Exact Mapping of the Stochastic Field Theory for Manna Sandpiles to Interfaces in Random Media

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Exact mapping of the stochastic field theory for Manna sandpiles to interfaces in random media

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We show that the stochastic field theory for directed percolation in presence of an additional conservation law (the C-DP class) can be mapped *exactly* to the continuum theory for the depinning of an elastic interface in short-range correlated quenched disorder. Along one line of parameters commonly studied, this mapping leads to the simplest overdamped dynamics. Away from this line, an additional memory term arises in the interface dynamics; we argue that it does not change the universality class. Since C-DP is believed to describe the Manna class of self-organized criticality, this shows that Manna stochastic sandpiles and disordered elastic interfaces (i.e. the quenched Edwards-Wilkinson model) share the same universal large-scale behavior.

Self-organized criticality (SOC) and scale-free avalanches arise in a variety of models: deterministic and stochastic sandpiles [1–6], propagation of epidemics [7], and elastic interfaces driven in random media [8–14]. In the last decade several authors found evidence that most of these models belong to a small number of common universality classes. A unifying framework was proposed based on the theory of *absorbing phase transitions* (APT) [15, 16]. These are non-equilibrium phase transitions between an active state and one –or many– absorbing states. The generic universality class is the *directed-percolation class* (DP) [19, 20]. The spreading exponents of the critical DP clusters are interpreted as avalanche exponents in the corresponding SOC system [16]. In presence of additional conservation laws, other classes may arise, e.g. the *conserved directed percolation* class (C-DP), with an infinite number of absorbing states. It is now often stated, though unproven, that the continuum fluctuation theory for C-DP is the effective field theory for Manna sandpiles [17].

Stochastic sandpiles are cellular automata where the toppling rule contains randomness, renewed at each toppling. A notable example is the Manna model [2, 4, 21]: *Randomly throw grains on a lattice. If the height at one point is* ≥ 2, *then move two grains from this site to randomly chosen neighboring sites.* Careful numerical studies [2, 22–25], showed that the Manna and the deterministic Bak-Tang-Wiesenfeld (BTW) models belong to different universality classes (see [5, 22, 26] for reviews). It was proposed in [27, 28] that the coarse-grained evolution equations for the Manna class identify with the stochastic continuum equations for the C-DP class:

$$\partial_t\rho(x,t)=a\rho(x,t)-b\rho(x,t)^2+D_\rho\nabla^2\rho(x,t) +\sigma\eta(x,t)\sqrt{\rho(x,t)}+\gamma\rho(x,t)\phi(x,t)\ , \qquad (1)$$

$$\partial_t\phi(x,t)=(D_\phi\nabla^2-m^2)\rho(x,t)\ . \qquad (2)$$

Here ρ is the local activity, and ϕ the local density of grains. The parameters b, D_ρ, D_ϕ are positive; $\eta(x,t)$ is a (centered) spacio-temporal white noise, $\langle\eta(x,t)\eta(x',t')\rangle=\delta^d(x-x')\delta(t-t')$. Clearly, $\rho(x,t)=0$ with arbitrary "background" field $\phi(x,t)$ forms an infinite set of (time-independent) absorbing states. The field $\phi(x,t)$ encodes the likeliness of absorbing configurations to propagate activity when perturbed. From (2) ϕ is a conserved field for $m=0$, reflecting conservation of the total number of grains. In [29, 30] it is claimed that all "stochastic models with an infinite number of absorbing states, in which the order-parameter evolution is coupled to a nondiffusive conserved field, define a unique universality class", the C-DP, as further supported in [31, 32]. The C-DP class is believed to contain conserved lattice-gas models, conserved threshold-transfer processes, and others [16, 30, 33]. On the other hand, there were early conjectures that sandpile models and disordered elastic manifolds belong to the same universality classes: The first claim relates the BTW model and elastic interfaces driven in a *periodic* disorder [9], reexamined recently [34]. It was followed by a conjecture [35] on the equivalence of the Oslo model [3] to an elastic string driven by its endpoint in a *non-periodic* quenched random field; the latter emerging from the stochastic noise in the Oslo model. Finally, it was conjectured that Manna sandpiles are equivalent to interfaces in random media [23, 36].

