Quanta of Geometry: Noncommutative Aspects

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Quanta of Geometry, Noncommutative Origin

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In the construction of spectral manifolds in noncommutative geometry, a higher degree Heisenberg commutation relation involving the Dirac operator and the Feynman slash of real scalar fields naturally appears and implies, by equality with the index formula, the quantization of the volume. We first show that this condition implies that the manifold decomposes into disconnected spheres which will represent quanta of geometry. We then refine the condition by involving the real structure and two types of geometric quanta, and show that connected spin-manifolds with large quantized volume are then obtained as solutions. The two algebras $M_2(\mathbb{H})$ and $M_4(\mathbb{C})$ are obtained which are the exact constituents of the Standard Model. Using the two maps from M_4 to S^4 the four-manifold is built out of a very large number of the two kinds of spheres of Planckian volume. We give several physical applications of this scheme such as quantization of the cosmological constant, mimetic dark matter and area quantization of black holes.

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Introduction.— To reconcile General Relativity with Quantum Mechanics it is natural to try to find a generalization of the Hiesenberg commutation relations $[p, q] = i\hbar$. One expects the role of the momentum p to be played by the Dirac operator, however, the role of the position variable q is more difficult to discover. In noncommutative geometry a geometric space is encoded by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where the algebra $\mathcal{A}$ is the algebra of functions which interacts with the inverse line element D by acting in the same Hilbert space $\mathcal{H}$, where D is an unbounded self-adjoint operator. There is, in the even-dimensional case, an additional decoration given by the chirality operator $\gamma = \gamma^*$, $\gamma^2 = 1$, $D\gamma = -\gamma D$ [1]. For a compact spin Riemannian manifold M the algebra $\mathcal{A}$ is the algebra of operator-valued functions on M, the Hilbert space $\mathcal{H}$ is the Hilbert space of L^2-spinors and the operator D is the Dirac operator. These operator theoretic data encodes not only the geometry (the Riemannian metric) but also the K-homology fundamental class of M which is represented by the K-homology class of the spectral triple. Among the operator theoretic properties fulfilled by the special spectral triples coming from Riemannian geometries, one of them called the *orientability condition* asserts that, in the even-dimensional case, one can recover the chirality operator γ as an expression of the form

$$\gamma = \sum a_0[D, a_1] \cdots [D, a_n], \qquad (1)$$

where the $a_j \in \mathcal{A}$ and the formal expression is a totally antisymmetric Hochschild cycle that represents the (oriented) volume form dv of the manifold [2] (for simplification, we have dropped the summation index appearing in this equation). Our goal in this letter is to show that the quantized Heisenberg commutation relations is a quantized form of the orientability condition. It was observed in [2] that in the particular case of even spheres the trace of the Chern character of an idempotent e, i.e. $e^2 = e$, leads to a remarkably simple operator theoretic equation which takes the form (up to a normalization factor $\frac{1}{2^{n/2} n!}$)

$$\langle Y\,[D, Y] \cdots [D, Y] \rangle = \sqrt{\kappa}\,\gamma \quad (n \text{ terms } [D, Y]) \qquad (2)$$

Here $\kappa = \pm 1$ and $C_\kappa \subset M_s(\mathbb{C})$, $s = 2^{n/2}$, is the Clifford algebra on $n+1$ gamma matrices Γ_A, $1 \leq a \leq n+1$[15]

$$\Gamma_A \in C_\kappa, \quad \left\{\Gamma^A, \Gamma^B\right\} = 2\kappa\,\delta^{AB},\ (\Gamma^A)^* = \kappa\Gamma^A$$

We let $Y \in \mathcal{A} \otimes C_\kappa$ be of the Feynman slashed form $Y = Y^A \Gamma_A$, and fulfill the equations

$$Y^2 = \kappa, \qquad Y^* = \kappa Y \qquad (3)$$

When we write $[D, Y]$ in (2), we mean $[D \otimes 1, Y]$. Finally $\langle\rangle$ applied to a matrix M_s of operators is its trace.

Note that here the components $Y^A \in \mathcal{A}$ but it is true in general that (3) implies that the components Y^A are self-adjoint commuting operators.

