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Reduction of SO(2) symmetry for spatially extended dynamical systems

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Spatially extended systems, such as channel or pipe flows, are often equivariant under continuous symmetry transformations, with each state of the flow having an infinite number of equivalent solutions obtained from it by a translation or a rotation. This multitude of equivalent solutions tends to obscure the dynamics of turbulence. Here we describe the 'first Fourier mode slice', a very simple, easy to implement reduction of SO(2) symmetry. While the method exhibits rapid variations in phase velocity whenever the magnitude of the first Fourier mode is nearly vanishing, these near singularities can be regularized by a time-scaling transformation. We show that after application of the method, hitherto unseen global structures, for example Kuramoto-Sivashinsky relative periodic orbits and unstable manifolds of travelling waves, are uncovered.

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Mounting evidence that exact coherent structures play a key role in shaping turbulent flows [1] necessitates developing new tools for elucidating how these structures are interrelated [2]. Unravelling these interrelations for flows which admit continuous symmetries requires special care. A solution to a problem in classical or quantum mechanics starts with the classification of problem's symmetries, followed by a choice of a basis invariant under these symmetries. For example, one formulates the two-body problem in three Cartesian coordinates, but when it comes to solving it, polar coordinates, with the phase along the symmetry direction as an explicit coordinate, are preferable. While a classification of problem's symmetries might be relatively straightforward, for high-dimensional nonlinear systems (fluids, nonlinear optical media, reaction-diffusion systems, *etc.*) a good choice of a symmetry-invariant frame is not as easy as transforming to polar coordinates. In this letter we describe the 'first Fourier mode slice', a simple method for reducing U(1) or SO(2) symmetry that has been tested on and works well for systems of dimensions ranging from 4 (for the 'two-mode system' [3]) to 10^5 (for fluid dynamics [4]).

The applications we have in mind are to solutions of spatially extended systems, such as Navier-Stokes equations for a velocity field u on a spatially periodic domain, where one starts the symmetry analysis by rewriting the equations in a Fourier basis,

$$u(x,\tau) = \sum_{k=-\infty}^{+\infty} \tilde{u}_k(\tau)\, e^{i q_k x} \,, \tag{1}$$

where $\tilde{u}_k = x_k + i\, y_k = |\tilde{u}_k| e^{i\phi_k}$, $q_k = 2\pi k/L$, L is the domain size, x is the spatial coordinate and τ is time. Thus a nonlinear PDE is converted to an infinite tower of ODEs. In computations this state space is truncated to $2m$ real dimensions [24], $a = (x_1, y_1, x_2, y_2, \ldots, x_m, y_m)^\mathsf{T}$. If the system has a translational symmetry, the complex Fourier modes (1) form a continuous family of states (a group orbit), equivalent under spatial translations $u(x,\tau) \to u(x+\delta x,\tau)$, and related by U(1) rotations

$$\tilde{u}_k \to \tilde{u}_k e^{ik\theta} \,, \quad \theta = 2\pi\delta x/L \,. \tag{2}$$

In other words, the formulation contains a redundant degree of freedom. Keeping such redundant degrees of freedom, as we shall illustrate here with the Kuramoto-Sivashinsky example, obscures the dynamics.

In this letter we shall assume that for a generic 'turbulent' state $u(x,\tau)$ the first Fourier mode never exactly vanishes, and define the symmetry-reduced Fourier modes $\hat{u}_k$ by fixing the phase of the first Fourier mode,

$$\hat{u}_k(\tau) = e^{-ik\phi_1(\tau)} \tilde{u}_k(\tau) \,, \tag{3}$$

The symmetry reduced Fourier modes $\hat{u}_k$ are invariant under the symmetry transformation (2) by construction. A phase-fixing transformation of this kind is very natural; the earliest example known to authors is the reduction of the S^1 symmetry of the complex Ginzburg-Landau equation by Luce [5]. When applied to spatiotemporally chaotic dynamics, however, the phase-fixing transformation (3) introduces what appear to be discontinuities in the flow. In this letter we show that a reexamination of the method of slices [6–11] leads to a regularization of such apparent singularities by means of a rescaled 'slice time'. This representation (from here on referred to as the 'first Fourier mode slice') reveals relations among important coherent structures of the flow, such as relative equilibria and relative periodic orbits, known to play an important role in shaping the state space of turbulent flows [1]. Here, for simplicity, we illustrate the first Fourier mode slice by applying it to the dynamics of

Kuramoto-Sivashinsky equation in one spatial dimension. As shown in ref. [4], the method is equally easily incorporated into spectral codes for $3D$ fluid flows in periodic domains, with no need for any further generalization.

