Effective Potentials and Morphological Transitions for Binary Black Hole Spin Precession

Michael Kesden, Davide Gerosa, Richard O'Shaughnessy, Emanuele Berti, and Ulrich Sperhake

numerically with time steps $t_{\mathrm{orb}} \ll \Delta t \lesssim t_{\mathrm{pre}}$, greatly reducing the computational cost of evolving spin directions for many orbital times. In this *Letter*, we show that the 2PN spin-precession equations [9] can be solved *analytically* in a suitably chosen frame for arbitrary mass ratios q and spins $\mathbf{S}_i$. The relative orientations of $\mathbf{L}$ and $\mathbf{S}_i$ are fully specified by the angles θ_i between $\mathbf{L}$ and $\mathbf{S}_i$ and the angle $\Delta\Phi$ between the spin components in the orbital plane; we provide parametric solutions for these angles in terms of S, the magnitude of the total spin $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$. These solutions improve our understanding of spin precession in much the same way that the solutions $r(f) = a(1 - e^2)/(1 + e\cos f)$ for Keplerian orbits provide additional insight beyond Newton's law $\ddot{\mathbf{r}} = -GM\hat{\mathbf{r}}/r^2$. We can use these solutions to *precession-average* the radiation-reaction equations for dE/dt and $d\mathbf{J}/dt$, allowing them to be numerically integrated with a time step $t_{\mathrm{pre}} \ll \Delta t' \lesssim t_{\mathrm{RR}}$. This greatly reduces the computational cost of evolving BBHs compared to the previous approach that integrated the orbit-averaged precession equations with the shorter time step $\Delta t \ll \Delta t'$. This improved efficiency is essential for transferring the BBH spins predicted at formation by population-synthesis models [10–18] to near merger, where spin directions affect gravitational waves (GWs) with frequencies in the sensitivity bands of current and future GW detectors [19–24]. Our new solutions may also facilitate the construction and interpretation of GW signals from BBHs in which both spins are misaligned. In particular, stellar-mass BBH spins depend on BH natal kicks and binary stellar evolution that can thus be constrained by ground-based GW detectors [13, 25]. Here-

Introduction. – The classic two-body problem was a major engine of historical progress in physics and astronomy. This problem can be solved analytically in Newtonian gravity; its solutions are the well known Keplerian orbits. The analogs to Newtonian point masses in general relativity are binary black holes (BBHs). Astrophysical BBHs have spins $\mathbf{S}_i$ [1] in addition to their masses m_i [the masses determine the total mass $M \equiv m_1 + m_2$, mass ratio $q \equiv m_2/m_1 \leq 1$, and symmetric mass ratio $\eta \equiv m_1 m_2/M^2 = q/(1 + q)^2$]. Full solutions to the two-body problem in general relativity must therefore include spin evolution in addition to orbital motion. Einstein's equations must be solved numerically [2–4] when the binary separation r is comparable to the gravitational radius $r_g \equiv GM/c^2$, but post-Newtonian (PN) approximations may be used when $r \gg r_g$. BBH evolution in the PN limit occurs on three distinct timescales: the orbital time $t_{\mathrm{orb}} \sim (r^3/GM)^{1/2}$ on which the binary separation $\mathbf{r}$ evolves, the precession time $t_{\mathrm{pre}} \sim c^2 r^{5/2}/[\eta(GM)^{3/2}] \sim (t_{\mathrm{orb}}/\eta)(r/r_g)$ on which the spin directions change, and the radiation-reaction time $t_{\mathrm{RR}} \sim E/|dE_{\mathrm{GW}}/dt| \sim c^5 r^4/[\eta(GM)^3] \sim (t_{\mathrm{orb}}/\eta)(r/r_g)^{5/2}$ on which the energy $E = -G\eta M^2/(2r)$ and orbital angular momentum $L = \eta(rGM^3)^{1/2}$ decrease.