In spectral geometry the metric dimension of the underlying space is defined by the growth of the eigenvalues of the Dirac operator. As shown in [2] for even n, equation (2), together with the hypothesis that the eigenvalues of D grow as in dimension n, imply that the volume, expressed as the leading term in the Weyl asymptotic formula for counting eigenvalues of the operator D, is "quantized" by being equal to the index pairing of the operator D with the K-theory class of $\mathcal{A}$ defined by the projection $e = (1 + \sqrt{\kappa}\,Y)/2$.

In this letter we shall take equation (2), and its two sided refinement (4) below using the real structure, as a geometric analogue of the Heisenberg commutation relations $[p, q] = i\hbar$ where D plays the role of p (momentum) and Y the role of q (coordinate) and use it as a starting point of quantization of geometry with quanta corresponding to irreducible representations of the operator relations. The above integrality result on the volume is a hint of quantization of geometry. We first use the one-sided (2) as the equations of motion of some field theory on M, obtained from the spectral data, and describe the solutions as follows. (For details and proofs see [3, 4]).

Let M be a spin Riemannian manifold of even dimension n and $(\mathcal{A}, \mathcal{H}, D)$ the associated spectral triple. Then a solution of the one-sided equation exists if and only if M breaks as the disjoint sum of spheres of unit volume. On each of these irreducible components the unit volume condition is the only constraint on the Riemannian metric which is otherwise arbitrary for each component [3].

Each geometric quantum is a topological sphere of arbitrary shape and unit volume (in Planck units). It would seem at this point that only disconnected geometries fit in this framework but in the NCG formalism it is possible to refine (2). It is the real structure J, an antilinear isometry in the Hilbert space $\mathcal{H}$ which is the algebraic counterpart of charge conjugation. This leads to refine the quantization condition by taking J into account in the two-sided equation[16]

$$\langle Z\,[D, Z] \cdots [D, Z]\rangle = \gamma \quad Z = 2EJEJ^{-1} - 1, \tag{4}$$

where E is the spectral projection for $\{1, i\} \subset \mathbb{C}$ of the double slash $Y = Y_+ \oplus Y_- \in C^\infty(M, C_+ \oplus C_-)$. It is the classification of finite geometries of [5] which suggested to use the direct sum $C_+ \oplus C_-$ of two Clifford algebras and the algebra $C^\infty(M, C_+ \oplus C_-)$. It turns out moreover that in dimension $n = 4$ one has $C_+ = M_2(\mathbb{H})$, the 2×2 matrix algebra whose elements are quaternions and $C_- = M_4(\mathbb{C})$ which is in perfect agreement with the algebraic constituents of the Standard Model [5]. One now gets two maps $Y_\pm : M \to S^n$ while, for $n = 2, 4$, (4) becomes,

$$\det\left(e^a_\mu\right) = \Omega_+ + \Omega_-. \tag{5}$$

where e^a_μ is the vierbein, with $\Omega_\pm$ the Jacobian of $Y_\pm$ (the pullback of the volume form of the sphere).

Let $n = 2$ or $n = 4$ then [3],

(i) *In any operator representation of the two sided equation* (4) *in which the spectrum of D grows as in dimension n the volume (the leading term of the Weyl asymptotic formula) is quantized.*

(ii) *Let M be a compact oriented spin Riemannian manifold of dimension n. Then a solution of* (5) *exists if and only if the volume of M is quantized to belong to the invariant $q_M \subset Z$ defined as the subset of Z*

$$\begin{aligned} q_M = \{\deg(\phi_+) + \deg(\phi_-) \mid \phi_\pm : M \to S^n\}, \\ |\phi_+|(x) + |\phi_-|(x) \neq 0, \forall x \in M, \end{aligned} \tag{6}$$

where deg is the topological degree of the smooth maps and $|\phi|(x)$ is the Jacobian of ϕ at $x \in M$.

The invariant q_M makes sense in any dimension. For $n = 2, 3$, and any M, it contains all sufficiently large integers. The case $n = 4$ is much more difficult, but the proof that it works for all spin manifold is given in [3].

