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Bending of Light in Quantum Gravity and the Equivalence Principle

N. E. J. Bjerrum-Bohr,[1,*] John F. Donoghue,[2,†] Barry R. Holstein,[2,‡] Ludovic Planté,[3,§] and Pierre Vanhove[3,4,¶]

[1] *Niels Bohr International Academy and Discovery Center, The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*

[2] *Department of Physics-LGRT, University of Massachusetts, Amherst, MA 01003 USA*

[3] *CEA, DSM, Institut de Physique Théorique, IPhT, CNRS, MPPU, URA2306, Saclay, F-91191 Gif-sur-Yvette, France*

[4] *Institut des Hautes Études Scientifiques Bures sur Yvette, F-91440, France*

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We consider the scattering of light-like matter in the presence of a heavy scalar object (such as the sun or a Schwarzschild black hole). By treating general relativity as an effective field theory we directly compute the non-analytic components of the one-loop gravitational amplitude for the scattering of massless scalars or photons from an external massive scalar field. These results allow a semi-classical computation of the bending angle for light-rays grazing the sun, including long-range $\hbar$ contributions. We discuss implications of this computation, in particular the violation of some classical formulations of the equivalence principle.

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Since the discovery of quantum mechanics and general relativity in the previous century it has been clear that these two theories have completely different notions of reality at a fundamental level. While deterministic physics is a crucial ingredient in general relativity, *i.e.*, particles follow field equations formulated as geodesic equations, in quantum mechanics such a concept has no meaning since one has to accept that space and momentum are mutually complementary concepts. The notion of a quantum field theory offers a middle ground to some extent by combining these concepts through field variables, but the traditional formulation of such a theory suffers from (non-renormalizable) divergences in the ultraviolet regime. Whatever the high energy theory of gravity turns out to be, it is intriguing that we can already answer a number of important questions simply by employing an effective field theory framework for general relativity, wherein the basic building block is the Einstein-Hilbert Lagrangian. In order to absorb ultraviolet divergences we include in the action all possible invariants allowed by the basic symmetries of the theory. This infinite set of corrections is usually seen as a signal of the loss of predictability and as a dependence on the high energy completion of the theory. However, at one-loop order something surprising happens that was first noticed by [1] and was exploited in [2, 3]—the basic Einstein-Hilbert term is sufficient to extract the long-range behavior of the theory. This feature was used to extract the quantum corrections to the Newtonian potential of a small mass attracted by a larger mass—

$$V(r) = -\frac{GMm}{r}\left(1 + \frac{3G(M+m)}{c^2 r} + \frac{41G\hbar}{10c^3 r^2}\right).$$

Here M is a large (scalar) object, say the sun, m is a small test-mass, r is the distance between the two objects, and G, c and $\hbar$, are Newton's constant, the speed of light and the Planck constant respectively. Since these initial computations there have appeared a number of papers computing various potentials [4], involving *e.g.*, fermionic and spin one matter. It has been explicitly demonstrated that the spin-independent components of one loop general relativity theory display universality both for the classical contribution as well as for the one-loop quantum correction [3, 4].

We will in this letter focus on a different problem, which has not yet been discussed in the literature, namely computing the leading quantum correction to the gravitational bending of light around the sun [5]. Our goal is to show that this quantity is readily calculable using modern field theory techniques. In doing so we find that the quantum corrections do not respect classical formulations of the equivalence principle. While the net effect is far too small to be seen experimentally, this quantum violation of the equivalence principle is an interesting phenomenon in its own right.

This letter is organized as follows. First we briefly review how to treat general relativity as an effective field theory coupled to photons and to light-like (massless) scalar matter. We work out amplitudes for the gravitational scattering of the photons as well as of massless scalar matter as a reference. As we will demonstrate, even at the quantum level of general relativity the universality of the couplings to energy-momentum holds largely unchanged. We show how our computation of the cross-section can be used to deduce a semi-classical deflection angle in which the post-Newtonian general relativistic corrections are reproduced and new quantum mechanical corrections are generated. Finally, we conclude and summarize our results.

