Nearly Linear Light Cones in Long-Range Interacting Quantum Systems

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Nearly-linear light cones in long-range interacting quantum systems

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In non-relativistic quantum theories with short-range Hamiltonians, a velocity v can be chosen such that the influence of any local perturbation is approximately confined to within a distance r until a time $t \sim r/v$, thereby defining a linear light cone and giving rise to an emergent notion of locality. In systems with power-law $(1/r^{\alpha})$ interactions, when α exceeds the dimension D, an analogous bound confines influences to within a distance r only until a time $t \sim (\alpha/v) \log r$, suggesting that the velocity, as calculated from the slope of the light cone, may grow exponentially in time. We rule out this possibility; light cones of power-law interacting systems are algebraic for $\alpha > 2D$, becoming linear as $\alpha \to \infty$. Our results impose strong new constraints on the growth of correlations and the production of entangled states in a variety of rapidly emerging, long-range interacting atomic, molecular, and optical systems.

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Though non-relativistic quantum theories are not explicitly causal, Lieb and Robinson [1] proved that an effective speed limit emerges dynamically in systems with short-ranged interactions, thereby extending the notion of causality into the fields of condensed matter physics, quantum chemistry, and quantum information science. Specifically, they proved that when interactions have a finite range or decay exponentially in space, the influence of a local perturbation decays exponentially outside of a space-time region bounded by the line $t = r/v$, which therefore plays the role of a light cone [Fig. 1(a)]. However, many of the systems to which non-relativistic quantum theory is routinely applied—ranging from frustrated magnets and spin glasses [2, 3] to numerous atomic, molecular, and optical systems [4–8]—possess power-law interactions, and hence do not satisfy the criteria set forth by Lieb and Robinson. Many questions about the fate of causality in such systems lack complete answers: Can information be transmitted with an arbitrarily large velocity [9], and if so, how quickly (in space or time) does that velocity grow? Under what circumstances does a causal region exist, and when it does, what does it look like [9–14]? The answers to these questions have far reaching consequences, for example imposing speed limits on quantum-state transfer [15] and on thermalization rates in many-body quantum systems [16], determining the strength and range of correlations in equilibrium [17], and constraining the complexity of simulating quantum dynamics with classical computers [18].

The results of Lieb and Robinson were first generalized to power-law $(1/r^{\alpha})$ interacting systems by Hastings and Koma [17], with the following picture emerging. For $\alpha > D$ [19], the influence of a local perturbation is bounded by a function $\propto e^{vt}/r^{\alpha}$, and while a light cone can still be defined as the boundary outside of which this function falls below some threshold value, yielding $t \sim \log r$, that boundary is *logarithmic* rather than linear [Fig. 1(b)]. Improvements upon these results exist,

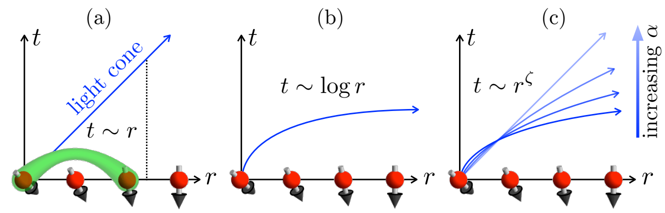

FIG. 1. (color online). (a) In a short-range interacting system, perturbing a single spin at $t = r = 0$ can only influence another spin (green connection) if it falls within a causal region bounded by a linear light cone ($t \sim r$) [1]. (b) Existing bounds for power-law interacting systems [12, 17] result in a logarithmic light cone ($t \sim \log r$) at large distances and times, and thus the maximum velocity grows exponentially in time. (c) We show that light cones of power-law interacting systems are necessarily polynomial, becoming increasingly linear for shorter-range interactions.

revealing, e.g., that the light-cone remains linear at intermediate distance scales [12], but all existing bounds consistently predict an asymptotically logarithmic light cone. An immediate and striking consequence is that the maximum group velocity, defined by the slope of the light cone, grows exponentially with time, thus suggesting that the aforementioned processes — thermalization, entanglement growth after a quench, etc. — may in principle be sped up *exponentially* by the presence of long-range interactions. In this manuscript, we show that this scenario is not possible. While light cones can potentially be sub-linear for any finite α, thus allowing a velocity that grows with time, for $\alpha > 2D$ they remain bounded by a polynomial $t \sim r^{\zeta}$, and $\zeta \leq 1$ approaches unity for increasing α [Fig. 3(c)]. Though the range of α over which our results are valid is reduced (relative to the results in Ref. [17]), they apply to a number of experimentally relevant systems, e.g. dipolar interactions in 1D (as can be realized with magnetic atoms, polar molecules, or trapped ions) or van-der-Waals-type interactions between Rydberg atoms in 1D or 2D.

