This is the accepted manuscript made available via CHORUS. The article has been published as:

Resource Quality of a Symmetry-Protected Topologically Ordered Phase for Quantum Computation

Jacob Miller and Akimasa Miyake

Phys. Rev. Lett. **114**, 120506 — Published 26 March 2015

DOI: 10.1103/PhysRevLett.114.120506

Resource quality of a symmetry-protected topologically ordered phase for quantum computation

Jacob Miller[*] and Akimasa Miyake[†]
Center for Quantum Information and Control, Department of Physics and Astronomy, University of New Mexico, Albuquerque, NM 87131, USA

We investigate entanglement naturally present in the 1D topologically ordered phase protected with the on-site symmetry group of an octahedron as a potential resource for teleportation-based quantum computation. We show that, as long as certain characteristic lengths are finite, all its ground states have the capability to implement any unit-fidelity one-qubit gate operation asymptotically as a key computational building block. This feature is intrinsic to the entire phase, in that perfect gate fidelity coincides with perfect string order parameters under a state-insensitive renormalization procedure. Our approach may pave the way toward a novel program to classify quantum many-body systems based on their operational use for quantum information processing.

PACS numbers: 03.67.Lx, 75.10.Pq, 64.70.Tg

Introduction.—Entanglement is ubiquitous in quantum many-body systems, and its complexity has drawn attention from interdisciplinary research fields, such as condensed-matter physics [1–4], quantum information processing (QIP) [5–7], and quantum simulation of quantum many-body systems [8–12]. A primary example is exotic ground states of topologically ordered phases [13–15], which arise from underlying nonlocal entanglement. It is widely known that braiding their excitations, known as anyons, could be used for topological quantum computation [16], and their intrinsic insensitivity against local noise could be used for quantum error correction [16, 17]. Many-body entanglement can be harnessed in a more direct way, and certain many-body states like 2D cluster states [18] and certain tensor network states [19–25] are quantum resources for measurement-based (or teleportation-based) quantum computation, in that universal quantum computation can be implemented on these states using only single-spin measurements.

Having in hand a long list of many-body entanglement useful for QIP, however, one may wonder "Is such computational usefulness robust in the same way that collective phenomena of quantum many-body systems do not depend on their microscopic details?" Phrased differently, "Can we define quantum phases useful for certain QIP tasks in the same way we define phase diagrams in condensed matter physics, which are typically characterized by order parameters?" There have been several attempts [26–34] to answer this affirmatively, but they unfortunately, with a few exceptions [30], were largely based on a limited class of states, using rather artificial Hamiltonians from a condensed matter physics perspective.

Here we tackle this challenge using the 1D counterpart of topologically ordered phases as a key building block for measurement-based quantum computation, taking advantage of recent characterizations of symmetry protected topologically ordered (SPTO) phases [35–38]. By inventing a physically-feasible renormalization procedure which extracts the robust, macroscopic features common among ground states within a phase, we prove that all the ground states in the 1D SPTO phase corresponding to octahedral on-site symmetry can be used to implement any one-qubit operations perfectly, as long as certain conditions on characteristic length scales are met. The leverage of a discrete symmetry is somehow reminiscent of magic states and their distillation [39] in the context of fault-tolerant, universal quantum computation. Furthermore, we show that the gate fidelity, which is a typical measure of resource quality in QIP, can be interpreted as an "operationally-motivated" order parameter of the phase, because it detects critical points of the phase in the same way as the conventional string order parameter widely used in condensed matter physics. As a whole, our results constitute the first solid evidence for quantum computationally useful phases of matter.

Matrix product states and 1D symmetry-protected topological orders.—The matrix product state (MPS) formalism [7, 40, 41] is an efficient means of describing the correlations in one-dimensional spin chains. A MPS description is given by associating a matrix, A_i, to every vector $|i\rangle$ of a single-spin basis $\{|i\rangle\}_{i=1}^{d}$. The amplitude associated with a basis vector $|i_1 i_2 \dots i_n\rangle$ is then given by

$$\langle i_1 i_2 \dots i_n|\psi\rangle = \mathrm{tr}\left(A_{i_1} A_{i_2} \cdots A_{i_n}\right). \qquad (1)$$

The correlation length of our MPS is denoted by ξ, and our MPS is short-range correlated if ξ is finite.