We begin by considering the Einstein-Hilbert Lagrangian coupled to QED and two neutral scalar fields

$$\mathcal{S} = \int d^4x\,\sqrt{-g}\left[\left(\frac{2}{\kappa^2}\,\mathcal{R} - \frac{1}{4}(\nabla_\mu A_\nu - \nabla_\nu A_\mu)^2\right)\right.$$

$$+ \left(-\frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}((\partial_\mu\phi)^2 - M^2\phi^2)\right) + S_{\rm EF}\Bigg], \quad (1)$$

where the covariant derivative is given by $\nabla_\mu A^\nu := \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\lambda} A^\lambda$ (we will be using the Feynman gauge) where $\Gamma^\lambda{}_{\mu\nu} = 1/2\, g^{\lambda\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$ are the Christoffel symbols and $\kappa^2 = 32\pi G_N/c^4$. The fields are denoted in the following way: gravitons—h, photons—γ, massless scalars—φ and massive scalars—ϕ.

In order to utilize this theory consistently, it is important to consider it as an effective field theory[1], by inclusion of a string of higher-order operators in the action. Divergences, being local, are absorbed into the coefficients of these local higher order operators. However, the long-range contributions correspond to non-analytic terms in momentum space or equivalently non-local behavior in coordinate space. These contributions are ultraviolet finite and follow uniquely from the vertices of $S_{\rm EF}$. For the purposes of evaluating only the longest range contributions, we need not display these higher order terms in the action.

The calculation is greatly simplified by two remarkable facts. One is that the on-shell gravitational tree-level amplitudes can be obtained as the square of gauge theory amplitudes [6, 7]. In our case the gravitational Compton amplitudes will be reduced to the product of QED Compton amplitudes [3, 8, 9]. The difficult calculations involving the triple graviton vertex can be avoided and are replaced by the much simpler QED vertices.

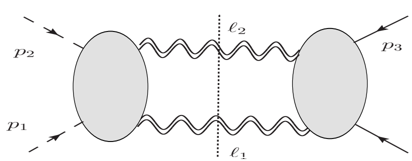

FIG. 1: The two gravitons cut for the amplitude between a massless particle (dashed line) and the massive scalar (solid line). The grey blob are tree-level gravitational Compton amplitudes.

The other great simplification is to use on-shell unitarity techniques [10], instead of Feynman diagrams. Calculating gravitational Feynman loops is a long and tedious process using the vertex rules of gravitational Lagrangian. Unitarity-based calculations construct the relevant amplitude from the discontinuity of the scattering process. The long range non-analytic terms in the one-loop amplitude can be readily calculated from these on-shell cuts using the property of unitarity, as was directly demonstrated in ref. [3]. Cutting the graviton internal lines (see figure 1), the integrand of the one-loop amplitude factorizes in terms of a product of relatively simple tree amplitudes, given in our case by the gravitational Compton amplitudes.

The corresponding cut graviton exchange amplitude takes the form

$$i\overset{1}{\mathcal{M}}{}^{[\eta(p_1)\eta(p_2)]}_{[\phi(p_3)\phi(p_4)]}\Big|_{\rm disc} = \int \frac{d^D\ell}{(2\pi)^4} \frac{\sum_{h_1,h_2} \overset{0}{\mathcal{M}}{}^{[h^{h_1}(\ell_1)h^{h_2}(-\ell_2)]}_{[\eta(p_1)\eta(p_2)]} \; \overset{0}{\mathcal{M}}{}^{[h^{h_1}(-\ell_1)h^{h_2}(\ell_2)]\,\star}_{[\phi(p_3)\phi(p_4)]}}{4\ell_1^2\ell_2^2}, \quad (2)$$

with the on-shell conditions $\ell_1^2 = \ell_2^2 = 0$ for the cut momenta of the internal graviton lines. For the photon case $\eta = \gamma$ and for the massless scalar case $\eta = \varphi$. Here $D = 4 - 2\epsilon$ and the $\star$ denotes conjugation. Our notation here is $\overset{0}{\mathcal{M}}$ for tree-level gravitational Compton amplitudes and $\overset{1}{\mathcal{M}}$ for one-loop amplitude of figure 1. We follow the notation and momentum conventions of [3] with all momenta defined as incoming and set $q := p_1 + p_2 = -p_3 - p_4 = \ell_2 - \ell_1$ with $t := q^2 = (p_1+p_2)^2$ and $p_3^2 = p_4^2 = M^2$. In the "all-incoming" convention t corresponds to the momentum transfer of a scattering process.