It is natural from the point of view of differential geometry, to consider the two sets of Γ-matrices and then take the operators Y and Y' as being the correct variables for a first shot at a theory of quantum gravity. Once we have the Y and Y' we can use them and get a map $(Y, Y') : M \to S^n \times S^n$ from the manifold M to the product of two n-spheres. Given a compact n-dimensional manifold M one can find a map $(Y, Y') : M \to S^n \times S^n$ which embeds M as a submanifold of $S^n \times S^n$. This is a known result, the strong embedding theorem of Whitney, [7], which asserts that any smooth real n-dimensional manifold (required also to be Hausdorff and second-countable) can be smoothly embedded in the real 2n-space. Of course $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n \subset S^n \times S^n$ so that one gets the required embedding. This result shows that there is no restriction by viewing the pair (Y, Y') as the correct "coordinate" variables.

Quantization of four-volume.— We now specialize to a four-dimensional Euclidean manifold and for simplicity consider only one set of maps, as this does not affect the analysis, and write $Y = Y^A\Gamma_A, \qquad A = 1, 2, ..., 5$, where Y^A are real and Γ_A are the Hermitian gamma matrices satisfying $\left\{\Gamma^A, \Gamma^B\right\} = 2\delta^{AB}$. The condition $Y^2 = 1$ implies

$$Y^A Y^A = 1, \tag{7}$$

where the index A is raised and lowered with δ^{AB}, thus defining the coordinates on the sphere S^4. Notice that Y^A are functions on the Euclidian manifold M_4 which depend on the coordinates x^μ. The Dirac operator on M_4 is

$$D = \gamma^\mu \left(\frac{\partial}{\partial x^\mu} + \omega_\mu\right), \tag{8}$$

where $\gamma^\mu = e^\mu_a \gamma^a$ and $\gamma^1\gamma^2\gamma^3\gamma^4 = \gamma$, and ω_μ is the connection, so that $[D, Y] = \gamma^\mu \frac{\partial Y^A}{\partial x^\mu}\Gamma_A$. Using the properties of gamma matrices one can check that the condition (2) reduces to

$$\det\left(e^a_\mu\right) = \frac{1}{4!}\epsilon^{\mu\nu\kappa\lambda}\epsilon_{ABCDE}Y^A\partial_\mu Y^B\partial_\nu Y^C\partial_\kappa Y^D\partial_\lambda Y^E. \tag{9}$$

Integrating over the volume of the manifold we find that

$$\begin{aligned} V = \int \tfrac{1}{4!}\epsilon^{\mu\nu\kappa\lambda}\epsilon_{ABCDE}Y^A\partial_\mu Y^B\partial_\nu Y^C\partial_\kappa Y^D\partial_\lambda Y^E d^4x \\ = \int \det\left(e^a_\mu\right) d^4x = \tfrac{8\pi^2 w}{3}. \end{aligned} \tag{10}$$

This number w is the number of components when using the one-sided equation (2) but using (4) one gets the sum of the degrees of the maps $Y_{\pm}: M \to S^n$ [6]. Thus, we conclude that in noncommutative geometry the volume of the compact manifold is quantized in terms of Planck units. This solves a basic difficulty of the spectral action [1] whose huge cosmological term is now quantized and no longer contributes to the field equations.

Gravitational action and cosmological constant.— Let us study consequences of the four volume quantization for Einstein gravity. For simplicity we shall utilize one set of maps $Y^A(x)$ since most of the details of what follows do not change when two sets $Y_{\pm}^A(x)$ are used instead. First we consider Euclidian compact spacetime and implement the kinematic constraints (7) and (9) in the action for gravity through Lagrange multipliers. This action then becomes

$$I = -\frac{1}{2}\int d^4x\sqrt{g}R + \frac{1}{2}\int d^4x\sqrt{g}\lambda'\left(Y^AY^A - 1\right) + \int d^4x\frac{\lambda}{2}\left(\sqrt{g} - \frac{\epsilon^{\mu\nu\kappa\lambda}}{4!}\epsilon_{ABCDE}Y^A\partial_\mu Y^B\partial_\nu Y^C\partial_\kappa Y^D\partial_\lambda Y^E\right), \tag{11}$$