In the presence of an on-site symmetry group G, G-invariant MPS's form distinct symmetry protected topological ordered (SPTO) phases, a classification of which was given in Refs. [36, 37]. Any transition between SPTO phases must be accompanied by either the introduction of long-range correlations or the breaking of on-site symmetry. This makes SPTO phase a robust property of many-body systems in the presence of symmetry. The group of π-rotations around the x, y, and z axes, $D_2 \simeq Z_2 \times Z_2$, defines two quantum phases, the trivial phase and the

[*] jmilla@unm.edu
[†] amiyake@unm.edu

D_2 SPTO phase. The archetypical member of the D_2 SPTO phase is the Affleck-Kennedy-Lieb-Tasaki (AKLT) state [42], whose MPS matrices are $A_\mu = \sigma_\mu$. μ labels the vectors in the spin-1 Pauli basis $\{|\mu\rangle\}_{\mu=1}^3$, defined by $S_\mu^{(1)}|\mu\rangle = 0$, with $S_\mu^{(1)}$ the spin-1 angular momentum operators. The σ_μ are the standard spin-$\frac{1}{2}$ Pauli operators.

Measurement-based quantum computation (MQC) [5, 6] is a convenient setting for quantum computation where the quantum nature of computation comes from the entanglement of an initial resource state. Through a sequence of single-spin measurements, an MQC protocol harnesses this entanglement to implement a quantum algorithm. In this paper, we focus on one-dimensional resource states, which are an essential building block for constructing universal resource states for quantum computation. As an illustration, we examine an MQC protocol utilizing the AKLT state [43]. If we measure a spin in our AKLT chain and obtain an outcome $|\psi_k\rangle = \sum_{\mu=1}^3 \psi_{k,\mu} |\mu\rangle$, then this results in an operator

$$A[\psi_k] = \sum_{\mu=1}^{3} \psi_{k,\mu}^* A_\mu = \sum_{\mu=1}^{3} \psi_{k,\mu}^* \sigma_\mu. \quad (2)$$

If we wish to implement a rotation by Θ around the z-axis, $U_\Theta = \exp(-i\frac{\Theta}{2}\sigma_z)$, a measurement outcome of $|\psi_{z,\Theta}\rangle = \cos\left(\frac{\Theta}{2}\right)|x\rangle - \sin\left(\frac{\Theta}{2}\right)|y\rangle$ will suffice, since

$$A[\psi_{z,\Theta}] = \sigma_x \left[\cos\left(\frac{\Theta}{2}\right) I - i \sin\left(\frac{\Theta}{2}\right)\sigma_z\right] \quad (3)$$

is indeed what we wanted, up to the σ_x term. This additional term is referred to as a byproduct operator, and can be dealt with as long as we maintain a record of the operator (See [6] for details).

Motivations of our work.—The above protocol characterizes one point within the D_2 SPTO phase, namely the AKLT state, as a resource state capable of generating arbitrary one-qubit operations. As stated in the Introduction, to explore whether such a resource characterization can be extended to the rest of the D_2 SPTO phase, we wish to invent a state-insensitive MQC protocol, in that an identical computation should be generated despite microscopic differences of ground states. An initiative along this direction was taken in [30], where all ground states of the 1D SO_3-invariant Haldane phase (or the so-called bilinear-biquadratic Hamiltonians) were studied using DMRG calculations. The perfect resource quality of these states for arbitrary single-qubit operations was demonstrated heuristically using a renormalization argument mapping any ground state towards the AKLT state. Later, Else *et. al.* [44] developed an algebraic characterization of the D_2 SPTO phase, which includes the SO_3-invariant Haldane phase, showing that any state within this phase can be used to implement a state-insensitive qubit teleportation operation. They obtain this result by showing that [45] for any spin-1 MPS within the D_2 SPTO phase, the component matrices associated with that state's MPS have the form

$$A_\mu = \sigma_\mu \otimes a_\mu. \quad (4)$$

The Hilbert spaces on the left and right side of the tensor product in Eq. (4) are called the protected space and the junk space, respectively. While the details of the junk operators, a_μ, vary from state to state, the structure of the protected space is common everywhere throughout the D_2 SPTO phase. Thus, if we measure our resource state in the Pauli basis, we will always end up teleporting the state of the protected space. In retrospect, this feature was first observed for certain ground states of the D_2 SPTO phase, like in the spin-1 XXZ Heisenberg model, as its so-called localizable entanglement diverges, and can thus be used to implement the identity channel [46, 47].