The relation between the gravitational and electrodynamic Compton processes is given by

$$i\overset{0}{\mathcal{M}}{}^{[h(k_1)h(k_2)]}_{[\eta(p_1)\eta(p_2)]} = \frac{\kappa^2}{4e^2}\, \frac{(p_1\cdot k_1)(p_1\cdot k_2)}{p_1\cdot p_2}\, \overset{0}{\mathcal{M}}{}^{\rm QED}_{S=0}\, \overset{0}{\mathcal{M}}{}^{\rm QED}_{\eta}\ , \quad (3)$$

and is derived in detail in [9]. Here $\mathcal{M}^{\rm QED}_\gamma = \mathcal{M}^{\rm QED}_{S=1}$ utilizes the Compton amplitude for the scattering of a photon from a massless charged spin-1 target while $\mathcal{M}^{\rm QED}_\varphi = \mathcal{M}^{\rm QED}_{S=0}$ employs the Compton amplitude of a photon from a massless charged spin-0 target. These tree-level relations connect one-loop gravitational physics with one-loop electrodynamics in a non-trivial and interesting way [3].

A final simplification is the use of the spinor-helicity formalism (see [11] for a review). While this notation is perhaps less familiar to some, it drastically reduces the form of the amplitudes which we now display. The only non-vanishing gravitational Compton helicity amplitudes involving photons γ and gravitons h are

$$i\overset{0}{\mathcal{M}}{}^{[h^+(k_1)h^-(k_2)]}_{[\gamma^+(p_1)\gamma^-(p_2)]} = \frac{\kappa^2}{4}\, \frac{[p_1\,k_1]^2\, \langle p_2\,k_2\rangle^2\, \langle k_2|p_1|k_1]^2}{(p_1\cdot p_2)(p_1\cdot k_1)(p_1\cdot k_2)}, \quad (4)$$

with $\overset{0}{\mathcal{M}}{}^{[h^+(k_1)h^-(k_2)]}_{[\gamma^-(p_1)\gamma^+(p_2)]}$ given by the above formula with p_1 and p_2 interchanged, and amplitudes with opposite helicity configurations are obtained by complex conjugation. For the tree-level massive scalar-graviton interaction amplitude we have

$$\begin{aligned} i\overset{0}{\mathcal{M}}{}^{[h^+(k_1)h^+(k_2)]}_{[\phi(p_1)\phi(p_2)]} &= \frac{\kappa^2}{4}\, \frac{M^4\,[k_1\,k_2]^4}{(k_1\cdot k_2)(k_1\cdot p_1)(k_1\cdot p_2)}, \\ i\overset{0}{\mathcal{M}}{}^{[h^-(k_1)h^+(k_2)]}_{[\phi(p_1)\phi(p_2)]} &= \frac{\kappa^2}{4}\, \frac{\langle k_1|p_1|k_2]^2\, \langle k_1|p_2|k_2]^2}{(k_1\cdot k_2)(k_1\cdot p_1)(k_1\cdot p_2)}. \end{aligned} \quad (5)$$

The tree-level amplitudes between the massless scalar φ and the graviton are obtained by setting $M = 0$. Amplitudes with opposite helicity configurations are obtained by complex conjugation.

The discontinuity integral of (2) is given by the sum of four box integrals with the same numerator factor

$$i\overset{1}{\mathcal{M}}{}^{[\eta(p_1)\eta(p_2)]}_{[\phi(p_3)\phi(p_4)]}\Big|_{\rm disc} = -\frac{\kappa^4}{4t^4}\sum_{h_1,h_2}\sum_{i=1}^{2}\sum_{j=3}^{4}\int\frac{d^D\ell}{(2\pi)^4}\frac{\mathcal{N}^{h_1\,h_2}}{\ell_1^2\ell_2^2(p_i\cdot\ell_1)(p_j\cdot\ell_1)}\,, \tag{6}$$

where h_1 and h_2 denote the helicities $(+/-)$ of the exchanged gravitons in the cut. With this construction one captures all the t-channel massless thresholds, which are the only terms of interest to us. The cut is evaluated as in [3], resulting in a very simple answer due to the dramatic simplification of the gravitational Compton tree-level amplitudes in (3): the singlet cut with $h_1 = h_2 = +$ or $h_1 = h_2 = -$ vanishes and the non-singlet cut is given by

