Emergent Supersymmetry at the Ising–Berezinskii-Kosterlitz-Thouless Multicritical Point

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Phys. Rev. Lett. **114**, 090404 — Published 6 March 2015

DOI: 10.1103/PhysRevLett.114.090404

Emergent supersymmetry at the Ising–Berezinskii-Kosterlitz-Thouless multicritical point

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(Dated: February 12, 2015)

We show that supersymmetry emerges in a large class of models in $1+1$ dimensions with both $\mathbb{Z}_2$ and $U(1)$ symmetry at the multicritical point where the Ising and Berezinskii-Kosterlitz-Thouless transitions coincide. To arrive at this result we perform a detailed renormalization group analysis of the multicritical theory including all perturbations allowed by symmetry. This analysis reveals an intricate flow with a marginally irrelevant direction that preserves part of the supersymmetry of the fixed point. The slow flow along this special line has significant consequences on the physics of the multicritical point. In particular, we show that the scaling of the $U(1)$ gap away from the multicritical point is different from the usual Berezinskii-Kosterlitz-Thouless exponential gap scaling.

Introduction.–Characterizing the transitions between distinct phases of matter is a perennial question in physics, but also one of its greatest successes due to the emergence of universal behavior at continuous phase transitions. At such transitions, the low energy physics is independent of microscopic details and can be understood in terms of its *universality class*. Famous examples of universality classes are the Ising transition associated to the breaking of a $\mathbb{Z}_2$ symmetry and the Berezinskii-Kosterlitz-Thouless (BKT) transition in systems with a $U(1)$ symmetry. In this paper we study the properties of the multicritical point where an Ising and BKT transition coincide in one-dimensional quantum or two-dimensional classical systems (see Fig. 1). We find that under certain conditions, a new universality class emerges at this multicritical point, which is characterized by emergent supersymmetry.

Supersymmetry pertains to the invariance of the system under a transformation that maps bosons into fermions and vice versa. It plays an important role in high energy physics, where it may provide a solution to the hierarchy problem. Supersymmetry can also play a role in condensed matter systems as an explicit or an emergent symmetry [1–11]. To the best of our knowledge, the Ising–BKT multicritical point is the first example of a condensed matter system where extended ($\mathcal{N} > 1$) supersymmetry emerges.

This multicritical point can, in principle, occur in any system with both a $\mathbb{Z}_2$ and a $U(1)$ symmetry. Possible realizations include spin-1 chains where the $SU(2)$ spin symmetry is broken down to a $U(1) \otimes \mathbb{Z}_2$ symmetry, and polar molecules [12] in an optical lattice confined to one dimension. In the latter example, if the dipoles are oriented by an external field, the strong repulsive interactions between the polar molecules give rise to a "zig-zag" instability towards spontaneously breaking a $\mathbb{Z}_2$ symmetry associated with reflection [13–16]. In addition, if the density of the molecules is commensurate with the optical lattice, a BKT superfluid to Mott insulator occurs at a critical strength of the lattice.

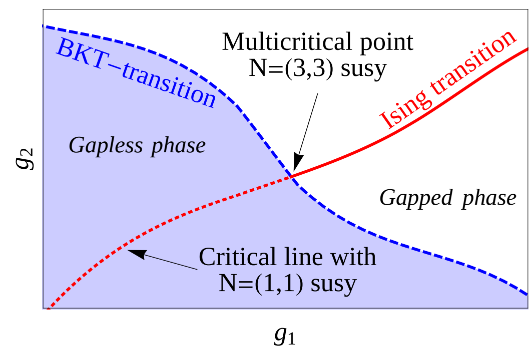


FIG. 1: Schematic phase diagram for two parameters, g_1 and g_2. The blue dashed line is the Berezinskii-Kosterlitz-Thouless transition, corresponding to $K = K_{\rm crit}$, separating the gapless quasi long-ranged order phase (shaded region, $K > K_{\rm crit}$) from the gapped disordered phase. The red line is the Ising transition. Inside the gapless phase the Ising transition can have $\mathcal{N} = (1,1)$ supersymmetry (dotted red line). The multicritical point is located at the intersection of the BKT and Ising transitions.