However, a simple argument given by Else *et. al.* [44] suggests that the resource characterization of the D_2 SPTO phase is limited to the identity channel (namely teleportation). If we perform some non-Pauli measurement, such as that in Eq. (3), we end up applying the operation

$$A[\psi_{z,\frac{\pi}{2}}] = \cos\left(\frac{\Theta}{2}\right) I \otimes a_x - i \sin\left(\frac{\Theta}{2}\right) \sigma_z \otimes a_y$$
$$\neq \left[\cos\left(\frac{\Theta}{2}\right) I - i \sin\left(\frac{\Theta}{2}\right)\sigma_z\right] \otimes a_x. \quad (5)$$

Because $a_x \neq a_y$ for arbitrary states, this operation generally won't have a well-defined effect on the protected space, and thus doesn't implement a state-insensitive unitary rotation within the D_2 SPTO phase.

Main Results.—Now we focus on MPS's invariant under on-site octahedral symmetry. This group can be generated by $\frac{\pi}{2}$-rotations around the x and z axes of the octahedron, and is actually isomorphic to the symmetric group of degree 4, S_4. Since the π-rotations in S_4 generate the group D_2, any state with S_4 symmetry also has D_2 symmetry. It can be shown that the classification of SPTO phases for on-site S_4 symmetry is identical to the case of D_2, and consequently, any MPS in the S_4 SPTO phase is automatically in the D_2 SPTO phase. This makes Eq. (4) applicable also to states in the S_4 SPTO phase, but the larger symmetry of S_4 imposes finer constraints on MPS's in the S_4 SPTO phase. We emphasize that this abstract characterization of SPTO phases is useful for making general statements, like the following two theorems, without specifying a system Hamiltonian or other microscopic details (although one could define a formal, local Hamiltonian for every MPS).

We study this S_4 SPTO phase by means of an operational "renormalization" protocol called z-buffering, which extracts macroscopic features common among ground states within the phase. This protocol, shown in Figure 1, consists of sequential *single-spin* measurements, with postselection for a desired measurement outcome which depends on the type of rotation we wish to implement. We first select a site, the computational site,

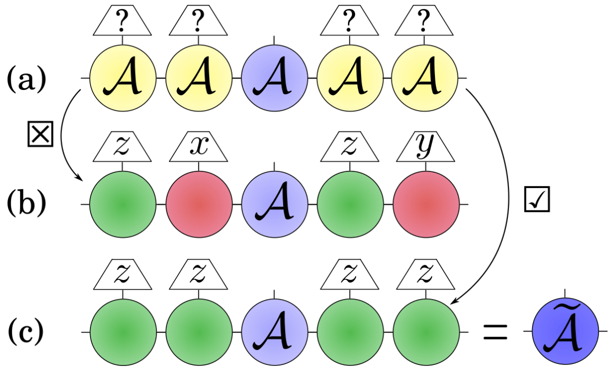

FIG. 1. Schematic of renormalization procedure to manifest the quality of resource states. (a) To perform z-buffering, we choose a computational site, and measure m surrounding sites in the Pauli basis. Here $m = 2$. (b) If our measurement fails to produce the all-$|z\rangle$ outcome, the computational site is measured in the Pauli basis, and we try again on another region. Since all of our measurement outcomes simply induce Pauli operations, the state of the protected space is (up to byproducts) unchanged. (c) If our measurement succeeds, the resource quality of our computational site is improved, at least when ζ_z is finite (Theorem 1).

which will eventually be used to generate the desired unitary rotation. Pauli measurements are then performed on the m sites on each side of this site. If we want to implement a z-axis rotation using the computational site, we postselect for the all-$|z\rangle$ outcome on these $2m$ buffering sites, a process called z-buffering. Similarly for x-axis rotations, x-buffering is utilized by postselecting the all-$|x\rangle$ outcome. The ability to perform z and x-axis rotations is all we need, since any single-qubit unitary gate can then be constructed using Euler angles.