The hierarchy $t_{\mathrm{orb}} \ll t_{\mathrm{pre}} \ll t_{\mathrm{RR}}$ implies that when considering evolution on one timescale, quantities evolving on a shorter (longer) timescale can be averaged (held constant). This has been used to derive orbit-averaged spin-precession equations $\dot{\mathbf{S}}_i = \bar{\mathbf{\Omega}}_i \times \mathbf{S}_i$ [5–8], where the precession frequencies $\bar{\mathbf{\Omega}}_i$ depend on the orbital angular momentum $\mathbf{L}$ and spins $\mathbf{S}_i$ but not on the instantaneous separation $\mathbf{r}$. These equations can be integrated

after we use geometrical units $G = c = 1$.

Precessional Solutions. – Consider the evolution of BBHs with misaligned spins on a circular orbit [26] on timescales $t_{pre} < t \ll t_{RR}$. We choose $\hat{\mathbf{z}}$ parallel to the total angular momentum $\mathbf{J}$, $\hat{\mathbf{x}}$ parallel to the component of the orbital angular momentum $\mathbf{L}$ perpendicular to $\mathbf{J}$, and $\hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$ to complete the orthonormal triad. Since $\mathbf{S} = \mathbf{J} - \mathbf{L}$, it too must lie in the xz–plane. Although $\mathbf{J}$ and thus $\hat{\mathbf{z}}$ are conserved on the precession time t_{pre}, $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ precess about $\hat{\mathbf{z}}$ with a frequency Ω_z. The angle θ_L between $\mathbf{J}$ and $\mathbf{L}$ is given by

$$\cos\theta_L = \frac{J^2 + L^2 - S^2}{2JL} \tag{1}$$

and depends exclusively on S and the constants L and J. We define a second orthonormal frame such that $\hat{\mathbf{z}}' = \hat{\mathbf{S}}$, $\hat{\mathbf{y}}' = \hat{\mathbf{y}}$, and $\hat{\mathbf{x}}' = \hat{\mathbf{y}}' \times \hat{\mathbf{z}}'$ completes the triad. $\mathbf{S}_1$ points in the direction (θ', φ') specified by traditional spherical coordinates in this second frame, where

$$\cos\theta' = \frac{S^2 + S_1^2 - S_2^2}{2SS_1} \tag{2}$$

also depends only on S, because S_1 and S_2 are conserved. Since $\mathbf{S}_2 = \mathbf{S} - \mathbf{S}_1$, the directions of all these angular momenta are specified in our initial (unprimed) frame by S and φ'.

The projected effective spin [27]

$$\xi \equiv M^{-2}[(1 + q)\mathbf{S}_1 + (1 + q^{-1})\mathbf{S}_2] \cdot \hat{\mathbf{L}} \tag{3}$$

is conserved by 2PN spin precession [9]. Radiation reaction (at 2.5PN) preserves the direction of $\mathbf{L}$ and thus ξ is further conserved on the radiation-reaction time t_{RR} as seen in our previous work [28]. Inserting expressions for $\mathbf{L}$ and $\mathbf{S}_i$ in terms of S and φ' into Eq. (3) yields

$$\xi(S, \varphi') = \{(J^2 - L^2 - S^2)[S^2(1+q)^2 - (S_1^2 - S_2^2)(1 - q^2)]$$
$$- (1 - q^2)A_1 A_2 A_3 A_4 \cos\varphi'\}/(4qM^2 S^2 L), \tag{4}$$

where

$$A_1 \equiv [J^2 - (L - S)^2]^{1/2}, \tag{5a}$$
$$A_2 \equiv [(L + S)^2 - J^2]^{1/2}, \tag{5b}$$
$$A_3 \equiv [S^2 - (S_1 - S_2)^2]^{1/2}, \tag{5c}$$
$$A_4 \equiv [(S_1 + S_2)^2 - S^2]^{1/2}. \tag{5d}$$