where $8\pi G = 1$. Notice that the last term is a four-form and represents the volume element of a unit four-sphere. It can be written in differential forms and is independent of variation of the metric. Variation of the action with respect to the metric gives

$$G_{\mu\nu} + \frac{1}{2}g_{\mu\nu}\lambda = 0, \tag{12}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor. Tracing this equation gives $\lambda = -\frac{1}{2}G$, and as a result equations for the gravitational field become traceless

$$G_{\mu\nu} - \frac{1}{4}g_{\mu\nu}G = 0. \tag{13}$$

Using the Bianchi identity these equations imply that $\partial_\mu G = 0$, and hence $G = 4\Lambda$, where Λ is the cosmological constant arising as a *constant of integration* (compare to [8]). Variation of the action with respect to Y^A does not lead to any new equations because the last term in equation (11) is a topological invariant if $Y^AY^A = 1$.

One immediate application is that, in the path integration formulation of gravity, and in light of having only the traceless Einstein equation (13), integration over the scale factor is now replaced by a sum of the winding numbers with an appropriate weight factor. We note that for the present universe, the winding number equal to the number of Planck quanta would be $\sim 10^{61}$ [9]

Three-volume quantization and mimetic matter.— In reality spacetime is Lorentzian and generically has one noncompact dimension corresponding to time. Therefore, the condition for the volume quantization is literally non-applicable there. However being implemented in the Euclidian action it leads nevertheless to the appearance of the cosmological constant as an integration constant even in the Lorentzian spacetime. To show this let us first make a Wick rotation and then decompactify M_4 to $\mathbb{R}\times S^3$. With this purpose we set $Y^5 = \eta X$ and one of the coordinates say $x^4 \to \eta t$ and take the limit $\eta \to 0$. In this limit equation (7) becomes $Y^aY^a = 1, \quad a = 1, \cdots, 4,$ and the constraint (9) turns to

$$\sqrt{g} = \lim_{\eta\to 0}\left(\frac{1}{4!}\kappa^4\epsilon^{\mu\nu\kappa\lambda}\epsilon_{ABCDE}Y^A\partial_\mu Y^B\partial_\nu Y^C\partial_\kappa Y^D\partial_\lambda Y^E\right) = \frac{1}{3!}\epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}\left(\partial_\mu X\right)Y^a\partial_\nu Y^b\partial_\kappa Y^c\partial_\lambda Y^d. \tag{14}$$

The Lorentzian action for the gravity is

$$I = -\frac{1}{2}\int d^4x\sqrt{-g}R + \frac{1}{2}\int d^4x\sqrt{-g}\lambda'\left(Y^aY^a - 1\right) + \int d^4x\frac{\lambda}{2}\left(\sqrt{-g} - \frac{1}{3!}\epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}\left(\partial_\mu X\right)Y^a\partial_\nu Y^b\partial_\kappa Y^c\partial_\lambda Y^d\right). \tag{15}$$

The equations of motion in this case are the same as before and the cosmological constant arises as a constant of integration. The variable X in (15) is *a priori* unrestricted. We will show now that the requirement of the volume quantization of S^3 in the mapping $M_4 \to \mathbb{R}\times S^3$ leads to the following normalization condition for this variable,

$$g^{\mu\nu}\partial_\mu X\partial_\nu X = 1. \tag{16}$$

Let us consider the $3+1$ splitting of space-time, so that

$$ds^2 = h_{ij}\left(dx^i + N^idt\right)\left(dx^j + N^jdt\right) - N^2dt^2 \tag{17}$$

where $N\left(x^i, t\right)$ and $N^i\left(x^i, t\right)$ are the lapse and shift functions respectively and $\sqrt{-g} = N\sqrt{h}$. Consider a hypersurfaces Σ defined by constant t. Taking

$$\partial_i X = 0, \qquad \partial_t X = N, \tag{18}$$

which satisfy (16) we have (for details see [4])

$$\left(N\sqrt{h}\right)_\Sigma = \frac{1}{3!}N\left(\epsilon^{ijk}Y^a\partial_i Y^b\partial_j Y^c\partial_k Y^d\epsilon_{abcd}\right), \tag{19}$$