If our desired outcome isn't obtained, we just measure the computational site in the Pauli basis and repeat this process on the next part of our spin chain, the state of our protected space simply being teleported by this undesired measurement outcome. Note that the probability of postselection is accounted for as overhead in the chain length, but this does not qualitatively change the resource quality (and its complexity), as long as it is finite. On the other hand, if our postselection succeeds, then the remaining computational state is renormalized by an amount depending on the ratio of m to a characteristic length scale, called the z-correlation length ζ_z, which governs this RG flow for each state. When ζ_z is finite, this RG flow generally terminates on a fixed point, which can be used to implement non-Pauli operations. The exception to this rule is for certain pathological states, where the act of z-buffering causes the state to become long-range correlated, in that the renormalized correlation length $\tilde{\xi}$ becomes infinite. This resource characterization is summarized in the following Theorem:

Theorem 1. *Consider any ground state of the 1D S_4 symmetry-protected topological ordered phase, which is characterized by a certain z-correlation length ζ_z and a renormalized correlation length $\tilde{\xi}$. As long as ζ_z and $\tilde{\xi}$ are both finite, the intrinsic entanglement of this state enables us to efficiently implement all one-qubit unitary operations under the setting of measurement-based quantum computation with arbitrarily high gate fidelity.*

The fact that our protocol enables the behavior described in Theorem 1 is proven in Appendix A. The main idea behind our proof [48] is that our MPS resource state, by virtue of being in the D_2 SPTO phase, will have SPTO degeneracy in the protected space, but generally not in the junk space. When we postselect for a repeated $|z\rangle$ outcome, we maintain this protected space degeneracy, but preferentially amplify a one-dimensional subspace of the junk space. After enough buffering, the junk space is sufficiently restricted to this one-dimensional subspace, corresponding to the largest eigenvalue λ_1 of a_z, so that our renormalized system can be treated effectively like the AKLT state. The length scale over which this happens, ζ_z, is set by the ratio of the largest to the second largest eigenvalue. The expected measurement overhead per gate required to achieve a gate fidelity $1-\epsilon$ is

$$\langle N \rangle = O\left(\zeta_z \left(\frac{1}{\epsilon}\right)^{4\zeta_z \log\left|\frac{1}{\lambda_1}\right|} \log\left(\frac{1}{\epsilon}\right)\right). \tag{6}$$

When the two largest eigenvalues of a_z become degenerate, corresponding to a divergence in ζ_z, z-buffering cannot completely restrict the junk space, and our RG flow stalls before reaching an AKLT-like state.

Theorem 1 says that the ground states of the S_4 SPTO phase generally share a common computational capability to implement perfect one-qubit gate operations. Since such capability is conveniently characterized in QIP by a measure called the gate fidelity, one could ask conversely "Could the gate fidelity be utilized as an alternative, operationally-motivated order parameter for quantum phases of matter?" Our second theorem below, proven in Appendix B, states a surprising correspondence between the gate fidelity and (a type of) so-called string order parameter [49], within the S_4 SPTO phase.

Theorem 2. *For any ground state in the 1D S_4 symmetry-protected topologically ordered phase with finite $\tilde{\xi}$, the gate fidelity of all one-qubit operations in measurement-based quantum computation is perfect if and only if the order parameters $\tilde{\mathcal{O}}_{D_4}^{(x)}$ and $\tilde{\mathcal{O}}_{D_4}^{(z)}$ take maximal values of $\frac{1}{2}$ when these quantities are evaluated upon completion of renormalization.*

Note that our order parameters $\tilde{\mathcal{O}}_{D_4}^{(x)}$ and $\tilde{\mathcal{O}}_{D_4}^{(z)}$ are specializations of the string order parameters $R_\infty(u)$ from [50] to the case of $\frac{\pi}{2}$-rotations about the x- and z-axes, u_{r_x} and u_{r_z}. In [50], these string order parameters are argued to be capable of detecting the presence of quantum phase transitions between different SPTO phases. Our order parameters are given by:

$$\tilde{\mathcal{O}}_{D_4}^{(\mu)} = \lim_{n\to\infty} \langle \psi_\mu | \left(u_{r_\mu}\right)^{\otimes n} |\psi_\mu\rangle. \tag{7}$$