Model and formalism.—We assume a generic spin model with time-independent Hamiltonian [20]

$$H = \frac{1}{2}\sum_{\mu,y,z} J_\mu(y,z) V_{y\mu} V_{z\mu}, \tag{1}$$

where $V_{y\mu}$ is a spin operator on site y with $\|V_{y\mu}\| = 1$ (where $\|O\|$ denote the operator norm of an operator O, which is the magnitude of its eigenvalue with largest absolute value). The non-negative coupling constants satisfy $\sum_\mu J_\mu(y,z) \equiv J(y,z) \le J/d(y,z)^\alpha$ for $y \neq z$, with $d(y,z)$ the distance between lattice sites y and z, and $J_\mu(y,y) = 0$. Our goal is to bound the size of an unequal-time commutator of two unity-norm operators A and B initially residing on sites i and j, respectively,

$$C_r(t) = \|[A(t), B]\| \le \mathscr{C}_r(t), \tag{2}$$

where $r = d(i,j)$. Since spin operators on different sites commute, $C_r(t)$ captures the extent to which an operator A has "spread" onto the lattice site j during the time evolution. As a result, it bounds numerous experimentally measurable quantities, for example connected correlation functions after a quantum quench [12–14, 21]. In general, a light cone can be defined by setting $\mathscr{C}_r(t)$ equal to a constant and solving for t as a function of r. A natural way to parametrize the shape of the light cone is to ask whether it can be bounded by the curve $r = t^\beta$ (with $\beta \ge 0$) in the large t limit, which is true whenever $\lim_{t\to\infty} \mathscr{C}_{t^\beta}(t) = 0$. Defining $1/\zeta$ to be the smallest value of β for which this limit vanishes, we can say that $t \sim r^\zeta$ is the tightest possible polynomial light cone. The original work by Lieb and Robinson proved that $\zeta = 1$ when interactions are finite-ranged or exponentially decaying. However, the generalization of their results to power-law interacting Hamiltonians [17] yields $\mathscr{C}_r(t) \sim e^{vt}/r^\alpha$, and thus $\lim_{t\to\infty} \mathscr{C}_{t^\beta}(t)$ never vanishes for finite β. Though Ref. [12] demonstrated that a linear light cone can still persist at intermediate distance scales, the true asymptotic shape of the light cone was nevertheless logarithmic. Thus the consensus of all previously available bounds is that $\zeta \to 0$, and the light cone is not bounded by a polynomial. In what follows, we first give a detailed physical picture (based on an interaction-picture representation of the short-range physics) of why a logarithmic light cone cannot exist, and then we present a formal proof that the light-cone is indeed algebraic. The technical details supporting our main formal results, Eqs. (10-12), are deferred to the supplemental material [22].

Strategy.—To prove the existence of a polynomial light cone, we begin by breaking H into a short-range and a long-range contribution, $H = H^{\rm sr} + H^{\rm lr}$, separated by a cutoff length scale χ. Defining $J_\mu^{\rm sr[lr]}(y,z) = J_\mu(y,z)$ if $d(y,z) \le \chi$ $[> \chi]$ and 0 otherwise, we can write

$$H^{\rm sr[lr]} = \sum_{\mu,y,z} J_\mu^{\rm sr[lr]}(y,z) V_{y\mu} V_{z\mu}. \tag{3}$$