Consider a first-order flow $\dot{a} = v(a)$ on state space $a \in \mathcal{M}$ obtained from the m-mode truncation of the Fourier expansion (1). Here the velocity function $v(a) = (\dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2 \ldots, \dot{x}_m, \dot{y}_m)^{\mathsf{T}}$ is the Fourier transform of the right side of the PDE for field $u(x,\tau)$. Translational symmetry in the configuration space implies that the dynamics satisfies the equivariance condition

$$v(a) = D(\theta)^{-1} v(D(\theta)a)\,, \qquad (4)$$

where

$$D(\theta) = \mathrm{diag}\,[\,R(\theta),\, R(2\theta),\, \ldots,\, R(m\theta)\,]\,, \qquad (5)$$

is a block-diagonal $[2m \times 2m]$ matrix representation of the SO(2) action and $R(k\theta)$ is the $[2\times 2]$ rotation matrix acting on the k-th Fourier mode. The generator of rotations is also a block-diagonal matrix, with $[2\times 2]$ infinitesimal generators of infinitesimal rotations T_k along its diagonal,

$$R(k\theta) = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix}, \quad T_k = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}. \qquad (6)$$

For visualization purposes we find it more convenient to work in the real SO(2) representation rather than the complex U(1) formulation (2). The group orbit $\mathcal{M}_a$ of a state space point a is the set of all points reachable from a by symmetry transformations, $\mathcal{M}_a = \{D(\theta)\,a \mid \theta \in [0, 2\pi)\}$. In the method of slices, one constructs a 'slice', a submanifold $\hat{\mathcal{M}} \subset \mathcal{M}$ that cuts each group orbit in an open neighborhood once and only once. The dynamics is then separated into the 'shape-changing' dynamics $\hat{a}(\tau) \in \hat{\mathcal{M}}$ within this submanifold, and a symmetry coordinate parametrized by the group parameter $\theta(\tau)$ (a 'moving frame' [12–14]) that reconstructs the original dynamics $a(\tau) \in \mathcal{M}$ by the group action $a(\tau) = D(\theta(\tau))\,\hat{a}(\tau)$. For the SO(2) case at hand, a one-parameter family of transformations, $\hat{\mathcal{M}}$ has one dimension less than $\mathcal{M}$.

There is a great deal of freedom in how one constructs a slice; in general one can pick any 'moving frame'. Computationally easiest way to construct a local slice is by considering a hyperplane of points $\hat{a}$ defined by

$$0 = \langle \hat{a} | t' \rangle\,, \quad \text{where} \quad \langle b | c \rangle = \sum_{\ell=1}^{2m} b_\ell c_\ell\,, \qquad (7)$$

is sketched in fig. 1. Here, $t' = T\hat{a}'$ is the group tangent (the direction of translations) evaluated at a reference state space point $\hat{a}'$, or 'template' [6]. The template is assumed not to lie in an invariant subspace, i.e., $D(\theta)\hat{a}' \neq \hat{a}'$ for all $D(\theta) \neq 1$.

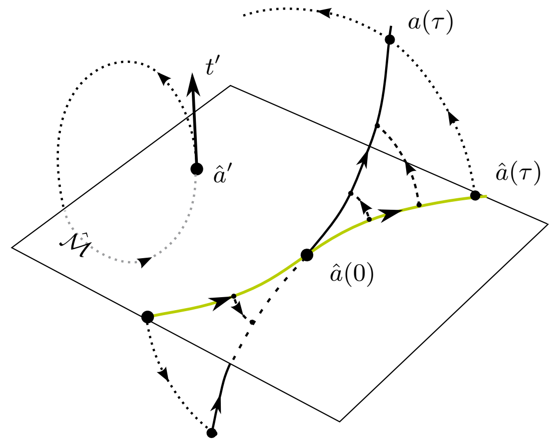

FIG. 1: (Color online) The slice hyperplane $\hat{\mathcal{M}}$, which passes through the template point $\hat{a}'$ and is normal to its group tangent t', intersects all group orbits (dotted lines) in an open neighborhood of $\hat{a}'$. The full state space trajectory $a(\tau)$ (solid black line) and the reduced state space trajectory $\hat{a}(\tau)$ (solid green line) belong to the same group orbit $\mathcal{M}_{a(\tau)}$ and are equivalent up to a 'moving frame' rotation by phase $\theta(\tau)$. Adapted from `ChaosBook.org.`