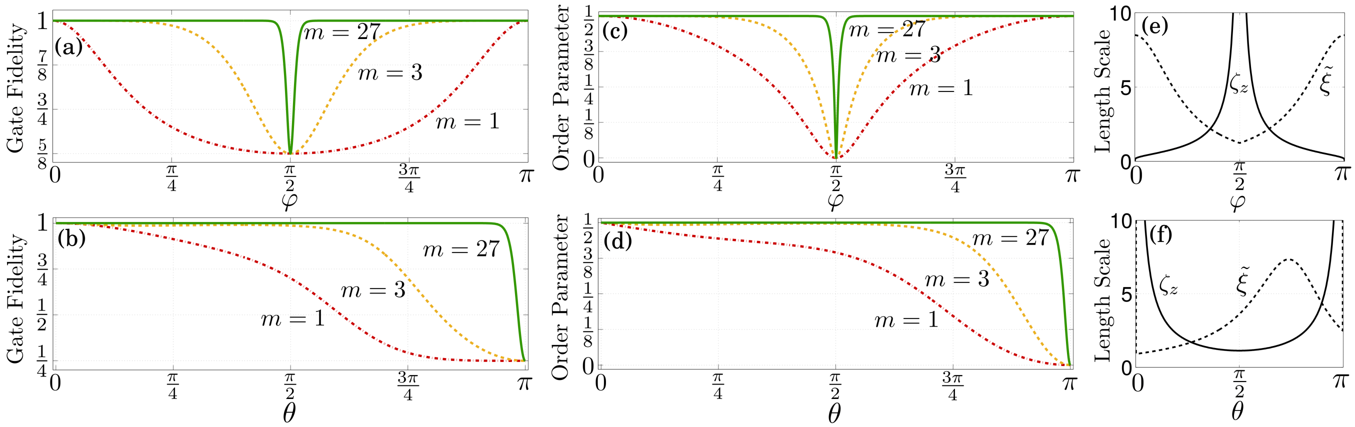

FIG. 2. (a & b) The gate fidelity for a protected space $\frac{\pi}{2}$-rotation about the z-axis, with resource states parameterized by φ, $\theta = \frac{\pi}{2}$ in (a), and by θ, $\varphi = \frac{\pi}{4}$ in (b). The renormalized gate fidelity tends toward unity everywhere except at the regions of divergent ζ_z, in agreement with Theorem 1. (c & d) The renormalized order parameter $\tilde{\mathcal{O}}^{(z)}_{D_4}$ for the same set of parameters as in (a) and (b), respectively. The RG limit of $\tilde{\mathcal{O}}^{(z)}_{D_4}$ is $\frac{1}{2}$ everywhere that the RG limit of the gate fidelity is 1, in agreement with Theorem 2. (e & f) The z-correlation length and the renormalized correlation length, ζ_z and $\tilde{\xi}$, shown for the same set of parameters as in (a) and (b), respectively. While both diverge at the poles of our parameter space, where our toy model is long-range correlated, the divergence of ζ_z at $\varphi = \frac{\pi}{2}$ is more surprising, and leads to a transition in the resource quality of our state there, as seen in (a).

The state $|\psi_\mu\rangle$ is the state of our many-body MPS after it has been mapped to the RG fixed point under μ-buffering, where μ is either x or z. While our bare spin chain possesses full S_4 symmetry, the process of renormalization breaks symmetry by picking out a preferred direction (the x or z-axis). Consequently, the symmetry group of $|\psi_\mu\rangle$ is reduced to $D_4^{(\mu)}$, which consists of the 8 rotations within S_4 that preserve this preferred axis.

Illustration of Our Results.—To demonstrate Theorems 1 and 2, we study the behavior of MPS's in the S_4 SPTO phase with a two-dimensional junk space. We have developed a general formalism based on representation theory [51], and can show that spin-1 MPS's of this form make up a two-parameter family that is isomorphic to a sphere. Choosing variables θ and φ, with $0 \leq \theta < \pi$ and $0 \leq \varphi < 2\pi$, gives a unique parameterization of this family of MPS's. Because S_4 symmetry includes D_2 symmetry, these MPS's have well-defined protected and junk spaces, with component matrices $A_\mu(\theta, \varphi) = \sigma_\mu \otimes a_\mu(\theta, \varphi)$, and

$$a_\mu(\theta, \varphi) = \frac{1}{\sqrt{3}} \left\{ \cos\left(\frac{\theta}{2}\right) I + e^{i\varphi} \sin\left(\frac{\theta}{2}\right) (\vec{n}_\mu \cdot \vec{\sigma}) \right\}. \quad (8)$$