Quite naturally, it was then proposed that C-DP and the depinning of an interface belong to the same universality class [23, 28, 31, 37, 38]. Until now this remarkable claim is mainly based on the numerical coincidence of critical exponents in simulations of discrete models, *believed* to belong to the respective universality classes [17, 23]. This coincidence of exponents and the convergence of these simulations was contested in [18] where it was proposed that Manna sandpiles are instead equivalent to DP. It is thus crucial to find a direct connection at the level of the continuum theories. The field theory of interfaces subject to disorder is well known, both for depinning [39–41] and avalanches [12, 13, 42]. It is described by functional RG (FRG), involving an infinite number (a function) of relevant couplings near its upper critical dimension $d_{\rm c}=4$ [43]. One would like to relate it to the C-DP field theory. Although it was realized that its renormalization is more complex than that of standard

DP which requires only a few couplings, the attempts to handle it were unsuccessful [33, 44]. Intriguingly, the full renormalized disorder correlator was measured numerically [45], and found indistinguishable from that of random interfaces obtained in [46].

The aim of this Letter is to provide an exact mapping in the continuum, between the C-DP class defined by Eqs. (1) and (2), and an interface driven in quenched disorder, with a specific, exponentially decaying, microscopic disorder correlator. Along a line in parameter space it maps C-DP to the simplest overdamped dynamics of the interface, thereby proving the long-sought equivalence of the two systems. Away from this line, the dynamics of the interface contains an additional memory kernel. As we show, it nevertheless falls into the same universality class as the simplest overdamped model, i.e. quenched Edward-Wilkinson (QEW).

Let us consider the two coupled equations of motion (1) and (2). For convenience we added a parameter m^2, since it appears in the interface model as an infrared regulator. Although we are interested in the limit $m \to 0$, it is useful to define the theory with $m > 0$, since this insures that the activity $\rho(x,t)$ will stop, even without grains leaving the system, which therefore can be taken infinitely large. To simplify the identification, note that by rescaling of space we can set $D_\rho \to 1$. By rescaling ϕ, we can then set $D_\phi \to 1$. Finally rescaling both ρ and ϕ, we can set $\sigma \to 1$. This simplifies the model to

$$\begin{aligned}\partial_t \rho(x,t) =& a\rho(x,t) - b\rho(x,t)^2 + \nabla^2 \rho(x,t) \\ &+ \eta(x,t)\sqrt{\rho(x,t)} + \gamma\rho(x,t)\phi(x,t) \qquad (3)\\ \partial_t \phi(x,t) =& (\nabla^2 - m^2)\rho(x,t) \ . \qquad (4)\end{aligned}$$

The activity variable $\rho(x,t) \geq 0$ for all times [47]. Note that $\gamma = 0$, $b > 0$, corresponds to directed percolation: In the absence of noise, i.e. in mean field, it exhibits a transition between $\rho > 0$ for $a > 0$ and $\rho = 0$ for $a \leq 0$. This transition exists in any d. The noise $\eta(x,t)$ becomes relevant for $d \leq d_c = 4$, a property shared by DP and C-DP; the latter has $\gamma > 0$ which we now examine.

As we will see below, $\gamma = b$ is special. We therefore set $b := \gamma + \kappa$. We define new variables, a *force* $\mathcal{F}(x,t)$ and a *velocity* $\dot{u}(x,t)$ (denoting ∂_t or a dot time derivatives):

$$\mathcal{F}(x,t) := \rho(x,t) - \phi(x,t) - \frac{a+m^2}{\gamma} \ , \qquad (5)$$

$$\rho(x,t) := \dot{u}(x,t) \ . \qquad (6)$$

The total number of topplings at position x since $t = 0$ is $u(x,t) - u(x,t=0) = \int_0^t \mathrm{d}t' \rho(x,t)$. The identification of u as a height for the associated elastic interface is standard [45], while the identification of $\mathcal{F}$ as a *force* is new. Clearly, the initial value of the field $u(x,t=0)$ does not carry any information for C-DP, while it does for the interface [48]. For notational simplicity we set $u(x,t=0) = 0$. All our results can be extended to the general case by replacing $u(x,t) \to u(x,t) - u(x,t=0)$. The equations of motion for $\mathcal{F}(x,t)$ and $\dot{u}(x,t)$ then are