A 2D classical model that exhibits the multicritical point is the generalized fully frustrated XY-model [17]. Note that the fully frustrated XY-model [2] and Ising-XY-model [18] do not accommodate the multicritical point, because for these models the BKT transition always occurs at a temperature below the Ising transition due to a subtle interplay between the Ising and BKT degrees of freedom at the microscopic level [19, 20].

We show that in a large class of systems this multicritical point is described by a critical theory with emergent Lorentz-invariant and emergent $\mathcal{N} = (3,3)$ supersymmetry. We arrive at this result by a renormalization group (RG) analysis of the decoupled multicritical theory and its symmetry-allowed perturbations. Interestingly, we find a very slow flow towards the decoupled Ising–BKT theory, as a result of marginally irrelevant operators. These operators give rise to qualitatively new behavior, such as unusual scaling of the correlation length.

Model.–We consider the Lagrangian density $\mathcal{L}_0 = \mathcal{L}_\Phi + \mathcal{L}_\chi$, where

$$\begin{aligned}\mathcal{L}_\Phi &= \frac{1}{2\pi K}\Big(\frac{1}{v}(\partial_\tau \Phi)^2 + v(\partial_x \Phi)^2\Big)\\ \mathcal{L}_\chi &= \chi_R(\partial_\tau + \frac{u}{\imath}\partial_x)\chi_R + \chi_L(\partial_\tau - \frac{u}{\imath}\partial_x)\chi_L. \end{aligned} \tag{1}$$

Here, v, K are the velocity and Luttinger parameter of the bosonic field, Φ, and u is the velocity of the left- and right-moving modes of the real (Majorana) fermionic field, χ. If Lorentz invariance is present, the fermion and boson velocities are equal, $u = v$. In this case, the resulting conformal field theory has an $\mathcal{N} = (1, 1)$ supersymmetry generated by the supercharges $\chi_L \partial_z \Phi_L$ and $\chi_R \partial_{\bar z} \Phi_R$, where $z = x + \imath v\tau$ and $\Phi(z, \bar z) = \Phi_L(z) + \Phi_R(\bar z)$. For the special case of $K = 4$, there is an additional extended $\mathcal{N} = (2, 2)$ supersymmetry generated by the supercharges $\chi_{L,R} \exp(\pm\imath\sqrt{2}\Phi_{L,R})$ for the left and right-movers (for details see the supplemental material [21]) [22–24].

Adding all perturbations to the fixed point action that are allowed if only $\mathbb{Z}_2$ and $U(1)$ symmetry are imposed, leads to

$$\mathcal{L}_{\rm int} = \imath m\chi_R\chi_L + g\cos(\sqrt{2}\Phi) - \lambda(\partial_x\Phi)\imath\chi_R\chi_L. \tag{2}$$

The first term, the mass term for the fermion, generates a relevant flow to the Ising ordered or disordered phase. At the multicritical point, we have $m = 0$. The cosine-term for the boson has scaling dimension $h = K/2$. At the multicritical point, that is for $K = K_{\rm crit} = 4$, this term is marginal. The last term is a marginal term that couples the boson to the fermion [25]. Finally, the velocities, u, v, are allowed to be different. Note that the λ-term and the velocity difference explicitly break Lorentz symmetry and will therefore be absent in Lorentz invariant systems, and in systems with a 90^o rotational symmetry that relates x and τ [2, 26].

RG analysis.–The renormalization group flow for $\mathcal{L} = \mathcal{L}_\Phi + \mathcal{L}_\chi + \mathcal{L}_{\rm int}$ with $g = 0$ was worked out in Ref. [25]. This applies to the Ising transition inside the gapless phase, since for $K > K_{\rm crit}$ the cosine term is irrelevant. For $m = 0$ and $u < v$ the RG flow is towards the Lorentz invariant, decoupled fixed point, i.e. $\lambda = 0$ and $u = v$. We point out that this implies the emergence of $\mathcal{N} = (1, 1)$ supersymmetry at the critical line inside the gapless phase, i.e. the red dotted line in Fig. 1 (see also [2]). In [27] we derived the flow equations to second order in the couplings, in the presence of the cosine term. In a scheme where the boson velocity is kept fixed and an anomalous dynamical exponent, $z = 1 + \lambda^2 K/(16uv)$, is introduced, one finds