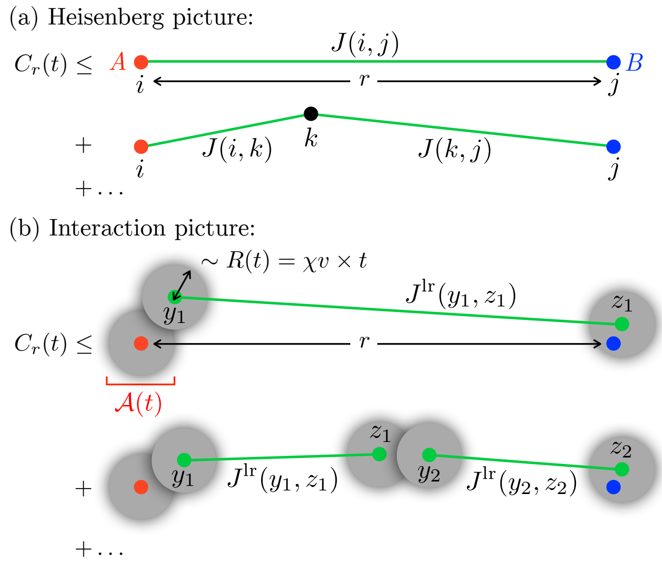

FIG. 2. (color online) Schematic illustration of a Lieb Robinson-type bound. (a) *Heisenberg picture*. The time evolution of an operator A is bounded by a series in which repeated applications of the Hamiltonian connect site i to site j. (b) *Interaction picture*. A similar series can be used to bound the dynamics induced by the interaction-picture time-evolution operator, but now the operator A and the interaction terms in the Hamiltonian are spread out over a light-cone radius of the short-range Hamiltonian.

We then move to the interaction picture [24] of $H^{\rm sr}$, where

$$C_r(t) = \|[\mathcal{U}^\dagger(t)\mathcal{A}(t)\mathcal{U}(t), B]\|. \tag{4}$$

Here $\mathcal{A}(t) = \exp(itH^{\rm sr}) A \exp(-itH^{\rm sr})$ [and all other script operators except $\mathcal{U}(t)$] is evolving under the influence of $H^{\rm sr}$, and the interaction-picture time evolution operator $\mathcal{U}(t)$ is a time-ordered exponential

$$\mathcal{U}(t) = T_\tau \exp\left(-i\int_0^t d\tau\, \mathcal{H}^{\rm lr}(\tau)\right), \tag{5}$$

where

$$\mathcal{H}^{\rm lr}(\tau) = \frac{1}{2}\sum_{\mu,y,z} J_\mu^{\rm lr}(y,z)\mathcal{V}_{y\mu}(\tau)\mathcal{V}_{z\mu}(\tau) \equiv \sum_{y,z} \mathcal{W}_{yz}(\tau). \tag{6}$$

The plan is now to treat the short-range physics, responsible for the time-dependence of interaction picture operators $\mathcal{A}(t)$ and $\mathcal{W}_{yz}(\tau)$, and the long-range physics, captured by the remaining interaction-picture time evolution operator $\mathcal{U}(t)$, with two independent bounds. The basic physical picture to have in mind is shown in Fig. 2. The original Lieb-Robinson approach is to work in the Heisenberg-picture, expressing $\mathscr{C}_r(t)$ as series of terms connecting sites i and j by repeated applications of H [1, 12, 17, 25] [Fig. 2(a)]. We will instead bound the dynamics induced by $\mathcal{U}(t)$ by a series of terms connecting sites i and j by repeated applications of $\mathcal{H}^{\rm lr}(\tau)$ [Fig. 2(b)]. Though $\mathcal{H}^{\rm lr}(\tau)$ is not a sum of local operators, the $\mathcal{V}_y(\tau)$ which comprise it are still approximately contained within a ball of (time-dependent) radius $R(t) = \chi v \times t$ [gray shaded disks in Fig. 2(b)], which is the light cone of the short-range Hamiltonian. Here v would be the Lieb-Robinson velocity for a nearest-neighbor Hamiltonian with coupling strength J, and must be multiplied by χ to account for the longest-range terms in $H^{\rm sr}$.