The A_i's are real in the allowed range $J_{min} \leq J \leq J_{max}$, $S_{min} \leq S \leq S_{max}$, where

$$J_{min} = L - S_1 - S_2, \tag{6a}$$
$$J_{max} = L + S_1 + S_2, \tag{6b}$$
$$S_{min} = \max\{|J - L|, |S_1 - S_2|\}, \tag{6c}$$
$$S_{max} = \min\{J + L, S_1 + S_2\}. \tag{6d}$$

Our approach needs to be modified for $L \leq S_1 + S_2$ ($J_{min} \leq 0$ above) [7, 29], but this does not occur until $r \leq r_{min} = [(1 + q^2)/q]^2 r_g$ for maximally spinning BBHs ($r_{min} = 4r_g$ for $q = 1$). Eq. (4) can be solved for $\cos\varphi'$ and then inserted into expressions for $\mathbf{L}$ and $\mathbf{S}_i$ to obtain the surprisingly simple relations

$$\cos\theta_1 = \frac{1}{2(1-q)S_1}\left[\frac{J^2 - L^2 - S^2}{L} - \frac{2qM^2\xi}{1+q}\right], \tag{7a}$$

$$\cos\theta_2 = \frac{q}{2(1-q)S_2}\left[-\frac{J^2 - L^2 - S^2}{L} + \frac{2M^2\xi}{1+q}\right], \tag{7b}$$

$$\cos\theta_{12} = \frac{S^2 - S_1^2 - S_2^2}{2S_1 S_2}, \tag{7c}$$

$$\cos\Delta\Phi = \frac{\cos\theta_{12} - \cos\theta_1 \cos\theta_2}{\sin\theta_1 \sin\theta_2} \tag{7d}$$

for $\cos\theta_i \equiv \hat{\mathbf{L}} \cdot \hat{\mathbf{S}}_i$, $\cos\theta_{12} \equiv \hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$, and the angle $\Delta\Phi$ between the spin components in the orbital plane. Note that S is the only variable in these expressions that evolves on t_{pre}.

The evolution of S is also surprisingly simple. If we set $\cos\varphi' = \mp 1$ in Eq. (4), we obtain two functions $\xi_\pm(S)$ that act like effective potentials for S. For given values of L and J, one of the A_i's vanishes at S_{min} and S_{max}, implying that $\xi_+(S_{min}) = \xi_-(S_{min})$, $\xi_+(S_{max}) = \xi_-(S_{max})$. Thus the two curves $\xi_\pm(S)$ form a closed loop in the $S\xi$–plane, as shown in Fig 1. The equation $\xi = \xi_\pm(S)$ has two roots $S_\pm(L, J, \xi)$ that determine the allowed range $S_- \leq S \leq S_+$. This is entirely analogous to how two roots $r_\pm(E, L)$ of the equation $E = V(r, L)$, where V is the effective potential for radial motion, determine pericenter and apocenter. The two roots are degenerate ($S_- = S_+$) at the maximum $\xi_{max}(L, J)$ of $\xi_+(S)$ and minimum $\xi_{min}(L, J)$ of $\xi_-(S)$, implying that $S = S_\pm$ remains constant – just as r remains constant for values of E and L corresponding to circular orbits, the minimum of the effective potential $V(r, L)$. These two configurations ($\xi = \xi_{min}$ and $\xi = \xi_{max}$) are precisely the $\Delta\Phi = 0°$ and $\Delta\Phi = \pm 180°$ spin-orbit resonances identified by Schnittman [30].