$$\begin{aligned}\frac{du}{d\ell} &= -\frac{u\lambda^2 K}{4}\Big(\frac{1}{(v+u)^2} - \frac{1}{4uv}\Big) \Rightarrow \frac{d\epsilon}{d\ell} = \frac{\lambda^2 K}{64 v^3}\epsilon^2\\ \frac{dm}{d\ell} &= m\left(1 - \frac{\lambda^2 K}{8}\left[\frac{1}{u(u+v)} + \frac{1}{(u+v)^2} - \frac{1}{2uv}\right]\right)\\ \frac{d\lambda}{d\ell} &= 0\\ \frac{dK}{d\ell} &= -K^2\Big(\frac{g^2\pi^2}{2v^2} - \frac{\lambda^2}{16uv}\Big)\\ \frac{dg}{d\ell} &= \frac{g}{2}(4-K)\end{aligned}$$

where $\epsilon = u - v \ll 1$. The velocity difference, ϵ, is relevant for $u > v$, and irrelevant for $u < v$. For $\lambda = 0$, the equations for g and K reduce to the Kosterlitz equations. There is a line of fixed points to second order in the couplings parametrized by

$$\lambda = 2\sqrt{2}\pi g,\ K = 4,\ m = 0,\ \text{and}\ u = v. \tag{3}$$

Remarkably, as shown in [27], precisely this line preserves part of the $\mathcal{N} = (2, 2)$ supersymmetry. Supersymmetry constrains the flow to be along the special line (3); to determine the direction of the flow for $u \le v$, it is therefore sufficient to compute $\beta(\lambda)$ to lowest non-vanishing order.

We proceed by fermionizing the boson for which we introduce a second, redundant bosonic field Φ_s, with velocity v_s and Luttinger parameter K_s, which is free and completely decoupled from the other fields. The theory is fermionized by introducing two flavours of complex fermions, $\psi_{a,p}$ where $a = 1, 2$ and $p = \pm$ for right- and left-movers, respectively. The precise relation between $\psi_{a,p}$ and the bosonic fields Φ, Φ_s is given in the supplemental material. We find that the fermionized action reads $\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_\chi + \mathcal{L}_{\rm int}$, with $\mathcal{L}_\chi$ as defined above,

$$\mathcal{L}_\psi = \sum_{p=\pm}\sum_{a=1,2} \psi^\dagger_{a,p}(\partial_\tau - \imath p v_F \partial_x)\psi_{a,p},$$

and

$$\begin{aligned}\mathcal{L}_{\rm int} = \imath m\chi_R\chi_L + 2\pi^2 g\left(\psi^\dagger_{1+}\psi_{1-} \psi^\dagger_{2+}\psi_{2-} + \psi^\dagger_{1-}\psi_{1+}\psi^\dagger_{2-}\psi_{2+}\right) + \bar g \sum_{a=1,2}\left(\psi^\dagger_{a,+}\psi_{a,+}\right)\sum_{a=1,2}\left(\psi^\dagger_{a,-}\psi_{a,-}\right)\\ -\sqrt{2}\pi\lambda \sum_{\substack{p=\pm\\ a=1,2}}\left(\psi^\dagger_{a,p}\psi_{a,p}\right)\imath\chi_R\chi_L + 2g_v\sum_{p=\pm}\psi^\dagger_{1,p}\psi_{1,p}\psi^\dagger_{2,p}\psi_{2,p} - \sum_{\substack{p=\pm\\ a=1,2}}\psi^\dagger_{a,p}(\imath p\epsilon_v\partial_x)\psi_{a,p} + \imath\epsilon_u(\chi_L\partial_x\chi_L - \chi_R\partial_x\chi_R),\end{aligned}$$

where the last three terms are a forward scattering term and velocity renormalizations for the fermion velocities v_F and u. These terms are included because they are generated by the interaction terms under RG. The parameters of the bosons are related to those of the complex fermions: $v = v_F + \epsilon_v + g_v/\pi$, $K = 4(1 - \bar{g}/(\pi v_F))$, $v_s = v_F + \epsilon_v - g_v/\pi$, and $K_s = 1$.