Our approach is motivated by the following observation: If we assume the existence of a logarithmic light cone, we can choose the cutoff χ to scale in such a way that $\mathscr{C}_r(t)$ does not grow exponentially in time, which contradicts the assumption. To see this, we first note that the existence of a logarithmic light cone allows us to choose χ to scale with *any* power of t while satisfying the following inequality along the light-cone boundary (at sufficiently long times),

$$R(t) = \chi v \times t \ll r \sim e^{vt}. \tag{7}$$

Physically, this inequality ensures that the point r falls well outside the short-range light-cone distance $R(t)$, and as a result both the operator $\mathcal{A}(t)$ and the $\mathcal{V}_y(\tau)$ comprising $\mathcal{H}^{\rm lr}(\tau \leq t)$ appear nearly local when viewed on the length scale r. We therefore expect that the time evolution induced by $\mathcal{U}(t)$ [Fig. 2(b)] should be qualitatively similar—up to the possibility of a different velocity—to that induced by U [Fig. 2(a)]. The velocity can be estimated by considering the following expansion of $A(t)$,

$$A(t) = \mathcal{A}(t) + i\sum_{yz}\int_0^t d\tau [\mathcal{W}_{yz}(\tau), \mathcal{A}(t)] + \dots. \tag{8}$$

Due to the quasi-locality of interaction-picture operators, a general commutator $[\mathcal{W}_{yz}(\tau), \mathcal{A}(t)]$ is exponentially suppressed unless either y or z resides within a distance $2R(t)$ of site i. Ignoring (for now) the exponentially small corrections from outside the short-range light cone, we can restrict the summation to run over y and z such that either $d(i, y) \leq 2R(t)$ or $d(i, z) \leq 2R(t)$, giving

$$\|\sum_{y,z}\int_0^t d\tau [\mathcal{W}_{yz}(\tau), \mathcal{A}(t)]\| \lesssim t \times R(t)^D \lambda_\chi \tag{9}$$

Here $\lambda_\chi = \sum_z J^{\rm lr}(y, z) \sim \chi^{D-\alpha}$, with $J^{\rm sr[lr]}(y, z) = \sum_\mu J_\mu^{\rm sr[lr]}(y, z)$. The coefficient of t on the right-hand side of Eq. (9) suggests a velocity $v_\chi \sim R(t)^D \lambda_\chi \sim t^D \chi^{2D-\alpha}$, which can be made small for large χ whenever $\alpha > 2D$.

An important achievement of this paper is a proof that the parametrically small velocity v_χ also controls the higher-order contributions from the interaction picture time-evolution operator. Therefore, in moving from the Heisenberg picture to the interaction picture, we are able (loosely speaking) to make the replacement $\mathscr{C}_r(t) \sim \exp(vt)/r^\alpha \to \exp(v_\chi t)/r^\alpha$. By letting χ grow with t in such a way that $v_\chi t$ stays constant in time [which can always be done in a manner consistent with Eq. (7)], the exponential time dependence is suppressed, violating our assumption of a logarithmic light cone. Indeed, as we will show, a proper scaling of χ will enable us to change the time dependence from exponential to algebraic, which in turn enables the recovery of a polynomial light-cone.