The dynamics within this slice hyperplane and the reconstruction equation for the phase parameter are given by

$$\hat{v}(\hat{a}) \;=\; v(\hat{a}) - \dot{\theta}(\hat{a})\,t(\hat{a})\,, \qquad (8)$$
$$\dot{\theta}(\hat{a}) \;=\; \langle v(\hat{a}) | t' \rangle / \langle t(\hat{a}) | t' \rangle\,, \qquad (9)$$

with $t(\hat{a}) = T\hat{a}$ the group tangent evaluated at the symmetry-reduced state space point $\hat{a}$. Eq. (8) says that the full state space velocity $v(\hat{a})$ is the sum of the in-slice velocity $\hat{v}(\hat{a})$ and the transverse velocity $\dot{\theta}(\hat{a})\,t(\hat{a})$ along the group tangent, and (9) is the reconstruction equation whose integral tracks the trajectory in the full state space (for a derivation and further references, see ref. [15]).

The phase velocity (9) becomes singular for $\hat{a}^*$ such that $t(\hat{a}^*)$ lies in the slice,

$$\langle t(\hat{a}^*) | t' \rangle = 0\,, \qquad (10)$$

or for $\hat{a}^*$ in an invariant subspace, where $t(\hat{a}^*) = 0$. The $(d-2)$-dimensional hyperplane of such points $\hat{a}^*$ forms the 'slice border', beyond which the slice does not apply. Both the slice hyperplane and its border depend on the choice of template $\hat{a}'$, with the resulting 'chart' in general valid only in some neighborhood of $\hat{a}'$. For a turbulent flow, symmetry reduction might require construction of a set of such local overlapping charts [10, 11]. However, as we now show for SO(2), a simple choice of template may suffice to avoid all slice border singularities in regions of dynamical interest.

We define the 'first Fourier mode slice' by choosing

$$\hat{a}' = (1, 0, 0, 0, \ldots)\,, \quad t' = (0, 1, 0, 0, \ldots)\,. \qquad (11)$$

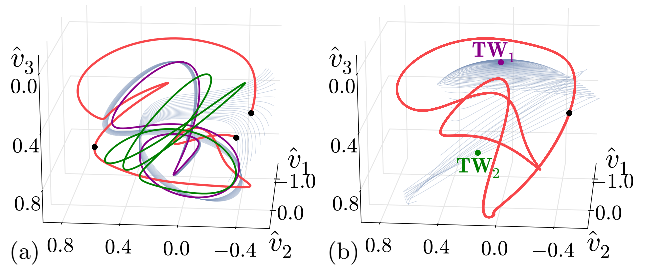

FIG. 2: (Color online) Kuramoto-Sivashinsky system (a) in the full state space: Unstable manifold of the relative equilibrium TW_1 (blue) traced out by integrating nearby points given by (17): 2 repeats of the $\tau_p = 33.5010$ relative periodic orbit (red), with instants $\tau = 0, \tau_p, 2\tau_p$ marked by black dots; group orbits (which are also the time orbits) of the TW_1 (magenta) and TW_2 (green). (b) In the symmetry reduced state space TW_1 and TW_2 orbits are reduced to single points, the unstable manifold is a smooth 2D surface, and the relative periodic orbit closes after a single period.

The slice determined by this template is the $(d-1)$-dimensional half-hyperplane

$$\begin{aligned} \hat{x}_1 &\geq 0\,, \quad \hat{y}_1 = 0\,, \\ \hat{x}_k, \hat{y}_k &\in \mathbb{R}\,, \quad \text{for all} \quad k > 1\,. \end{aligned} \tag{12}$$

The condition $\hat{y}_1 = 0$ follows from the slice condition (7), whereas $\hat{x}_1 \geq 0$ ensures a single intersection for every group orbit. This choice of slice corresponds to (3), fixing the phase of the first Fourier mode. Now that we have the equations (8) and (9) for the dynamics in the slice hyperplane we see why this phase fixing transformation can run into singularities: slice border (10) for the template (11) is located at $\hat{x}_1 = 0$, and the denominator of (9) approaches zero as the trajectory approaches the slice border. We regularize this singularity by defining the in-slice time as $d\hat{\tau} = d\tau/\hat{x}_1$, and rewriting (8) and (9) as

$$d\hat{a}/d\hat{\tau} = \hat{x}_1 v(\hat{a}) - \dot{y}_1(\hat{a})\, t(\hat{a})\,, \tag{13}$$

$$d\theta(\hat{a})/d\hat{\tau} = \dot{y}_1(\hat{a})\,. \tag{14}$$

The phase velocity (14) is now the non-singular, full state space velocity component $\dot{y}_1$ orthogonal to the slice, and the full state space time is the integral

$$\tau(\hat{\tau}) = \int_0^{\hat{\tau}} d\hat{\tau}'\, \hat{x}_1(\hat{\tau}')\,. \tag{15}$$