We recover the RG equations to second order in the fermionized model taking a field theoretic approach (see supplemental material) and proceed to compute the beta-function for λ to third order in the couplings on the special line (3). We argue below that this suffices to establish stability of the fixed point. Schematically, we first perturbatively compute the 4-point function, $G_\lambda^{(4)} = \langle \psi_{1+}^\dagger \psi_{1+} \chi_R \chi_L \rangle$, to third order in the couplings. We then introduce counterterms for the divergent diagrams by imposing the renormalization condition at the renormalization scale M. Finally, we use Callan-Symanzik (CS) equation for the 4-point function to obtain the beta-function:

$$\left[M \frac{\partial}{\partial M} - \sum_i \beta(g_i) \frac{\partial}{\partial g_i} + 2\gamma_\psi + 2\gamma_\chi \right] G_\lambda^{(4)} = 0, \quad (4)$$

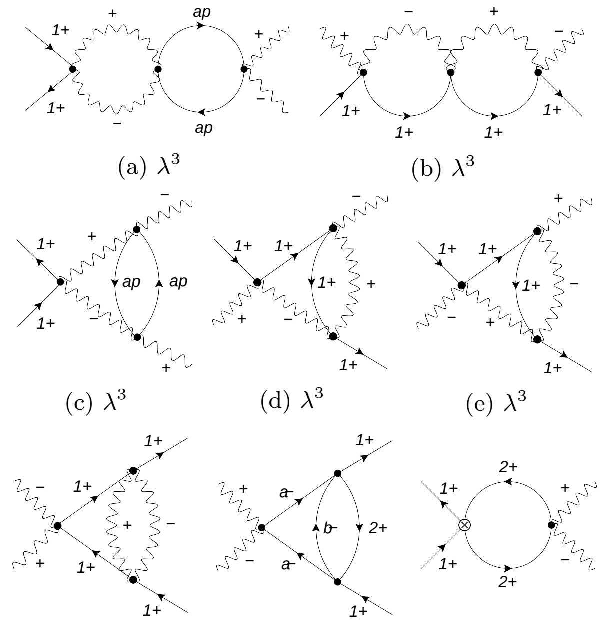
where g_i are all the couplings, $\beta(g_i) = -\partial g_i / \partial(\ln M)$ and $\gamma_{\psi,\chi}$ are the field renormalizations. Note that we have defined the beta-functions with the sign convention commonly used in the condensed matter community (see, e.g., [28]), as opposed to that typically used in field theory textbooks (e.g. [29]).

The computation of the counterterm for λ, δ_λ, to third order turns out to be a great challenge in bookkeeping. Figures 2a-2g show all the divergent diagrams for $\bar{g} = 0$ and $g = \lambda/(2\sqrt{2}\pi)$. Roughly speaking, the loops containing a propagator of both a left- $(+)$ and a right-moving $(-)$ field are divergent. Diagrams 2a and 2b are easily computed by generalizing the second order computations. There are essentially two kinds of the slightly more subtle nested diagrams: the diagrams where the inner loop is divergent (figs. 2e-2g) and those where the outer loop is divergent (figs. 2c and 2d). Especially computing the former, one has to proceed with some caution. We discuss this in the supplemental material.

Eventually, we obtain $\delta_\lambda = 3\lambda^3 (4v_F)^{-2} \left[\frac{2}{\varepsilon} - \log(M^2) \right]$ and working out the CS equation (4), we find

$$\beta(\lambda) = M \frac{\partial}{\partial M} \delta_\lambda - 2\lambda(\gamma_\psi + \gamma_\chi) = -\frac{\lambda^3}{2v_F^2}, \quad (5)$$

where we used $\gamma_\psi = 0$ for $g = \lambda/(2\sqrt{2}\pi)$ and $\bar{g} = 0$. We thus find that λ is marginally irrelevant and the flow is towards $\lambda = 0$ as we go to lower energy scales. Note that for $u \neq v$ there will be corrections to the flow in λ, but since they are of the form $\epsilon \mathcal{O}(\lambda^3)$, where ϵ is the velocity difference as above, these terms are subleading for small ϵ. The supersymmetry on the special line $g = \lambda/(2\sqrt{2}\pi)$, $\bar{g} = 0$ implies that this line is preserved under the RG flow; i.e., on this line we have $\beta(g) = \beta(\lambda)/(2\sqrt{2}\pi)$ and $\beta(\bar{g}) = 0$ to third order in the couplings.