The BBH spins $\mathbf{S}_i$ and orbital angular momentum $\mathbf{L}$ are shown at $S = S_\pm$ for three different values of ξ but the same L and J in Fig. 2. These vectors are coplanar at $S_\pm$, since these points lie on the curves $\xi_\pm(S)$ defined such that $\cos\varphi' = \mp 1$; we must therefore have $\Delta\Phi = 0°$ or $180°$ at $S_\pm$. There are three possibilities as S increases from S_- to S_+ and returns to S_-: (1) $\Delta\Phi$ begins at $0°$, decreases to a minimum $-\Delta\Phi_*$, returns to $0°$ at S_+, increases to a maximum $+\Delta\Phi_*$, then returns to $0°$ back at S_-, (2) $\Delta\Phi$ begins at $-180°$, increases to $0°$ at S_+, then continues to increase to $+180°$ back at S_-, and (3) $\Delta\Phi$ begins at $180°$, increases to a maximum $180° + \Delta\Phi_*$, returns to $180°$ at S_+, decreases to a minimum $180° - \Delta\Phi_*$, then returns to $180°$ back at S_-. These three possibilities (libration about $\Delta\Phi = 0°$, circulation, and libration
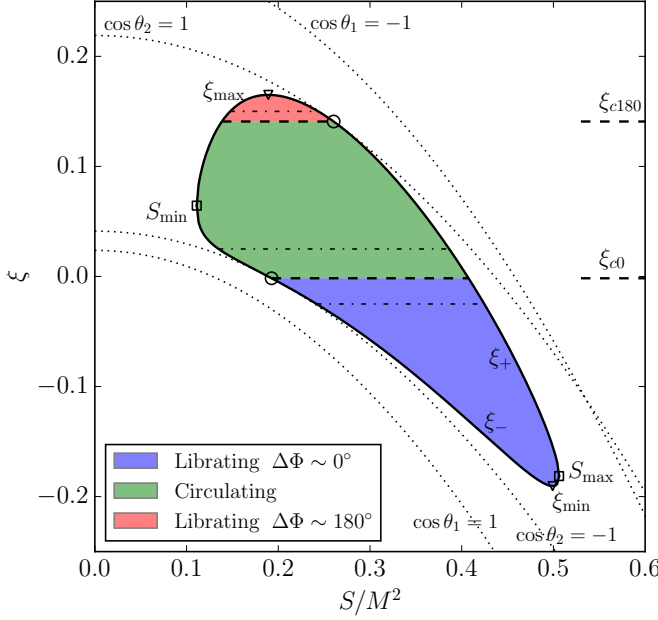
FIG. 1. Effective potentials $\xi_\pm(S)$ for the spin precession of BBHs with maximal spins, mass ratio $q = 0.8$, $L = 0.781M^2$ ($r = 10M$), and $J = 0.85M^2$. These two functions form a loop enclosing the allowed values of S and ξ. Since ξ is conserved during the inspiral, S oscillates between the two roots $S_\pm$ of the equation $\xi = \xi_\pm(S)$ on the precession time. The two roots are degenerate at $\xi_{\min}$ and $\xi_{\max}$, implying that S is constant: these configurations correspond respectively to the $\Delta\Phi = 0°(\Delta\Phi = \pm180°)$ spin-orbit resonances of Schnittman [30]. The four dotted curves are the contours $\cos\theta_i = \pm1$ given by Eqs. (7a) and (7b); transitions between BBHs for which $\Delta\Phi$ circulates and those for which it librates about $0°$ $(\pm180°)$ occur where these curves are tangent to the potentials $\xi_\pm(S)$, as indicated by the lower (upper) dashed line $\xi = \xi_{c0}$ (ξ_{c180}). The three dot-dashed lines correspond to the three BBH systems shown in Fig. 2 as representative of each morphology.

about $\Delta\Phi = 180°$) are shown in the left, center, and right panels of Fig. 2. The libration amplitude $\Delta\Phi_*$ depends on L, J, and ξ.

