 shows the data for the $\mathbb{Z}_3$ Read-Rezayi state. We again obtain excellent agreement for the topological entanglement entropies with the CFT predictions of $\gamma_{\mathrm{vac.}} = 1.44768$ and $\gamma_{\mathrm{qh}} = 0.96647$ for the ground-state and quasihole sectors, respectively.

In Fig. 3 we present the results for the non-unitary Gaffnian state, both in the ground-state and with a σ particle in side A. We find a clear area law in both cases. Upon extrapolation, the topological entanglement entropy is equal (within numerical error) between these two cases, suggesting that the quantum dimension of the σ particle is 1 ± 0.02. This would correspond to an abelian particle although its fusion rules [18] are clearly non-abelian. The value of the topological entanglement entropy is also within numerical error of $\log \sqrt{5} = 0.804719$ - the value for an abelian state at filling $2/5$, even though the state is non-abelian and the value of the total quantum dimension computed from the S-matrix leads to a clearly different value of 1.44768 for the vacuum sector and a quantum dimension of the σ particle of $(1+\sqrt{5})/2$. Note that a similar result was observed for classical stringnet models[28], where the constant correction to the entanglement only probes the abelian sector. Moreover, due to the small range of accessible perimeter values, we cannot probe any logarithmic correction to the area law which might emerge in some critical model[29, 30]. While the resemblance of the entanglement entropy with that of an abelian 2/5 state might not be numerically surprising - considering the extremely large ($> 93\%$) overlap of this state with the abelian Jain state at the same filling factor - the abelian quantum dimension of the purportedly non-abelian quasiholes is impossible to reconcile. This is strong evidence that the Gaffnian state is not gapped in the bulk, which casts a doubt on the validity of our results for the Gaffnian entanglement entropy and quantum dimensions. Indeed our calculation involves making the cylinder infinitely long before extrapolating for large perimeters L. For a gapless state it is unclear whether this yields the same result as sending both length and perimeter to infinity while keeping a finite aspect ratio. Nevertheless this is an indication of the gapless nature of the Gaffnian, which we are going to confirm and quantify by looking for long range correlations.

We now numerically compute the bulk correlation length in the Laughlin, Moore-Read and Gaffnian states. This correlation length is intrinsically related to the gap of the transfer matrix[31]. The correlation function of an operator $\mathcal{O}(x)$ takes the form:

$$\langle \mathcal{O}^\dagger(x)\mathcal{O}(0)\rangle - \langle \mathcal{O}^\dagger(x)\rangle\langle \mathcal{O}(0)\rangle \propto e^{-|x|/\zeta} \qquad (3)$$

where the correlation length $\zeta(L) = 2\pi l_B^2/(L\log(\frac{\lambda_1}{\lambda_2}))$ can be expressed in terms of the largest and second largest eigenvalues of the MPS transfer matrix $\lambda_1(L)$ and $\lambda_2(L)$. L is the circumference of the cylinder where the correlation function is computed and l_B is the magnetic length. In all the FQH states we considered we observed that the MPS transfer matrix was always gapped at finite L, which leads to a finite correlation length $\xi(L)$. When describing a gapped physical system, the correlation length ζ must remain finite in the thermodynamic limit $L \to \infty$. In Fig. 4 we show the correlation lengths for the $\nu = 1/3$ and $\nu = 1/5$ Laughlin states. As expected, they quickly saturate to a constant value, confirming the gapped nature of the Laughlin states. Fig. 4 also provides the correlation length of the Moore-Read wavefunction. In this situation, the MPS transfer matrix is block diagonal in the abelian $(1, \psi)$ and non-abelian (σ) sector [18]. Hence two gaps exist, corresponding to the two correlation lengths of the abelian and non-abelian sectors. Both correlations lengths are finite in the thermodynamic limit and have roughly the same value as observed in Fig. 4. Ref. 32 has found a correlation length of resp. $\zeta \simeq 2.7 l_B$ for the Moore-Read state in the vacuum sector, in agreement with our results.

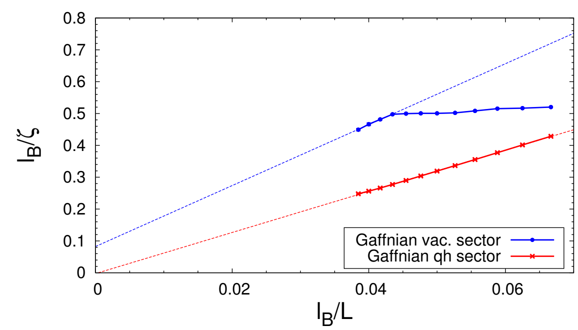

FIG. 5. Correlation lengths ζ of the Gaffnian state for both the vacuum and the quasihole sectors as a function of cylinder perimeter L. A linear fit as a function of $1/L$ for the correlation lengths $\zeta_{\rm qh}$ in the quasihole sector leads to an divergent extrapolated value. In the vacuum sector, due to the level crossing, it is difficult to make any reliable extrapolation.