and therefore $\int\limits_\Sigma \sqrt{h}d^3x = w(\frac{4}{3}\pi^2)$, where w is the winding number for the mapping $\Sigma \to S^3$. Thus we have shown that (16) implies quantization of the volume of of compact 3d hypersurfaces in 4d spacetime. This condition can be understood as a restriction of the maps $Y^A(x)$ along directions orthogonal to the hypersurface Σ, to be length-preserving. To incorporate this condition in the action (15) we add to it the term

$$+\int d^4x\sqrt{-g}\lambda''\left(g^{\mu\nu}\partial_\mu X\partial_\nu X - 1\right) \tag{20}$$

which corresponds to mimetic dark matter [12, 13]. Thus the resulting action describes both dark matter and dark energy. Both substances arise automatically when the kinematic 4d and 3d compact volume-quantization in noncommutative geometry is incorporated in the gravity action. We note that the same field X could be used when we consider two different set of maps $Y_{+}^{A}(x)$ and $Y_{-}^{A}(x)$ from Σ to S^3.

Area quantization and black holes.— We can determine the conditions under which the area of any compact 2d submanifold of the 4d manifold must also be quantized. One can show (see [4] for details) that by writing $Y^4 = \eta X^1$, $Y^5 = \eta X^2$ and rescaling the coordinates transverse to the 2d hypersurface Σ as $x^\alpha \to \eta x^\alpha$, we get

$$\sqrt{{}^{(2)}g} = \det\left(e_i^a\right) = \frac{1}{2!}\epsilon^{ij}\epsilon_{ABC}Y^A\partial_i Y^B \partial_j Y^C, \tag{21}$$

provided that the area preserving condition on the hypersurface

$$\det\left(g^{\mu\nu}\partial_\mu X^m \partial_\nu X^n\right)|_\Sigma = 1, \qquad m, n = 1, 2, \tag{22}$$

is satisfied, where the index A in Y^A now $= 1, 2, 3$ and x^i are coordinates on the hypersurface . Hence, the area of a two-dimensional manifold is quantized

$$\begin{aligned} S &= \int \sqrt{{}^{(2)}g}\, d^2x \\ &= \int \frac{1}{2!}\epsilon^{\mu\nu}\epsilon_{ABC}Y^A\partial_\mu Y^B\partial_\nu Y^C d^2x = 4\pi n, \end{aligned} \tag{23}$$

where n is the winding number for the mapping of two-dimensional manifold to the sphere S^2, defined by $Y^AY^A = 1$.

This can have far-reaching consequences for black holes and de Sitter space. In particular, the area of the black hole horizon must be quantized in units of the Planck area (see also [10]). Because the areas of the black hole of mass M is equal to $A = 16\pi M^2$, this implies the mass quantization $M_n = \frac{\sqrt{n}}{2}$.

As it was shown in [11] the Hawking radiation in this case can be considered as a result of quantum transitions from the level n to the nearby levels $n-1, n-2, ...$ As a result even for large black holes the Hawking radiation is emitted in discrete lines and the spectrum with the thermal envelope is not continuous. The distance between the nearby lines for the large black holes is of order

$$\omega = M_n - M_{n-1} \simeq \frac{1}{4\sqrt{n}} = \frac{1}{8M}, \tag{24}$$

and proportional to the Hawking temperature, while the width of the line is expected to be at least ten times less than the distance between the lines [11]. Note that taking the minimal area to be α larger than the Planck area changes the distance between the lines by a factor α. Within the framework discussed, area quantization could thus have observable implications for evaporating black holes. Applying the same reasoning to the event horizon in de Sitter universe we find that the cosmological constant in this case must be quantized as $\Lambda_n = \frac{3}{n}$. It is likely that this quantization can have drastic consequences for the inflationary universe, in particular, in the regime of self-reproduction (see also [14]).

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[15] It is $n+1$ and not n where Γ_{n+1} is up to normalization the product of the n others.

[16] The γ involved here commutes with the Clifford algebras and does not take into account an eventual $\mathbb{Z}/2$-grading γ_F of these algebras, yielding the full grading $\gamma \otimes \gamma_F$.