(a) λ^3 (b) λ^3 (c) λ^3 (d) λ^3 (e) λ^3 (f) λ^3 (g) λg^2 (h) $\delta_{g_v}\lambda$

FIG. 2: The divergent diagrams that contribute to the λ-counterterm at third order, where $a = 1, 2$, $p = \pm$ and $b = 1, 2$ such that $b \neq a$. The last diagram (2h) contains a second order counterterm that partially cancels the divergent diagrams. Note that since we take $\bar{g} = 0$, we suppressed all diagrams containing $\bar{g}$-vertices. Finally, we have also suppressed the diagrams whose divergences are clearly taken care of by the velocity counterterms.

We analysed the stability of the fixed point by considering small perturbations away from the special line (see supplemental material). The results are summarized in the schematic flow diagram shown in figure 3. We establish that there is an attractive plane for which the flow is towards the special line and eventually terminates at the fixed point. Furthermore, this plane is a separatrix; below the plane the flow is towards $\lambda = g = 0$, and $\bar{g}$ finite, while above the plane the flow is towards increasing g and $\bar{g}$. This corresponds to the gapless and gapped phase in the $U(1)$ sector, respectively, i.e. the dashed and drawn red lines in Fig. 1 (remember that $m = 0$). It follows that the decoupled, Lorentz invariant fixed point with $\mathcal{N} = (3, 3)$ supersymmetry can be reached upon tuning to the attractive plane and $m = 0$, which requires to fine tune two parameters (as expected for the multicritical point).

Interestingly, although the Ising and $U(1)$ sectors decouple at the infra-red, the physics of the multicritical point is quite different from the usual Ising and BKT physics due to the slow flow towards the fixed point. In particular, we find that the correlation length in the $U(1)$

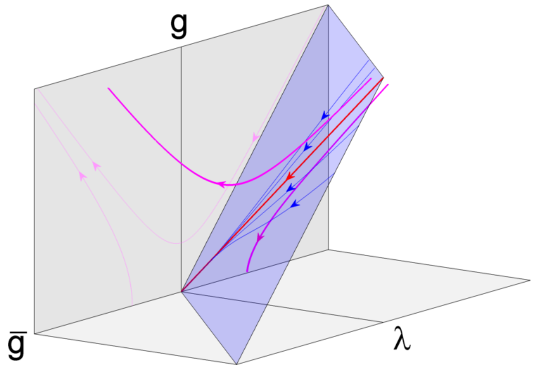


FIG. 3: Schematic flow diagram in the $(g, \bar{g}, \lambda)$-space. The red line is the special line along which there is a slow flow towards the fixed point. The blue lines represent the flow lines in the attractive plane. In purple we show the flow for relevant perturbations towards the gapped (line above the plane) and gapless (line below the plane) phases in the $U(1)$ sector. In the $\lambda = 0$-plane the flow lines are governed by the usual Kosterlitz equations.

	multicritical point	critical Ising line
$v_{\rm phys}, u_{\rm phys}$	$v_F \ell^{-1/4} \to 0$	$v_F e^{-a_1 \ell^{1/5}} \to 0$
γ	$(\ln \frac{1}{T})^{1/4} \to \infty$	$e^{a_2 (\ln \frac{1}{T})^{1/5}} \to \infty$
Δ_m	$\lvert m\rvert (\ln \frac{1}{\lvert m\rvert})^{-1/4}$	$\lvert m\rvert e^{-a_3 (\ln \frac{1}{\lvert m\rvert})^{1/5}}$

TABLE I: We show the behavior of the physical velocities, $v_{\rm phys}$ and $u_{\rm phys}$, the specific heat coefficient, γ, and the Ising gap, Δ_m, asymptotically close to the multicritical point (middle column) and the critical Ising line inside the gapless phase (right column). These expressions are valid for $T, |m|, 1/\ell \ll 1$, where T is the temperature, m the Ising mass and all three parameters are measured in dimensionless units. The a_i are positive constants whose values can be found in Ref. 25.