Derivation.—In order to formalize the above picture, we must first take a step back and treat the interaction-picture operators more carefully. First, we denote the set of points within a radius $R_\ell(t) \equiv R(t) + \ell\chi$ of the point i by $\mathscr{B}(i, R_\ell(t))$, and the complement of this set by $\overline{\mathscr{B}}(i, R_\ell(t))$. Now we can obtain an approximation to $\mathcal{A}(t)$, supported entirely on $\mathscr{B}(i, R_\ell(t))$, by integrating over all unitaries on $\overline{\mathscr{B}}(i, R_\ell(t))$ with respect to the Haar measure [21, 22], $\mathcal{A}(\ell, t) = \int_{\overline{\mathscr{B}}(i,R_\ell(t))} d\mu(U) U\mathcal{A}(t)U^\dagger$. It is important to note that for large ℓ, $\mathcal{A}(\ell, t)$ is a good approximation to $\mathcal{A}(t)$ *at all times*, since its time-dependent support radius $R_\ell(t)$ remains a distance $\ell\chi$ outside of the short-range light cone. Because $\mathcal{A}(\ell, t)$ tends to $\mathcal{A}(t)$ as $\ell \to \infty$, we can rewrite $\mathcal{A}(t) = \sum_{\ell=0}^\infty \mathcal{A}^\ell(t)$, with $\mathcal{A}^0(t) = \mathcal{A}(0, t)$ and $\mathcal{A}^{\ell>0}(t) = \mathcal{A}(\ell, t) - \mathcal{A}(\ell - 1, t)$. Each operator $\mathcal{A}^\ell(t)$ is supported on $\mathscr{B}(i, R_\ell(t))$, and is expected to become small for large ℓ, since both $\mathcal{A}(\ell, t)$ and $\mathcal{A}(\ell - 1, t)$ are becoming better approximations to $\mathcal{A}(t)$, and hence must be approaching each other. Formally, by applying a standard short-range Lieb-Robinson bound to $H^{\rm sr}$, one can show that $\|\mathcal{A}^\ell(t)\| \leq ce^{-\ell}$, with c a constant [22]. The ability to write $\mathcal{A}(t)$ as the sum of a sequence of operators with increasing support but exponentially decreasing norm is the mathematical basis for the intuition that interaction-picture operators are quasi-local. A similar construction enables us to write $\mathcal{W}_{yz}(\tau) = \sum_{m,n} \mathcal{W}_\xi(\tau)$, where the index $\xi = \{y, z, m, n\}$ describes the location $y[z]$ and support $m[n]$ of the operators $\mathcal{V}_y^m(\tau)[\mathcal{V}_z^n(\tau)]$ comprising $\mathcal{W}_\xi(\tau)$. Once again, the size of these operators decreases exponentially in the radius of their support [22],

$$\|\mathcal{W}_\xi(\tau)\| \leq c^2 J^{\rm lr}(y, z)e^{-(m+n)}/2, \tag{10}$$

but algebraically in the separation $d(y, z)$.

Now we would like to constrain the time evolution due to $\mathcal{U}(t)$, which further expands the support of $\mathcal{A}(t)$ in Eq. (4). As suggested in Fig. 2, our bound is comprised of terms in which sites i and j are connected by repeated applications of the interaction-picture Hamiltonian $\mathcal{H}^{\rm lr}(\tau)$. Employing a generalization of the techniques originally used by Lieb and Robinson, we obtain [22]

$$C_r(t) \leq \sum_{\ell=0}^\infty \|[\mathcal{A}^\ell(t), B]\| + 4c\sum_{a=1}^\infty \frac{t^a}{a!}\mathcal{J}_a(i, j), \tag{11}$$

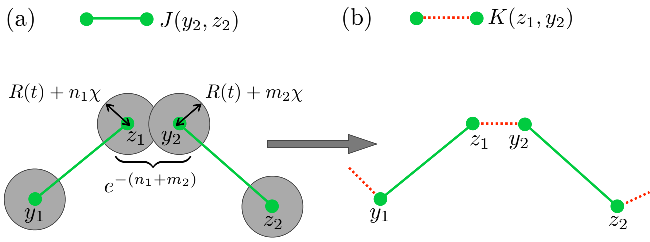

FIG. 3. (color online). (a) Schematic representation of the term $\ldots\|\mathcal{W}_{\xi_1}\|D(\xi_1,\xi_2)\|\mathcal{W}_{\xi_2}\|\ldots$ in Eq. (12). Each green line represents a single term in the interaction-picture Hamiltonian, and the operators at the endpoints are supported over a ball or radius $R(t)+m\chi$ (grey disks). (b) In deriving a bound, the additional summations over the sizes of the supports of each operator generate exponentially decaying connections between successive terms [Eq. (13)].

where

$$\mathcal{J}_a(i,j) = 4^a \sum_{\ell,\xi_1,\ldots,\xi_a} e^{-\ell} D_{\rm i}(\xi_1)\|\mathcal{W}_{\xi_1}\|D(\xi_1,\xi_2)\|\mathcal{W}_{\xi_2}\| \times \ldots$$
$$\cdots \times \|\mathcal{W}_{\xi_{a-1}}\|D(\xi_{a-1},\xi_a)\|\mathcal{W}_{\xi_a}\|D_{\rm f}(\xi_a). \quad (12)$$