For example, the full state space period $\tau_p = \tau_p(\hat{\tau}_p)$ of a relative periodic orbit $a(\tau_p) = g_p\, a(0)$ is the integral (15) over one period $\hat{\tau}_p$ in the slice.

We illustrate the utility of first Fourier mode slice by applying it to the Kuramoto-Sivashinsky system on a periodic domain in one spatial dimension,

$$u_t = -\tfrac{1}{2}(u^2)_x - u_{xx} - u_{xxxx}\,,$$

a model PDE extensively studied as it exhibits spatiotemporal chaos [16]. The relative equilibrium (traveling wave) TW_i and relative periodic orbit solutions that we use in this example are described in ref. [17], where the domain size has been set to $L = 22$, large enough to exhibit complex spatiotemporal dynamics. In terms of complex Fourier modes (1) the Kuramoto-Sivashinsky equation takes form:

$$\dot{\tilde{u}}_k = (q_k^2 - q_k^4)\,\tilde{u}_k - i\frac{q_k}{2}\sum_{m=-\infty}^{+\infty} \tilde{u}_m \tilde{u}_{k-m}\,. \tag{16}$$

In the real representation $\tilde{u}_k = x_k + i\, y_k$, Kuramoto-Sivashinsky equation is equivariant under SO(2) rotations (5). We have adapted the ETDRK4 method [18, 19] for numerical integration of the symmetry reduced equations (13), where we set $\tilde{u}_0 = 0$ and truncate the expansion (16) to $m = 15$ Fourier modes, so the state space is 30-dimensional, $a = (x_1, y_1, x_2, y_2, ..., x_{15}, y_{15})^\mathsf{T}$.

As an illustration of symmetry reduction, we trace out a segment of the unstable manifold of the relative equilibrium TW_1 by integrating n trajectories for time τ, with initial conditions $\hat{a}_1, \cdots, \hat{a}_n$ on the tangent vector $\hat{e}_1$,

$$\hat{a}_\ell = \hat{a}_{TW_1} + \epsilon\, e^{\ell\delta} \hat{e}_1\,, \quad \text{where} \quad \delta = 2\pi\mu^{(1)}/n\,\omega^{(1)}\,. \tag{17}$$

Here $\hat{a}_{TW_1}$ is the point of intersection of the TW_1 orbit with the slice hyperplane, n we set to 20, integration time we set to $\tau = 115$, ϵ is a small parameter that we set to 10^{-6}, and $\hat{e}_1 = \text{Re}\,\hat{V}_1/|\text{Re}\,\hat{V}_1|$. The unstable manifold of TW_1 is four-dimensional, with $\hat{V}_1$, $\hat{V}_2$ the expanding complex stability eigenvectors of TW_1 with eigenvalues $\lambda^{(j)} = \mu^{(j)} \pm i\,\omega^{(j)}$. Here we present the two-dimensional submanifold associated with the most expanding complex eigenvector $\hat{V}_1$. Fig. 2 shows the state space projections of the unstable manifold of TW_1, along with the $\tau_p = 33.5010$ relative periodic orbit and the relative equilibrium TW_2. The coordinate axes are projections (v_1, v_2, v_3) onto three orthonormal vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ constructed from Re $\hat{V}_1$, Im $\hat{V}_1$ and Re $\hat{V}_2$ via Gram-Schmidt orthogonalization. It is clear from fig. 2 (a) that without the symmetry reduction, the TW_1 unstable manifold is dominated by the drifts along its group orbit. In the symmetry reduced state space $\hat{\mathcal{M}}$, fig. 2 (b), the dynamically important, group-action transverse part of the unstable manifold of TW_1 is revealed. While the drifts along the symmetry direction complicate the relative periodic orbit in fig. 2 (a), the same orbit closes onto itself after one repeat within the slice hyperplane, fig. 2 (b). Likewise, TW_2, which is topologically a circle but appears convoluted in the projection of fig. 2(a), is reduced to a single equilibrium point. The stage is now set for a construction of symbolic dynamics for the flow by means of Poincaré sections and return maps [20].