$$\begin{aligned}\partial_t \mathcal{F}(x,t) =& -\gamma \mathcal{F}(x,t)\dot{u}(x,t) - \kappa \dot{u}(x,t)^2 \\ &+ \eta(x,t)\sqrt{\dot{u}(x,t)} \ , \qquad (7)\\ \partial_t \dot{u}(x,t) =& [\nabla^2 - m^2]\dot{u}(x,t) + \partial_t \mathcal{F}(x,t) \ . \qquad (8)\end{aligned}$$

The problem is defined with initial data $\dot{u}(x,t=0)$ and $\mathcal{F}(x,t=0)$. The second equation (8) can be integrated into

$$\partial_t u(x,t) = [\nabla^2 - m^2]u(x,t) + \mathcal{F}(x,t) + f(x) \ , \qquad (9)$$

$$f(x) := \dot{u}(x,0) - \mathcal{F}(x,0) = \phi(x,0) + \frac{a+m^2}{\gamma} \ . \qquad (10)$$

Eq. (9) describes the motion of an elastic interface subject to a known time-independent external force $f(x)$, and a space-time dependent force $\mathcal{F}(x,t)$. Because of the term m^2, the interface also sees a quadratic well. Integration of Eq. (4) shows that the change in the background field, $\phi(x,t) - \phi(x,0)$, can be interpreted as the sum of the elastic force plus the force from the quadratic well, acting on the interface. Eq. (7) determines $\mathcal{F}(x,t)$ *as a stochastic functional* of the field $u(x,t)$, depending on the noise. It is formally written as $\mathcal{F}(x,t) \equiv \mathcal{F}[u,\eta](x,t)$. Once $\mathcal{F}(x,t)$ is known, substituting it into Eq. (9) yields an elastic manifold in a random medium. As we show now, $\mathcal{F}(x,t)$ can be written *explicitly*. Eq. (7) is linear in $\mathcal{F}$ with two source terms, hence its solution is

$$\mathcal{F}(x,t) = e^{-\gamma u(x,t)}\mathcal{F}(x,t=0) + \mathcal{F}_{\rm dis}(x,t) + \mathcal{F}_{\rm ret}(x,t) \ . \qquad (11)$$

The first term depends on the initial condition, and decays to zero if the interface moves by more than $1/\gamma$; it can be ignored in the steady state. The second term can be interpreted as a quenched random pinning force. It arises from the noise in Eq. (7), is independent of κ, and is the only term when $\kappa = 0$ (then $\mathcal{F}_{\rm ret} = 0$) i.e. for $\gamma = b$. It can be written as $\mathcal{F}_{\rm dis}(x,t) = F\big(u(x,t),x\big)$, where for each x, $F(u,x)$ is an Orstein-Uhlenbeck process [49], solution of the stochastic equation

$$\partial_u F(u,x) = -\gamma F(u,x) + \tilde{\eta}(x,u) \ , \qquad (12)$$

with initial data $F(0,x) = 0$, and $\tilde{\eta}(x,u)$ a white noise, uncorrelated in x and u. A pedestrian way to derive Eq. (12) is to write the white noise $\eta(x,t) = \mathrm{d}B_x(t)/\mathrm{d}t$ in Eq. (7) in terms of independent one-sided Brownians $B_x(t)$ indexed by x, with $B_x(0) = 0$, and integrate the linear equation as

$$\begin{aligned}\mathcal{F}_{\rm dis}(x,t) &= \int_0^t \mathrm{d}t' \frac{\mathrm{d}B_x(t')}{\mathrm{d}t'} \sqrt{\dot{u}(x,t')}e^{-\gamma[u(x,t)-u(x,t')]} \\ &= e^{-\gamma u(x,t)} \int_0^{u(x,t)} e^{\gamma u}\mathrm{d}\tilde{B}_x(u) = F\big(u(x,t),x\big) \ . \qquad (13)\end{aligned}$$