Eq. (7) implies that BBHs with $\xi = \xi_{\min}$ are trapped in the $\Delta\Phi = 0°$ resonance. Comparing the left and center panels of Fig. 2, we see that the transition between BBHs with $\Delta\Phi = 0°$ and those with $\Delta\Phi = \pm180°$ at S_- [(1) → (2) above] occurs at the value $\xi \equiv \xi_{c0}$ at which $\mathbf{L}$ is aligned with either $\mathbf{S}_1$ or $-\mathbf{S}_2$ at S_-. This transition is marked by the lower dashed line separating the blue and green regions in Fig. 1. As ξ increases further, we see by comparing the center and right panels of Fig. 2 that we eventually reach a value $\xi \equiv \xi_{c180}$ at which $\Delta\Phi$ transitions from $0°$ to $180°$ at S_+ [(2) → (3) above]. This transition is marked by the upper dashed line separating the green and red regions in Fig. 1. These morphological transitions correspond to the quasistable equilibria noted by Schnittman [30]. Finally, as ξ continues to increase

the amplitude of the oscillations in S decreases, until the $\Delta\Phi = \pm180°$ resonance is reached at $\xi_{\max}$.

Although S parameterizes spin directions much like the true anomaly parameterizes Keplerian orbits, one may also want the time-dependent solutions $S(t)$. The spin-precession equations [8, 9, 31, 32] imply

$$\frac{dS}{dt} = -\frac{3(1-q^2)}{2q}\frac{S_1 S_2}{S}\frac{(\eta^2 M^3)^3}{L^5}\left(1 - \frac{\eta M^2\xi}{L}\right)$$
$$\times \sin\theta_1 \sin\theta_2 \sin\Delta\Phi \qquad (8)$$

where again the right-hand side depends only on S when we use Eq. (7). Oscillations in S have a precessional period $\tau(L, J, \xi) = 2\int_{S_-}^{S_+} dS/|dS/dt|$. The basis vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ precess about $\hat{\mathbf{z}}$ at a rate

$$\Omega_z = \frac{J}{2}\left(\frac{\eta^2 M^3}{L^2}\right)^3\left\{1 + \frac{3}{2\eta}\left(1 - \frac{\eta M^2\xi}{L}\right)\right.$$
$$- \frac{3(1+q)}{2qA_1^2 A_2^2}\left(1 - \frac{\eta M^2\xi}{L}\right)[4(1-q)L^2(S_1^2 - S_2^2)$$
$$\left. - (1+q)(J^2 - L^2 - S^2)(J^2 - L^2 - S^2 - 4\eta M^2 L\xi)]\right\},$$
$$(9)$$

implying that they precess through an angle $\alpha(L, J, \xi) = 2\int_{S_-}^{S_+}(\Omega_z\, dS)/|dS/dt|$ in each precessional period.

Gravitational Inspiral. – Although L and J are conserved on t_{pre}, they vary on the longer radiation-reaction timescale t_{RR}. At lowest PN order, the orbit-averaged angular momentum flux is given by the well known quadrupole formula [26] $d\mathbf{J}/dt = -(32/5)(\eta M^2/L)^8(\eta\mathbf{L}/M)$, implying $dL/dt = \hat{\mathbf{L}}\cdot d\mathbf{J}/dt$ and $dJ/dt = \hat{\mathbf{J}}\cdot d\mathbf{J}/dt$. This expression for dL/dt is independent of S, but that for dJ/dt is not. However, if the above precession angle $\alpha \neq 2\pi n$ for integer n, the average of $d\mathbf{J}/dt$ over many precession periods will be parallel to $\mathbf{J}$. Using the monotonically decreasing L to parameterize the inspiral we obtain the *precession-averaged* result

$$\left\langle\frac{dJ}{dL}\right\rangle_{\mathrm{pre}} = \frac{2}{\tau}\int_{S_-}^{S_+}\frac{\cos\theta_L\, dS}{dS/dt}$$
$$= \frac{1}{2LJ}\left[J^2 + L^2 - \frac{2}{\tau}\int_{S_-}^{S_+}\frac{S^2\, dS}{dS/dt}\right], \quad (10)$$