Here $D(\xi_1,\xi_2)$ is unity whenever $\mathscr{B}(z_1, R_{n_1}(t)) \cap \mathscr{B}(y_2, R_{m_2}(t)) \neq \varnothing$ and vanishes otherwise, thus constraining the points z_1 and y_2 in the progression $\ldots\|\mathcal{W}_{\xi_1}\|D(\xi_1,\xi_2)\|\mathcal{W}_{\xi_2}\|\ldots$ to be near each other, as shown in Fig. 3(a). Similarly, $D_{\rm i}(\xi_1)$ is unity when $\mathscr{B}(i, R_\ell(t)) \cap \mathscr{B}(y_1, R_{m_1}(t)) \neq \varnothing$ and vanishes otherwise, while $D_{\rm f}(\xi_a)$ is unity when $j \in \mathscr{B}(z_a, R_{n_a}(t))$ and vanishes otherwise, thus constraining the first interaction $\mathcal{W}_{\xi_1}$ to originate from near the point i, and the final one $\mathcal{W}_{\xi_a}$ to terminate near the point j.

Equation (12) can be simplified by first carrying out the summation over indices $m_1,\ldots,m_a$ and $n_1,\ldots,n_a$, which were necessary to account for the exponentially decaying contribution to interaction-picture operators outside the short-range light cone. For example, considering the intersection shown in Fig. 3(a), one can show that

$$\sum_{n_1,m_2} \|\mathcal{W}_{\xi_1}\|D(\xi_1,\xi_2)\|\mathcal{W}_{\xi_2}\| \leq \quad (13)$$
$$\kappa^2 \frac{c^2 e^{-m_1}}{2} J^{\rm lr}(y_1,z_1)K(z_1,y_2)J^{\rm lr}(y_2,z_2)\frac{c^2 e^{-n_2}}{2},$$

(with κ a constant), where $K(z_1,y_2)$ decays exponentially in $d(z_1,y_2)$ [22], directly reflecting the quasi-locality of the interaction-picture operators. Using this inequality repeatedly in Eq. (12) we obtain

$$\mathcal{J}_a(i,j) \leq \kappa^2(2\kappa^2c^2)^a \sum_{\substack{y_1,\ldots,y_a\\ z_1,\ldots,z_a}} K(i,y_1)J^{\rm lr}(y_1,z_1)K(z_1,y_2)\times$$
$$\cdots \times J^{\rm lr}(y_{a-1},z_{a-1})K(z_{a-1},y_a)J^{\rm lr}(y_a,z_a)K(z_a,j). \quad (14)$$

Every term in Eq. (14) connects sites i and j by repeated applications of K's and J's, which capture, respectively, physics below and above the cutoff length-scale χ [see Fig. 3(b)]. The summations over indices $y_1,\ldots,y_a$ can then be carried out by bounding the discrete convolution $\sum_{y_2} K(z_1,y_2)J^{\rm lr}(y_2,z_2) \leq (2\kappa\lambda_\chi)F(z_1,z_2)$ to give [22]

$$\mathcal{J}_a(i,j) \leq 2\kappa^2(4\kappa^3c^2\lambda_\chi)^a \sum_{z_1,\ldots,z_a} F(i,z_1)\times\ldots\times F(z_a,j). \quad (15)$$

Because K decays exponentially while J decays only algebraically, their convolution is dominated [at large $d(z_1,z_2)$] by terms where y_2 is much closer to z_1 than to z_2, and hence F inherits the long-distance algebraic decay of $J^{\rm lr}$ [26],

$$F(z_1,z_2) = \begin{cases} 1; & d(z_1,z_2) \leq 6R(t) \\ [6R(t)/d(z_1,z_2)]^\alpha; & d(z_1,z_2) > 6R(t). \end{cases} \quad (16)$$