$$\mathcal{N}^{+-} + \mathcal{N}^{-+} = \Re\mathrm{e}\left[\left(\mathrm{tr}_-(\ell_1 \not{p}_1 \ell_2 \not{p}_3)\right)^4\right]. \tag{7}$$

for the massless scalar-massive scalar amplitude and

$$\mathcal{N}^{+-}+\mathcal{N}^{-+} = \Re\mathrm{e}\left[\frac{\left(\mathrm{tr}_-(\ell_2\not{p}_2\ell_1\not{p}_3)\mathrm{tr}_+(\ell_2\not{p}_3\ell_1\not{p}_1\not{p}_3\not{p}_2)\right)^2}{\langle p_1|p_3|p_2]^2}\right], \tag{8}$$

for the photon-massive scalar amplitude where $\mathrm{tr}_\pm(\cdots) := \mathrm{tr}(\frac{1\pm\gamma_5}{2}\cdots)$. Performing standard tensor integral reductions [12] into scalar boxes, scalar bubbles and scalar triangle integrals, the amplitude is decomposed in terms of integral functions with a massless t-channel cut

$$\begin{aligned}\frac{i}{4\kappa^4}\overset{1}{\mathcal{M}}{}^{[\eta(p_1)\eta(p_2)]}_{[\phi(p_3)\phi(p_4)]}\Big|_{\rm disc} &= bo^\eta(t,u)\, I_4(t,u) + bo^\eta(t,s)\, I_4(t,s)\\ &+\; t^\eta_{12}(t)\, I_3(p_1,p_2,0) + t^\eta_{34}(t)\, I_3(p_3,p_4,M^2)\\ &+\; bu^\eta(t)\, I_2(t,0)\,.\end{aligned} \tag{9}$$

Here $I_4(t,u)$ and $I_4(t,s)$ are the scalar box integrals given in §4.4.6 of [13], $I_3(t)$ is the massless triangle integral with vanishing internal masses, $I_3(t,m)$ the finite massive triangle integral and $I_2(t)$ is the massless scalar bubble integral both given in Eq. (III.17) of [3]. [In the massless ($M \to 0$) limit this computation reproduces the graviton cut given by Dunbar and Norridge [14, eq. (4.10)].]

The integral reduction yields as well massive bubbles, tadpoles and analytic pieces that do not possess a massless t-channel cut. Such pieces are not completely determined from the cut and are not of interest to our analysis since they do not contribute to the long range interactions at low-energy.

Computation of the cut discontinuity can be accomplished using traditional methods and is greatly simplified by the use of on-shell identities. We will elsewhere present the details of these computations and here quote only the leading (non-analytic) results required to perform the analysis of the cross-section and the semi-classical bending angle.

In the leading low-energy ($\omega \ll M$) limit, where ω is the frequency of the photon, the total amplitude sum of the tree-level and one-loop contributions $i\mathcal{M} = \frac{i}{\hbar}\overset{0}{\mathcal{M}} + i\overset{1}{\mathcal{M}}$ take the very striking form

$$\begin{aligned}i\mathcal{M}^{[\eta(p_1)\eta(p_2)]}_{[\phi(p_3)\phi(p_4)]} &= \frac{\mathcal{N}^\eta}{\hbar}\Big[\kappa^2\frac{(2M\omega)^2}{4t}\\ &+\; \hbar\frac{\kappa^4}{4}\Big(4(M\omega)^4(I_4(t,u)+I_4(t,s)) + 3(M\omega)^2 t I_3(t)\\ &-\; 15(M^2\omega)^2 I_3(t,M) + bu^\eta(M\omega)^2 I_2(t)\Big)\Big]\,,\end{aligned} \tag{10}$$

where $\mathcal{N}^\varphi = 1$ for the massless scalar, while for the photon, we have $\mathcal{N}^\gamma = (2M\omega)^2/(2\langle p_1|p_3|p_2]^2)$ for the $(+-)$ photon helicity contribution and its complex conjugate for the $(-+)$ photon helicity contribution. The photon amplitude vanishes for the polarization configurations $(++)$ and $(--)$ is a direct consequence of the properties of the tree-amplitudes in eq. (4) . Notice that $|\mathcal{N}^\gamma|^2 \to 1$ in the low-energy limit and that this prefactor does not affect the cross-section. The coefficients of the bubble contributions are $bu^\varphi = 3/40$ and $bu^\gamma = -161/120$.