The force $F(u,x)$ is the solution of the Orstein-Uhlenbeck process (12) in terms of the white noises $\tilde{\eta}(x,u) =$

$\mathrm{d}\tilde{B}_x(u)/\mathrm{d}x$. It can be written as a (time-changed) Brownian, $F(u,x) = \frac{e^{-\gamma u}}{\sqrt{2\gamma}}\tilde{B}_x\big(e^{2\gamma u}-1\big)$. The second line in (13) is obtained noting that under a time change $\mathrm{d}u = \dot{u}(x,t)\mathrm{d}t$ each Brownian $B_x(t)$ is changed into another Brownian $\tilde{B}_x(u)$ with $\tilde{B}_x(0) = 0$, as $\sqrt{\dot{u}(x,t')}\mathrm{d}B_x(t') = \mathrm{d}\tilde{B}_x\big(u(x,t')\big)$. One then uses the identity $\int_0^v f(u)\mathrm{d}B_x(u) = \tilde{B}_x(\int_0^v f(u)^2\mathrm{d}u)$ for test functions $f(u)$, from the scale invariance of Brownian motion.

Hence, neglecting the first (decaying) term in Eq. (11), we showed that for $\gamma = b$ C-DP maps onto

$$\partial_t u(x,t) = [\nabla^2 - m^2]u(x,t) + F(u(x,t),x) + f(x)\ . \quad (14)$$

This is an interface driven in a quenched random force field $F(u,x)$, which is Gaussian, specified by its correlator, calculated from the formulae above, using $\overline{B_x(u)B_{x'}(u')} = \delta(x-x')\min(u,u')$. The Orstein-Uhlenbeck process becomes stationary when the interface has been driven on distances larger than $1/\gamma$:

$$\begin{aligned}\overline{F(u,x)F(u',x')} &= \delta^d(x-x')\frac{\mathrm{e}^{-\gamma|u-u'|} - \mathrm{e}^{-\gamma(u+u')}}{2\gamma} \\ &\to_{\gamma u,\gamma u'\gg 1}\delta^d(x-x')\Delta_0(u-u') \quad (15)\end{aligned}$$

with $\overline{F(u,x)} = 0$. The *bare disorder correlator* of the random pinning force thus is

$$\Delta_0(u) = \frac{\mathrm{e}^{-\gamma|u|}}{2\gamma}\ . \quad (16)$$

It is short-ranged, and as a peculiarity has a linear cusp. Usually one considers smooth microscopic disorder, i.e. an analytic $\Delta_0(u)$, which under RG (i.e. coarse graining) develops a cusp linked to the existence of many metastable states and avalanches beyond the Larkin scale $L_\mathrm{c} \sim 1/m_\mathrm{c}$ [50]. A cusp in the microscopic disorder means that there are avalanches of arbitrarily small sizes. On the other hand, any short-ranged force-force correlator flows at large scale, under coarse-graining, to the same renormalized disorder correlator, the universal *depinning* fixed point [50]. Its upper critical dimension is $d_\mathrm{c} = 4$, implying that C-DP has $d_\mathrm{c} = 4$. The fixed-point function has been calculated analytically in an $\varepsilon = d_\mathrm{c} - d$ expansion [41] and measured numerically [46]. It determines the two independent exponents of the depinning transition, the roughness exponent ζ of the field $u \sim L^\zeta$, $\zeta > 0$ for $d < d_\mathrm{c}$, and the dynamic exponent z, $t \sim L^z$, $z < 2$ for $d < d_\mathrm{c}$, and their ε-expansions [41].