The solutions of Kuramoto-Sivashinsky system are conventionally visualized in the configuration space, as

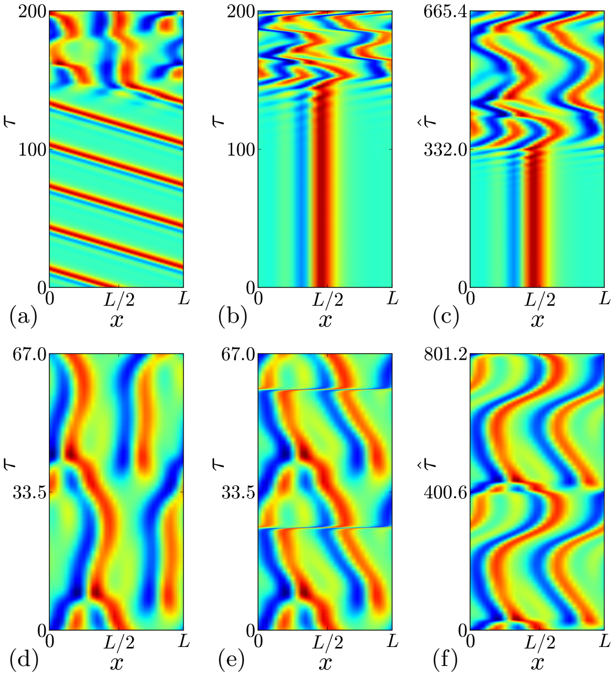

FIG. 3: (Color online) Traveling wave TW_1 with phase velocity $c = 0.737$ in configuration space: (a) the full state space solution, (b) symmetry-reduced solution with respect to the lab time, and (c) symmetry-reduced solution with respect to the in-slice time. Relative periodic orbit $\tau_p = 33.50$ in configuration space: (d) the full state space solution, (e) symmetry-reduced solution with respect to the lab time, and (f) symmetry-reduced solution with respect to the in-slice time.

time evolution of color-coded value of the function $u(x, t)$. Fig. 3 (b,e) illustrates that a relative equilibrium and a relative periodic orbit become an equilibrium and a periodic orbit after symmetry reduction. Fig. 3 (a,b,c) shows that a numerical trajectory eventually diverges from the unstable relative equilibrium and falls onto the strange attractor. The sharp shifts along x direction in fig. 3 (e) correspond to the time intervals where trajectory has a nearly vanishing first Fourier mode. Plotted as the function of the in-slice time $\hat{\tau}$ in fig. 3 (f), these rapid episodes are well resolved.

While the first Fourier mode slice resolves the reduced flow arbitrarily close to the $\hat{x}_1 = 0$ slice border, by sampling it with the in-slice time, this symmetry reduction scheme works only as long as the amplitude of the first mode is nonzero. For turbulent flows the first Fourier mode slice appears empirically valid for regions of dynamical interest; in all our numerical simulations of long-time ergodic trajectories of Kuramoto-Sivashinsky system (as well as of Navier-Stokes equations [4]) we have never encountered exactly vanishing first mode.

In summary, we recommend that the 'first Fourier mode slice', a very simple symmetry reduction prescription (3), easily implemented numerically, be used to reduce the U(1) or SO(2) symmetry of spatially extended systems, such as shear flows in periodic domains. For example, Avila *et al.* [21] have recently shown that localized relative periodic orbits have features strikingly similar to turbulent puffs. The first Fourier mode slice visualisations of the state space, such as fig. 2 (b), should help illuminate details of the role such solutions play in transition to turbulence.

We are indebted to Xiong Ding, Ashley P. Willis and Francesco Fedele for stimulating discussions, to Daniel Borrero-Echeverry and Hugues Chaté for a critical reading of the manuscript, and to the anonymous referee, whose suggestions had led to many improvements to this letter. P. C. thanks the family of late G. Robinson, Jr. and NSF DMS-1211827 for support. Matplotlib library [22] was used to produce the figures in this letter. The Matlab code used in our computations is available on a GitHub repository [23].

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[24] The state space becomes $(2m+1)$-dimensional if one includes the 0th Fourier mode $\tilde{u}_0$, which is an invariant of translations (2). By Galilean invariance of the Kuramoto-Sivashinsky equation, which we use for the demonstrations in this letter, $\tilde{u}_0$ is a conserved quantity. Conventionally the average $\int_0^L u\,dx$ is set equal to zero, i.e., $\tilde{u}_0 = 0$.