that is indeed independent of S. At higher PN order, $d\mathbf{J}/dt$ is spin-dependent and thus $\langle dJ/dL\rangle_{\mathrm{pre}}$ is not simply a time-weighted average of $\cos\theta_L$ [8]. This equation allows J to be evolved numerically with a time step $t_{\mathrm{pre}} \ll \Delta t' \lesssim t_{\mathrm{RR}}$ consistent with the timescale on which it is varying. Unless we need to keep track of the precessional phase, we can use this precession-averaged equation combined with the orbit-averaged solutions of the previous section to evolve BBH spin directions far more efficiently than the conventional approach relying exclusively on the orbit-averaged equations $\dot{\mathbf{S}}_i = \bar{\boldsymbol{\Omega}}_i\times\mathbf{S}_i$. Preliminary results [29] indicate that as L and J evolve,
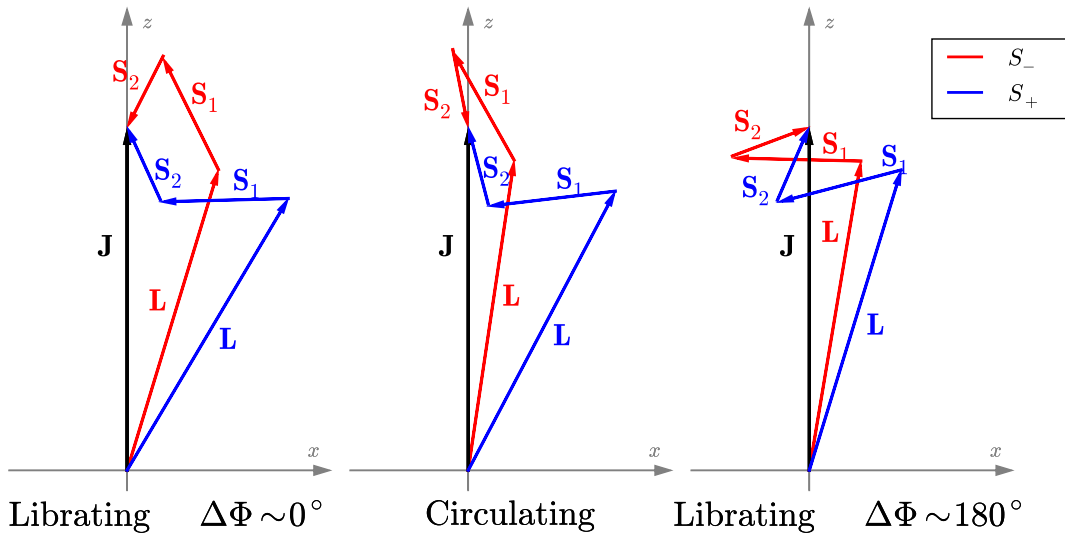
FIG. 2. The three morphologies of BBH spin precession. The angular momenta $\mathbf{J}$, $\mathbf{L}$, and $\mathbf{S}_i$ are all in the xz–plane at $S = S_\pm$. In all three panels the BBHs have maximal spins, $q = 0.8$, $L = 0.781M^2$ ($r = 10M$), and $J = 0.85M^2$ as in Fig. 1. The left, middle, and right panels correspond to $\xi = -0.025$, 0.025 and 0.15, respectively. If the components of $\mathbf{S}_i$ perpendicular to $\mathbf{L}$ are aligned with each other at both roots $S_\pm$, $\Delta\Phi$ librates about $0°$. If they are aligned at one root and anti-aligned at the other, $\Delta\Phi$ circulates. If they are anti-aligned at both roots, $\Delta\Phi$ librates about $180°$.

circulating BBHs $[\xi_{c0}(L, J) \leq \xi \leq \xi_{c180}(L, J)]$ can be captured into one of the two librating morphologies $[\xi < \xi_{c0}(L, J)$ or $\xi > \xi_{c180}(L, J)]$ consistent with earlier studies [30].