Let us now discuss the correspondence between the active-absorbing phase transitions for C-DP and depinning. For simplicity consider a spatially uniform initial condition $\phi(x,t=0) = \phi$, s.t. the initial driving force acting on the interface in Eq. (10) is uniform, $f(x) = f$. We now set the control parameter $m \to 0$ so that there is a globally active phase corresponding to an interface moving at constant steady-state mean velocity $\overline{\dot{u}(x,t)} = v \sim (f - f_\mathrm{c})^\beta > 0$, if $f > f_\mathrm{c}$. Here f_c is the depinning threshold force, in principle calculable once the correlator Δ_0 is known. Translating to C-DP it implies an active phase with $\rho > 0$, when $a + \gamma\phi > \gamma f_\mathrm{c}$, and a phase transition where ρ vanishes with the same exponent β as a function of the distance to criticality. Due to a symmetry of the interface problem, $\beta = \nu(z-\zeta) = \frac{z-\zeta}{2-\zeta}$. This gives $\rho = \dot{u} \sim t^{-\theta}$ at criticality with $\theta = 1 - \frac{\zeta}{z}$, e.g. as response to a (large) specially uniform perturbation at $t = 0^+$, in the limit of $v \to 0^+$. In the language of APT [16] this is a *steady-state exponent*.

Let us now consider the protocol for avalanches in the absorbing phase, near criticality. In the sandpile model (e.g. in numerical simulations for Manna) one usually starts from an initial condition with non-vanishing activity $\rho(x,0) = \dot{u}(x,0) \geq 0$, either adding a single grain, or adding grains in an extended region. This generates an avalanche which stops when $\rho(x,t) = 0$ for all x. For the elastic manifold it is equivalent to having the interface at rest up to time $t = 0$, and then to increase the force by $\dot{u}(x,0)$. This is repeated until one reaches the steady state. It is known for interfaces that under this procedure the system reaches the *Middleton attractor*, a sequence of well-characterized metastable states between successive avalanches [54]. Avalanches with this statistics have well-defined exponents [13, 55, 56].

To summarize, along the line $\gamma = b$, we presented an exact mapping in any dimension, between the C-DP Eqs. (1)-(2) and a driven interface with overdamped dynamics, subject to a *quenched* random force $F(u(x,t),x)$ with correlations (16), in a parabolic well. This confirms the beautiful numerical observation of Ref. [45] that Manna sandpiles, the Oslo model, C-DP and disordered elastic manifolds have the same renormalized effective disorder correlator. If one accepts that the Manna class coincides with C-DP, it establishes the long sought mapping to disordered elastic interfaces [57]. Our exact mapping extends beyond the stationary state and allows to study the evolution from any initial state.

Some remarks are in order: The interface equation (14) with the choice of correlator (15) possesses a special Markovian property, which it inherits from the force evolution equation (7) (for $\kappa = 0$), and which allows it to be solved without storing the full random-force landscape. The latter is constructed as the avalanche proceeds, hence is determined only for $u \leq u(x,t)$. This property was noted in [56, 58] and can be used for efficient numerics [56, 59].

The limit $\gamma \to 0$ is also of interest. If one keeps $\kappa = 0$, i.e. $b \to 0$, one sees from (10) and (11) that in that limit

$$\dot{u}(x,t) - \dot{u}(x,t=0) = [\nabla^2 - m^2]u(x,t) + \tilde{B}_x\big(u(x,t)\big) \quad (17)$$

This is the *Brownian force model* (BFM): It provides the mean-field theory for avalanches of an interface [13, 60, 61], hence also for C-DP, in $d \geq 4$. If we keep $b > 0$, the limit $\gamma \to 0$ is towards DP.

Let us finally discuss C-DP for $\kappa \neq 0$, i.e. away from the line $\gamma = b$ in Eq. (3). If the new source term $\kappa\dot{u}^2$, which appears in Eq. (7) for $\partial_t \mathcal{F}$, were directly inserted into Eq. (8) for $\dot{u}$, the mapping to the interface would fail, as such a term is relevant [62]. Fortunately, it is *screened* by the disorder, and is only marginal. To show this, consider the additional contribution to Eq. (11),

$$\mathcal{F}_{\rm ret}(x,t) = -\kappa \int_0^t \mathrm{d}t'\, \dot{u}(x,t')^2\, e^{-\gamma[u(x,t)-u(x,t')]} \ . \quad (18)$$

Integrating by part, and inserting into Eq. (9) we obtain

$$\begin{aligned}\frac{b}{\gamma}\partial_t u(x,t) =& [\nabla^2 - m^2]u(x,t) + F(u(x,t),x) + f(x) \\ &+ \frac{\kappa}{\gamma}\int_0^t \mathrm{d}t' \ddot{u}(x,t') e^{-\gamma[u(x,t)-u(x,t')]} \\ &+ \left[\frac{b}{\gamma}\dot{u}(x,0) - f(x)\right] e^{-\gamma u(x,t)} \ . \quad (19)\end{aligned}$$