The Pauli-type operators $\vec{n}_\mu \cdot \vec{\sigma}$ form a triad defined by

$$-\frac{1}{2}\sigma_x + \frac{\sqrt{3}}{2}\sigma_y \ , \ -\frac{1}{2}\sigma_x - \frac{\sqrt{3}}{2}\sigma_y \ , \ \sigma_x \ , \quad (9)$$

for $\mu = x, y, z$ respectively. A numerical calculation of the gate fidelity, order parameter, and relevant length scales of states throughout the parameter space is shown in Figure 2. We can see that the RG flow induced by z-buffering improves the gate fidelity of a $\frac{\pi}{2}$-rotation, an illustration by the "most non-Pauli" z-axis rotation, almost everywhere in our toy model. The points at which the gate fidelity is not improved are precisely those with divergent ζ_z, in agreement with Theorem 1. Furthermore, we see remarkable similarity between the plots showing gate fidelity and those showing $\tilde{\mathcal{O}}^{(z)}_{D_4}$ in Figure 2, both of which improve as the degree of z-buffering is increased. After sufficient renormalization (i.e., at $m = \infty$), the gate fidelity achieves its maximum value precisely when $\tilde{\mathcal{O}}^{(z)}_{D_4} = \tilde{\mathcal{O}}^{(x)}_{D_4} = \frac{1}{2}$, as stated in Theorem 2.

There are a few singular states in our parameter space with regard to their behavior under renormalization. As shown in Figure 2, the region with $\varphi = \pm\pi$ and any θ, as well as the poles at $\theta = 0, \pi$, have divergent ζ_z. This can be understood by noticing that a_z is unitary at these points, so that z-buffering just acts as a change of basis on the junk space. Interestingly, the original correlation length ξ, does not diverge at $\varphi = \pm\pi$, so that this is a new kind of singular state only detected by our operationally motivated classification of quantum many-body states. In contrast, states at the poles ($\theta = 0, \pi$) are not within the S_4 SPTO phase, because the original MPS's are long-range correlated, having a divergent ξ. There is another singular state at $(\theta, \varphi) = (2\arctan(2), 0)$, whose pathological behavior is discussed in Appendix C.

Conclusion.— We proved two theorems to demonstrate the intrinsic, quantum computational usefulness of the 1D S_4 SPTO phase as a "universal" quantum channel. We think that our physically feasible renormalization procedure, called z-buffering, is interesting on its own, because our state-insensitive protocol indicates that it is possible to harness such intrinsic capability of the phase without knowledge of microscopic details, at least as long as the states are guaranteed to be in the phase. As an outlook, since it is plausible that resource states for universal computation should generally possess such universal-channel capability in two or higher dimensions, our work is expected to serve as a stepping stone in the search for universal resource states in naturally-occurring quantum many-body systems.

The work was supported in part by National Science Foundation grants PHY-1212445 and PHY-1314955.