The condition $\alpha(L, J, \xi) = 2\pi n$ corresponds to a newly identified resonance between precession about $\mathbf{J}$ and precession in the meridional plane (the xz–plane in our basis). The direction of $\mathbf{J}$ evolves rapidly at these resonances, since $\langle d\mathbf{J}/dt \rangle_{\mathrm{pre}} \nparallel \mathbf{J}$. This could affect the direction of the spin of the final black hole, which is often assumed to point in the direction of $\mathbf{J}$ at merger [28, 33, 34], and could also leave an observational signature if a resonance occurs within the sensitivity band of GW detectors. Preliminary results [35] suggest that generic BBHs often pass through these resonances as they inspiral.

Discussion. – We have derived new analytic solutions for BBH spin precession by recognizing that L, J, and ξ remain constant on the precession time t_{pre}. These solutions provide new insights into this deeply fundamental problem in general relativity and allow us to *precession-average* the evolution equations for L and J on the radiation-reaction time t_{RR}. These precession-averaged equations give us the ability to efficiently evolve BBH spin directions from formation to near merger, which is essential to the study of both stellar-mass and supermassive BBHs. Our previous work [13, 25] revealed that initial spin directions imprinted by the astrophysics of BBH formation leave detectable GW signatures. The new solutions derived in this *Letter* will greatly expand our capability to explore such formation models. Supermassive BBH spins will also precess many times before

merger [36]; these solutions will help us predict final-spin distributions for different models of supermassive black-hole growth [14–18] as well as final-kick distributions that depend sensitively on BBH spin directions at merger [37–41]. Finally, our new solutions may help in the construction and interpretation of GWs from generic double-spin binaries, a timely development given the likely first direct detection of GWs later this decade.

* kesden@utdallas.edu
† d.gerosa@damtp.cam.ac.uk
‡ rossma@rit.edu
§ eberti@olemiss.edu

¶ U.Sperhake@damtp.cam.ac.uk

[1] R. P. Kerr, Physical Review Letters **11**, 237 (1963).

[2] F. Pretorius, Physical Review Letters **95**, 121101 (2005), gr-qc/0507014.

[3] M. Campanelli, C. O. Lousto, P. Marronetti, and Y. Zlochower, Physical Review Letters **96**, 111101 (2006), gr-qc/0511048.

[4] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, and J. van Meter, Physical Review Letters **96**, 111102 (2006), gr-qc/0511103.

[5] B. M. Barker and R. F. Oconnell, General Relativity and Gravitation **11**, 149 (1979).

[6] K. S. Thorne and J. B. Hartle, Phys. Rev. D **31**, 1815 (1985).

[7] T. A. Apostolatos, C. Cutler, G. J. Sussman, and K. S. Thorne, Phys. Rev. D **49**, 6274 (1994).

[8] L. E. Kidder, Phys. Rev. D **52**, 821 (1995), gr-qc/9506022.

[9] É. Racine, Phys. Rev. D **78**, 044021 (2008), arXiv:0803.1820 [gr-qc].

[10] R. O'Shaughnessy, J. Kaplan, V. Kalogera, and K. Belczynski, ApJ **632**, 1035 (2005), astro-ph/0503219.

[11] K. Belczynski, R. E. Taam, E. Rantsiou, and M. van der Sluys, ApJ **682**, 474 (2008), astro-ph/0703131.

[12] T. Fragos, M. Tremmel, E. Rantsiou, and K. Belczynski, ApJ **719**, L79 (2010), arXiv:1001.1107 [astro-ph.HE].

[13] D. Gerosa, M. Kesden, E. Berti, R. O'Shaughnessy, and U. Sperhake, Phys. Rev. D **87**, 104028 (2013), arXiv:1302.4442 [gr-qc].

[14] S. A. Hughes and R. D. Blandford, ApJ **585**, L101 (2003), astro-ph/0208484.

[15] E. Berti and M. Volonteri, ApJ **684**, 822 (2008), arXiv:0802.0025.

[16] E. Barausse, MNRAS **423**, 2533 (2012), arXiv:1201.5888.