The remaining summation over indices $z_1,\ldots,z_a$ can be carried out (as in Ref. [17]) by invoking a so-called reproducibility condition, valid for power-law decaying functions. In particular, we find $\sum_{z_2} F(z_1,z_2)F(z_2,z_3) \leq gR(t)^D F(z_1,z_3)$ [22], where g is a constant and the factor of $R(t)^D$ enters because $F(z_1,z_2)$ decays algebraically only for $d(z_1,z_2) \gtrsim R(t)$. Utilizing this condition repeatedly in Eq. (15), we obtain [for $r > 6R(t)$]

$$\mathcal{J}_a(i,j) \leq \kappa^2\,(R(t)/r)^\alpha \times (v_\chi)^a, \quad (17)$$

where further numerical pre-factors have been absorbed into κ^2, and $v_\chi = \vartheta R(t)^D \lambda_\chi$ is a cutoff-dependent velocity (with ϑ a constant) with the scaling predicted by Eq. (9). Plugging Eq. (17) into Eq. (11), we obtain our final bound [27]

$$C_r(t) \leq \mathscr{C}_r(t) \equiv 2c\kappa\left(e^{vt-r/\chi} + 2\kappa\frac{e^{v_\chi t}}{[r/R(t)]^\alpha}\right). \quad (18)$$

The first term is the bound one would obtain for the finite-range Hamiltonian $H^{\rm sr}$. The second term contains the effect of $H^{\rm lr}$, which leads to a bound similar to that of Ref. [17], except with a velocity that is parametrically small in the cutoff χ, and a distance r that is rescaled by the radius R of the short-range light cone.

Light cone shape.—Equation (18) can now be minimized with respect to the cutoff χ, which we accomplish by letting χ scale with time as a power-law ($\chi \propto t^\gamma$), which enforces the scaling $R(t) \sim t^{\gamma+1}$ and $v_\chi t \sim t^{(1+D)+\gamma(2D-\alpha)}$. The exponential time dependence can be suppressed by keeping $v_\chi t \sim 1$, which requires $\gamma = (1+D)/(\alpha-2D)$. Dropping pre factors (since we only care about asymptotics at large r and t), we obtain

$$\mathscr{C}_r(t) \sim \exp[vt - r/t^\gamma] + \frac{t^{\alpha(1+\gamma)}}{r^\alpha}. \quad (19)$$

Thus, as argued earlier, the cutoff can be chosen to scale with time in such a way that the long-range contribution to the bound (scaling in space as $r^{-\alpha}$) has an *algebraic* rather than exponential time dependence. If we now make the substitution $r = t^{\beta}$, we see that $\lim_{t\to\infty} \mathscr{C}_{t^{\beta}}(t)$ vanishes whenever $\beta > 1/\zeta$, with

$$1/\zeta = 1 + (1 + D)/(\alpha - 2D). \tag{20}$$

Thus the light cone is bounded by a power law $t = r^{\zeta}$ ($0 < \zeta < 1$) whenever $\alpha > 2D$, and becomes increasingly linear ($\zeta \to 1$) as α grows larger.

As discussed in the introduction, our results impose stringent constraints on the growth of entanglement after a quantum quench. In addition, our bound implies much stricter constraints on *equilibrium* correlation functions than were previously known [17]. In particular, it follows from Eqs. (19) and (20) that correlations in the ground state of H decay at long distances as $1/r^{\alpha}$, so long as the spectrum of H remains gapped [28] (in fact, when combined with the results of Ref. [12], our results could be used to show that ground-state correlation functions exhibit a hybrid exponential-followed-by-algebraic decay, as observed recently in Refs. [12, 29, 30]). Understanding what happens to the light cone in the intermediate regime $D < \alpha < 2D$, where our results do not apply but Ref. [17] continues to predict a logarithmic light cone, would be an interesting direction for future investigation. We also note that, while we have ruled out the possibility of a logarithmic light cone in favor of one that is a nearly-linear polynomial, it is possible that *any* sub-linearity of the light cone is impossible above some critical α.