Note that the boundary term in the integration by part has changed the friction coefficient by κ/γ. This equation of motion is equivalent to the C-DP Eqs. (1)-(2) for $\rho(x,t) = \dot{u}(x,t)$ with initial data $\dot{u}(x,0)$, $\phi(x,0)$. It is a salient result of our letter. Note that it results from a simple change of variables, which maps a system with *annealed noise*, the C-DP, to a system with *quenched noise*, the interface; as such it bears some analogy to the Cole-Hopf transformation used to solve the Kardar-Parisi-Zhang (KPZ) equation.

The first line in Eq. (19) describes the standard overdamped equation of motion of the interface, with the same random force $F(u,x)$ as before, but a new friction coefficient b/γ. The third line depends on the initial condition. It rapidly decays to zero, and can be neglected in the stationary regime. The second line is a new memory term. We argue that it is marginally irrelevant: Consider the large-γ limit, and replace $e^{-\gamma z} \to \frac{1}{\gamma}\delta(z)$, hence $e^{-\gamma[u(x,t)-u(x,t')]} \to \frac{1}{\gamma\dot{u}(x,t)}\delta(t-t')$. The second line of (19) then becomes $\frac{\kappa}{\gamma^2}\partial_t \ln \dot{u}(x,t) + \mathcal{O}(\gamma^{-3})$ where each new power in the $1/\gamma$ expansion comes with a power of $1/u \sim L^{-\zeta}$ and is more and more irrelevant [64]. Hence we conclude that the universality class of C-DP and of QEW should be the same, even for $b \neq \gamma$.

The present work calls for further studies: First, Eq. (19) can be analyzed using FRG to confirm our conclusions and explore this unusual interface dynamics. Our work opens the way to study, within a common RG framework, a variety of models ranging from interfaces to absorbing phase transitions. It can be extended to long-range elasticity (long-range toppling), or to a variety of perturbations. The simplest one is to add $m^2\dot{w}(x,t)$ to each of the Eqs. (1)-(2) in order to reproduce the standard driving for the interface [12]. Another extension is the crossover to DP as both γ and b are small.

Second, Eq. (19) permits to study initial conditions, hence to disentangle transients from properties of the Middleton attractor. That allows to treat avalanches with localized seeds in the context of APTs, used to define *spreading exponents*. E.g. the survival probability in C-DP, $P^{\rm surv}_{\text{C-DP}}(t) \sim t^{-\delta}$ is related to the avalanche-duration distribution at depinning, $P_{\rm dep}(T) \sim T^{-\alpha}$, via $\delta = \alpha - 1 = (d-2+\zeta)/z$. We checked that indeed $\delta = 0.17$ and 0.48 in $d = 1$ and 2, both for depinning, see table 2 of [55], and Manna sandpiles [17, 63].

Third, since our mapping is local in space, it can be extended to finite-size systems at $m = 0$. Imposing $\rho(x,t) = \phi(x,t) = 0$ at the boundary corresponds to the common choice to let grains "fall off". Here it implies $u(x,t) = \dot{\mathcal{F}}(x,t) = 0$ at the boundary.

Finally one should understand the cusp of Ref. [45] in a more general setting. Challenging questions are whether the quenched KPZ class and the DP with quenched disorder [16] can be treated similarly.

In conclusion, we provide an *exact mapping* from the field theory of a reaction-diffusion system with a conservation law, the C-DP system of Eqs. (1)-(2), to a continuum model of an interface driven in a random landscape. Using universality we show that the C-DP class and, if we accept its equivalence to the Manna class, also Manna stochastic sandpiles, as well as the quenched Edwards-Wilkinson model belong to a single, and hence *very large* universality class which spans self-organized criticality, avalanches in disordered systems, and reaction-diffusion models. This points towards a unified field theory for these systems using functional RG. It also defines a framework for probabilists to put this claim on *rigorous* grounds, as was recently done for the KPZ class [65, 66].

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