In Fig. 5 we plot the correlation length for the non-unitary Gaffnian state for both sectors. For the quasihole (the field of scaling dimension $-1/20$ in the neutral CFT[18]) in the thermodynamic limit the correlation length diverges, signaling gaplessness. In the vacuum sector, the correlation length does not have a smooth behavior. While for small cylinder perimeter L, the correlation length seems to clearly extrapolate to a finite value, for larger L the slope changes dramatically. This is due to a level crossing in the second transfer matrix eigenvalue λ_2. Note that the system size where the level crossing occurs is out of reach of previous techniques which are limited to sizes around $\simeq 17 l_B$. While other works using the exact expression of the Gaffnian quasihole states in terms of Jack polynomials[15] have indicated that the state fails to screen the non-abelian quasihole, this is the first clear calculation of the gapless nature of the non-unitary Gaffnian state.

In this paper we have computed the topological entanglement entropy for the Moore-Read, $\mathbb{Z}_3$ Read-Rezayi and Gaffnian states for both the vacuum and the quasihole sectors. While for unitary states the total quantum dimension and the quantum dimension of the individual quasiholes matches the CFT predictions, for the Gaffnian state, we find quantum dimensions identical to those of the abelian FQH state at identical filling. We also computed the correlation lengths in these states and find that the unitary states have finite correlation lengths in the thermodynamic limit, while the non-unitary Gaffnian has diverging correlation length in at least the quasihole sector, signaling gaplessness.[33]

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[1] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[2] B. I. Halperin, Helv. Phys. Acta **56**, 75 (1983).
[3] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
[4] G. Moore and N. Read, Nuclear Physics B **360**, 362 (1991).
[5] S. H. Simon, E. H. Rezayi, N. R. Cooper, and I. Berdnikov, Phys. Rev. B **75**, 075317 (2007).
[6] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. B **77**, 184502 (2008).
[7] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **100**, 246802 (2008).
[8] B. Estienne, N. Regnault, and R. Santachiara, Nuclear Physics B **824**, 539 (2010).
[9] S. H. Simon, E. H. Rezayi, and N. Regnault, Phys. Rev. B **81**, 121301 (2010).
[10] N. Read, Phys. Rev. B **79**, 245304 (2009).
[11] N. Read, Phys. Rev. B **79**, 045308 (2009).
[12] P. Bonderson and C. Nayak, Phys. Rev. B **87**, 195451 (2013).
[13] N. Regnault, M. O. Goerbig, and T. Jolicoeur, Phys. Rev. Lett. **101**, 066803 (2008).
[14] C. Toke and J. K. Jain, Phys. Rev. B **80**, 205301 (2009).
[15] B. A. Bernevig, P. Bonderson, and N. Regnault, ArXiv e-prints (2012), arXiv:1207.3305 [cond-mat.mes-hall].
[16] M. P. Zaletel and R. S. K. Mong, Phys. Rev. B **86**, 245305 (2012).
[17] B. Estienne, Z. Papić, N. Regnault, and B. A. Bernevig, Phys. Rev. B **87**, 161112 (2013).
[18] B. Estienne, N. Regnault, and B. A. Bernevig, ArXiv e-prints (2013), arXiv:1311.2936 [cond-mat.str-el].
[19] M. Fannes, B. Nachtergaele, and R. Werner, Communications in mathematical physics **144**, 443 (1992).
[20] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, Quantum Info. Comput. **7**, 401 (2007).
[21] M. Levin and X.-G. Wen, Phys. Rev. Lett. **96**, 110405 (2006).
[22] A. Kitaev and J. Preskill, Phys. Rev. Lett. **96**, 110404 (2006).
[23] O. S. Zozulya, M. Haque, K. Schoutens, and E. H. Rezayi, Phys. Rev. B **76**, 125310 (2007).
[24] D. Bianchini, O. Castro-Alvaredo, B. Doyon, E. Levi, and F. Ravanini, Journal of Physics A: Mathematical and Theoretical **48**, 04FT01 (2015).
[25] P. Calabrese and J. Cardy, J. Stat. Mech. , P06002 (2004).
[26] M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993).
[27] A pictorial representation of the difference between the usual DMRG cutoff and our cutoff, as well as several numerical benchmarks, are presented the Supplementary Material.
[28] M. Hermanns and S. Trebst, Phys. Rev. B **89**, 205107 (2014).
[29] E. Fradkin and J. E. Moore, Phys. Rev. Lett. **97**, 050404 (2006).
[30] M. Oshikawa, ArXiv e-prints (2010), arXiv:1007.3739 [cond-mat.stat-mech].
[31] M. Fannes, B. Nachtergaele, and R. Werner, Communications in Mathematical Physics **144**, 443 (1992).
[32] M. Baraban, G. Zikos, N. Bonesteel, and S. H. Simon, Phys. Rev. Lett. **103**, 076801 (2009).
[33] See Supplemental Material, which includes Refs. [18, 34–37].
[34] H. Li and F. D. M. Haldane, Phys. Rev. Lett **101**, 010504 (2008).
[35] E. Verlinde, Nuclear Physics B **300**, 360 (1988).
[36] T. Gannon, Nuclear Physics B **670**, 335 (2003).
[37] P. Fendley, M. Fisher, and C. Nayak, Journal of Statistical Physics **126**, 1111 (2007).