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[1] E. H. Lieb and D. W. Robinson, Comm. Math. Phys. **28**, 251 (1972).
[2] K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).
[3] M. A. Ruderman and C. Kittel, Phys. Rev. **96**, 99 (1954).
[4] M. Saffman, T. G. Walker, and K. Mølmer, Rev. Mod. Phys. **82**, 2313 (2010).
[5] R. Islam, C. Senko, W. C. Campbell, S. Korenblit, J. Smith, A. Lee, E. E. Edwards, C.-C. J. Wang, J. K. Freericks, and C. Monroe, Science **340**, 583 (2013).
[6] S. Gopalakrishnan, B. L. Lev, and P. M. Goldbart, Phys. Rev. Lett. **107**, 277201 (2011).
[7] K. Aikawa, A. Frisch, M. Mark, S. Baier, A. Rietzler, R. Grimm, and F. Ferlaino, Phys. Rev. Lett. **108**, 210401 (2012).
[8] B. Yan, S. A. Moses, B. Gadway, J. P. Covey, K. R. A. Hazzard, A. M. Rey, D. S. Jin, and J. Ye, Nature (London) **501**, 521 (2013).
[9] J. Eisert, M. van den Worm, S. R. Manmana, and M. Kastner, Phys. Rev. Lett. **111**, 260401 (2013).
[10] P. Hauke and L. Tagliacozzo, Phys. Rev. Lett. **111** (2013).
[11] J. Juenemann, A. Cadarso, D. Perez-Garcia, A. Bermudez, and J. J. Garcia-Ripoll, Phys. Rev. Lett. **111**, 230404 (2013).
[12] Z.-X. Gong, M. Foss-Feig, S. Michalakis, and A. V. Gorshkov, Phys. Rev. Lett. **113**, 030602 (2014).
[13] P. Richerme, Z.-X. Gong, A. Lee, C. Senko, J. Smith, M. Foss-Feig, S. Michalakis, A. V. Gorshkov, and C. Monroe, arXiv:1401.5088 (2014).
[14] P. Jurcevic, B. P. Lanyon, P. Hauke, C. Hempel, P. Zoller, R. Blatt, and C. F. Roos, arXiv:1401.5387 (2014).
[15] S. Bose, Contemp. Phys. **48**, 13 (2007).
[16] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. **83**, 863 (2011).
[17] M. Hastings and T. Koma, Comm. Math. Phys. **265**, 781 (2006).
[18] M. M. Wolf, F. Verstraete, M. B. Hastings, and J. I. Cirac, Phys. Rev. Lett. **100**, 070502 (2008).
[19] For $\alpha < D$ it is known that a light-cone does not exist (in general) [9].
[20] The results presented are significantly more general than Eq. (1) suggests, and can easily be generalized to fermionic models, or Hamiltonians with arbitrary single-particle terms or time dependence.
[21] S. Bravyi, M. B. Hastings, and F. Verstraete, Phys. Rev. Lett. **97**, 050401 (2006).
[22] (See Supplemental Material at link xxx, which includes Ref. [23], for details omitted in the main text).
[23] S. Bachmann, S. Michalakis, B. Nachtergaele, and R. Sims, Comm. Math. Phys. **309**, 835 (2012).
[24] B. Nachtergaele, H. Raz, B. Schlein, and R. Sims, Commun. Math. Phys. **286**, 1073 (2009).
[25] B. Nachtergaele and R. Sims, Contemp. Math. **529**, 141 (2010).
[26] As α gets larger, larger separations are required for the algebraic decay to dominate over the exponential decay. Strictly speaking, this consideration imposes that Eq. (18) is only valid whenever $vt > \alpha \log \alpha$. This restriction is irrelevant, since we are ultimately concerned with the asymptotic light-cone shape at large r and t, however the $\alpha \to \infty$ limit could in principle be taken at finite r and t by using the techniques of Ref. [12].
[27] Note that the first term comes from bounding the first summation in Eq. (11). The exponential decrease with r/χ occurs because each $\mathcal{A}^{\ell}(t)$ has a finite radius of support; the first non vanishing $[\mathcal{A}^{\ell}(t), B]$ occurs when $\ell \geq r/\chi - vt$, in which case $\|\mathcal{A}^{\ell}(t), B]\| \leq 2ce^{vt - r/\chi}$.
[28] Z.-X. Gong, M. Foss-Feig, C. W. Clark, and A. V. Gorshkov, (in preparation) (2015).
[29] J. Schachenmayer, B. P. Lanyon, C. F. Roos, and A. J. Daley, Phys. Rev. X **3**, 031015 (2013).
[30] D. Vodola, L. Lepori, E. Ercoles, A. V. Gorshkov, and G. Pupillo, arXiv:1405.5440 (2014).