[17] A. Sesana, E. Barausse, M. Dotti, and E. M. Rossi, ApJ **794**, 104 (2014), arXiv:1402.7088.

[18] D. Sijacki, M. Vogelsberger, S. Genel, V. Springel, P. Torrey, *et al.*, (2014), arXiv:1408.6842 [astro-ph].

[19] G. M. Harry and LIGO Scientific Collaboration, Classical and Quantum Gravity **27**, 084006 (2010).

[20] C. S. Unnikrishnan, International Journal of Modern Physics D **22**, 1341010 (2013).

[21] K. Somiya, Classical and Quantum Gravity **29**, 124007 (2012), arXiv:1111.7185 [gr-qc].

[22] M. Punturo, M. Abernathy, F. Acernese, B. Allen, N. Andersson, K. Arun, F. Barone, B. Barr, M. Barsuglia, Beker, *et al.*, Classical and Quantum Gravity **27**, 194002 (2010).

[23] P. Amaro-Seoane *et al.*, GW Notes, Vol. 6, p. 4-110 **6**, 4 (2013), arXiv:1201.3621 [astro-ph.CO].

[24] E. Barausse, J. Bellovary, E. Berti, K. Holley-Bockelmann, B. Farris, B. Sathyaprakash, and A. Sesana, ArXiv e-prints (2014), arXiv:1410.2907 [astro-ph.HE].

[25] D. Gerosa, R. O'Shaughnessy, M. Kesden, E. Berti, and U. Sperhake, Phys. Rev. D **89**, 124025 (2014), arXiv:1403.7147 [gr-qc].

[26] P. C. Peters and J. Mathews, Physical Review **131**, 435 (1963).

[27] T. Damour, Phys. Rev. D **64**, 124013 (2001), gr-qc/0103018.

[28] M. Kesden, U. Sperhake, and E. Berti, Phys. Rev. D **81**, 084054 (2010), arXiv:1002.2643 [astro-ph.GA].

[29] D. Gerosa, M. Kesden, E. Berti, R. O'Shaughnessy, and U. Sperhake, (2014), in preparation.

[30] J. D. Schnittman, Phys. Rev. D **70**, 124020 (2004), astro-ph/0409174.

[31] G. Faye, L. Blanchet, and A. Buonanno, Phys. Rev. D **74**, 104033 (2006), gr-qc/0605139.

[32] L. Blanchet, A. Buonanno, and G. Faye, Phys. Rev. D **74**, 104034 (2006), gr-qc/0605140.

[33] A. Buonanno, L. E. Kidder, and L. Lehner, Phys. Rev. D **77**, 026004 (2008), arXiv:0709.3839.

[34] E. Barausse and L. Rezzolla, ApJ **704**, L40 (2009), arXiv:0904.2577 [gr-qc].

[35] X. Zhao and M. Kesden, (2014), in preparation.

[36] M. C. Begelman, R. D. Blandford, and M. J. Rees, Nature **287**, 307 (1980).

[37] J. A. González, M. Hannam, U. Sperhake, B. Brügmann, and S. Husa, Physical Review Letters **98**, 231101 (2007), gr-qc/0702052.

[38] M. Campanelli, C. O. Lousto, Y. Zlochower, and D. Merritt, Physical Review Letters **98**, 231102 (2007), gr-qc/0702133.

[39] M. Campanelli, C. Lousto, Y. Zlochower, and D. Merritt, ApJ **659**, L5 (2007), gr-qc/0701164.

[40] M. Kesden, U. Sperhake, and E. Berti, ApJ **715**, 1006 (2010), arXiv:1003.4993 [astro-ph.CO].

[41] E. Berti, M. Kesden, and U. Sperhake, Phys. Rev. D **85**, 124049 (2012), arXiv:1203.2920 [astro-ph.HE].

[42] J. D. Hunter, Computing in Science and Engineering **9**, 90 (2007).