sector diverges in an unusual way as one approaches the multicritical point from the gapped phase:

$$\xi(\delta g_0) = \xi_0 e^{\frac{1}{4}\left(\ln\left[\frac{\lambda_0}{\delta g_0}\right]\right)^2}, \qquad (6)$$

where ξ_0 is a constant, λ_0 is the initial value of λ along the special line, and $\delta g_0 > 0$ is the distance from the special line in the g-direction (see the supplemental material for details). Note that the coefficient of the logarithm squared of 1/4 is universal. The scaling of the correlation length we obtain is a remarkable result that should be contrasted with the result for the conventional BKT transition, where the correlation length diverges as $\xi(\delta g_0) = \xi_0 \exp(c_1/\sqrt{\delta g_0})$, with c_1 a dimensionful non-universal constant, and for a conventional phase transition driven by a relevant operator with scaling dimension $\Delta < 2$, where the correlation length scales as $\xi(\delta g_0) \sim (\delta g_0)^{1/(\Delta-2)}$. The result we obtain resides between these two well-known cases: the divergence is faster than any power law, but subexponential in the sense that $\log \xi$ grows slower than any power law; in particular, it is slower than at a traditional BKT transition.

To infer the gap scaling from the correlation length we need to take into account that the λ-term gives rise to an anomalous dynamical critical exponent, $z(\ell)$, that depends on the RG scale, $\ell = -\ln M$. The relative factor of $e^{\ell(z(\ell)-1)}$ between the rescaling of the correlation length and the gap has to be integrated over the entire RG trajectory, but since z flows to $z = 1$ very slowly, the result actually vanishes. This situation is also encountered in Ref. 25 and the consequences on the physical velocities, the specific heat and the Ising gap are nicely discussed. Here the situation is very similar, the main difference is the scaling of λ. Consequently, we find $z - 1 \sim \ell^{-1}$ at the multicritical point, while $z - 1 \sim \ell^{-4/5}$ at the Ising transition inside the gapless $U(1)$ phase [25]. At the multicritical point, we thus find that the vanishing of the physical velocities, the specific heat coefficient divergence and the Ising gap suppression take on a different form as summarized in table I, showing also the results from Ref. 25 for comparison.

Furthermore, for the scaling of the $U(1)$ gap we find

$$\Delta(\delta g_0) \approx \Delta_0 \frac{1}{\sqrt{\ln\left[\frac{\lambda_0}{\delta g_0}\right]}} e^{-\frac{1}{4}\left(\ln\left[\frac{\lambda_0}{\delta g_0}\right]\right)^2}.$$

Like the correlation length, the gap shows neither power law nor exponential scaling, but something in between. Finally, we expect that order parameter correlation functions will receive logarithmic corrections to the usual scaling due to the slow flow. To compute these corrections is beyond the scope of this paper, since one needs the field renormalizations to third order in the couplings.

Conclusions– we have shown that an $\mathcal{N} = (3, 3)$ supersymmetry emerges in a class of lattice models. The requirements are: 1) the model has a $U(1)$ and a $\mathbb{Z}_2$ symmetry, 2) the system is tuned to the multicritical point where the Ising and BKT transitions coincide, and 3) the bare velocity of the fermionic degree of freedom is smaller than or equal to the bare velocity of the bosonic degree of freedom. Apart from the emergent extended supersymmetry, we find that the RG flow towards the fixed point is extremely slow. Consequently, the Ising–BKT multicritical point with emergent supersymmetry lies in a new universality class that is characterized, in particular, by a superpolynomial but subexponential scaling of the corrlation length in the $U(1)$ sector. It would be interesting to see if there are other multicritical points where supersymmetry emerges, in particular, also in higher dimensions and investigate if a deeper connection exists between supersymmetry and multicriticality.

ACKNOWLEDGEMENTS

L.H. was partially funded by the John Templeton foundation and by a DOE early career grant. E.B. was supported by the Israel Science Foundation, by the Israel-US Binational Science Foundation, by the Minerva Foundation, by a Marie Curie CIG grant, and by the Robert Rees Fund. L.H. and E.B. also acknowledge support from the NSF grant no. DMR-1103860.

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