---

[1] H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).
[2] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Phys. Rev. B **81**, 064439 (2010).
[3] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **82**, 155138 (2010).
[4] G. De Chiara, L. Lepori, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. **109**, 237208 (2012).
[5] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).
[6] R. Raussendorf, D. E. Browne, and H. J. Briegel, Phys. Rev. A **68**, 022312 (2003).
[7] G. Vidal, Phys. Rev. Lett. **91**, 147902 (2003).
[8] J. I. Cirac and P. Zoller Nat. Phys. **8**, 264 (2012).
[9] S. Korenblit *et al*, New J. Phys. **14**, 095024 (2012).
[10] I. M. Georgescu, S. Ashhab, and F. Nori, Rev. Mod. Phys. **86**, 153 (2014).
[11] J. Eisert, M. Friesdorf, and C. Gogolin, ArXiv:1403.6481.
[12] I. Cohen and A. Retzker, Phys. Rev. Lett. **112**, 040503 (2014).
[13] X.-G. Wen, *Quantum Field Theory of Many-body Systems* (Oxford University Press, Oxford, 2007).
[14] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
[15] X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
[16] A. Kitaev, Ann. Phys. **303**, 2 (2003).
[17] A. Kitaev and C. Laumann, *Topological phases and quantum computation* (Oxford University Press, Oxford, 2010), pp. 101–125.
[18] H. J. Briegel and R. Raussendorf, Phys. Rev. Lett. **86**, 910 (2001).
[19] F. Verstraete and J. I. Cirac, Phys. Rev. A **70**, 060302 (2004).
[20] D. Gross and J. Eisert, Phys. Rev. Lett. **98**, 220503 (2007).
[21] D. Gross, J. Eisert, N. Schuch, and D. Pérez-García, Phys. Rev. A **76**, 052315 (2007).
[22] J.-M. Cai, A. Miyake, W. Dür, H. J. Briegel, Phys. Rev. A **82**, 052309 (2010).
[23] A. Miyake, Ann. Phys. **326**, 1656 (2011).
[24] T.-C. Wei, I. Affleck, and R. Raussendorf, Phys. Rev. Lett. **106**, 070501 (2011).
[25] Y. Li *et al*, Phys. Rev. Lett. **107**, 060501 (2011).
[26] D. E. Brown, M. B. Elliott, S. T. Flammia, S. T. Merkel, A. Miyake, and A. J. Short, New. J. Phys. **10**, 023010 (2008).
[27] A. C. Doherty and S. D. Bartlett, Phys. Rev. Lett. **103**, 020506 (2009).
[28] S. D. Barrett, S. D. Bartlett, A. C. Doherty, D. Jennings, and T. Rudolph, Phys. Rev. A **80**, 062328 (2009).
[29] S. O. Skrøvseth and S. D. Bartlett, Phys. Rev. A **80**, 022316 (2009).
[30] S. D. Bartlett, G. K. Brennen, A. Miyake and, J. M. Renes, Phys. Rev. Lett. **105**, 110502 (2010).
[31] A. S. Darmawan, G. K. Brennen, and S. D. Bartlett, New J. Phys. **14**, 013023 (2012).
[32] D. V. Else, S. D. Bartlett, and A. C. Doherty, New J. Phys. **14**, 113016 (2012).
[33] K. Fujii and T. Morimae, Phys. Rev. A **85**, 010304(R) (2012).
[34] K. Fujii, Y. Nakata, M. Ohzeki, and M. Murao, Phys. Rev. Lett. **110**, 120502 (2013).
[35] Z. C. Gu and X. G. Wen, Phys. Rev. B **80**, 155131 (2009).
[36] X. Chen, Z. C. Gu, and X. G. Wen, Phys. Rev. B **83**, 035107 (2011).
[37] N. Schuch, D. Pérez-García, and J. I. Cirac, Phys. Rev. B **84**, 165139 (2011).
[38] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. B **85**, 075125 (2012).
[39] S. Bravyi and A. Kitaev, Phys. Rev. A **71**, 022316 (2005).
[40] M. Fannes, B. Nachtergaele, and R. F. Werner, Commun. Math. Phys. **144**, 443 (1992)
[41] D. Pérez-García, F. Verstraete, M. M. Wolf, and J. I. Cirac, Quantum Inf. Comput. **7**, 401 (2007).
[42] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Comm. Math. Phys. **115**, 477 (1988).
[43] G. K. Brennen and A. Miyake, Phys. Rev. Lett. **101**, 010502 (2008).
[44] D. V. Else, I. Schwarz, S. D. Bartlett, and A. C. Doherty, Phys. Rev. Lett. **108**, 240505 (2012).
[45] Their actual result is more general, but it is only the spin-1 D_2 version of their result that is necessary for what follows.
[46] M. Popp, F. Verstraete, M. A. Martín-Delgado, and J. I. Cirac, Phys. Rev. A **71**, 042306 (2005).
[47] L. Campos Venuti and M. Roncaglia, Phys. Rev. Lett. **94**, 207207 (2005).
[48] While the intuition is essentially the same, the case of non-normal a_z (i.e., $[a_z, a_z^\dagger] \neq 0$) needs some elaboration.
[49] M. den Nijs and K. Rommelse, Phys. Rev.B **40**, 4709 (1989).
[50] D. Pérez-García, M. M. Wolf, M. Sanz, F. Verstraete, and J. I. Cirac, Phys. Rev. Lett. **100**, 167202 (2008).
[51] J. Miller and A. Miyake (to be published).
[52] See Supplemental Material for our proofs of Theorems 1 and 2, and a discussion of $\tilde{\xi}$. The supplemental material includes Refs. [53, 54].
[53] M. Sanz, M. M. Wolf, D. Pérez-García, and J. I. Cirac, Phys. Rev. A **79**, 042308 (2009).
[54] J. Novotný, G. Alber, and I. Jex, J. Phys. A: Math. Theor. **42**, 282003 (2009).