It is a striking example of the universality of the gravitational couplings that all coefficients–except those for the bubble–are identical (for the leading contribution) for the scalar and photon scattering.

From this result we can compute the leading contribution to the amplitude (expanding all integrals in terms of leading order contributions as done in [2, 3]):

$$\begin{aligned}i\mathcal{M}^{[\eta(p_1)\eta(p_2)]}_{[\phi(p_3)\phi(p_4)]} &\simeq \frac{\mathcal{N}^\eta}{\hbar}(M\omega)^2\\ &\times\Big[\frac{\kappa^2}{t} + \kappa^4\frac{15}{512}\frac{M}{\sqrt{-t}}\\ &+\hbar\kappa^4\frac{15}{512\pi^2}\log\left(\frac{-t}{M^2}\right) - \hbar\kappa^4\frac{bu^\eta}{(8\pi)^2}\log\left(\frac{-t}{\mu^2}\right)\\ &+\hbar\kappa^4\frac{3}{128\pi^2}\log^2\left(\frac{-t}{\mu^2}\right) + \kappa^4\frac{M\omega}{8\pi}\frac{i}{t}\log\left(\frac{-t}{M^2}\right)\Big].\end{aligned} \tag{11}$$

Where μ^2 is an arbitrary mass scale parameter used in dimensional regularization. The two terms in the second line correspond, respectively, to the leading Newtonian contribution and first post-Newtonian correction [1–3]. The next three logarithmic terms represent quantum gravity corrections. The first term on the third line corresponds to the quantum correction to the metric evaluated in [15]. The second term on the third line arises from the one-loop ultraviolet divergence of the amplitude and is the only contribution depending on the spin of the massless field. On the fourth line the first term, involves a new form not found in the previous analysis. Finally, the last term, arising from the discontinuity of the box integral, contributes to the phase of the amplitude and is not directly observable. For this reason it will not be considered further.

It is very interesting that in the low-energy limit the one-loop amplitudes for the massless scalar and for the photon involve the same coefficients except for the $bu^\eta\,\log(-t/\mu^2)$ contribution from the massless bubble. This means that these massless particles feel the same gravitational interaction from the massive object except for this quantum contribution. Since the matter content and properties are different for the scalar and photonic theories, obtaining a universal result for the bubble coefficient should not be expected. [The arguments in [3, 4] imply that the amplitude for a massless spin-$\frac{1}{2}$ scattering on the sun will differ as well only by the bubble contribution.]

Note that, because of the vanishing of the photon scattering amplitudes for the helicity configurations $(++)$ and $(--)$, the amplitude is the same in the plane of scattering or along its orthogonal component, which explicitly rules out the possibility of birefringent effects.

We do not know of a fully quantum treatment of the bending of light which is capable of describing the one-loop amplitude. However, in order to try to understand the impact of the above corrections, we can proceed by defining, in the small momentum transfer limit $t \simeq -\vec{q}^2$, a semi-classical potential for a massless scalar and photon interacting with a massive scalar object by use of the Born approximation

$$\begin{aligned} V_\eta(r) &= \frac{\hbar}{4M\omega}\int \mathcal{M}^{[\eta(p_1)\eta(p_2)]}_{[\phi(p_3)\phi(p_4)]}(\vec{q})\, e^{i\vec{q}\cdot r}\frac{d^3q}{(2\pi)^3} \\ &\simeq -\frac{2GM\omega}{r} + \frac{15}{4}\frac{(GM)^2\omega}{r^2} \\ &\quad + \frac{8bu^\eta - 15}{4\pi}\frac{G^2M\omega\hbar}{r^3} + \frac{12G^2M\omega\hbar}{\pi}\frac{\log\frac{r}{r_o}}{r^3}. \end{aligned}$$

where r_0 is an infrared scale.

Using naïvely the semi-classical formula for angular deflection given in [16, chap. 21]–[17] and the form of the above potential we find for the bending angle of a photon and for a massless scalar

$$\begin{aligned} \theta_\eta &\simeq -\frac{b}{\omega}\int_{-\infty}^{+\infty}\frac{V'_\eta(b\sqrt{1+u^2})}{\sqrt{1+u^2}}du \qquad (12) \\ &\simeq \frac{4GM}{b} + \frac{15}{4}\frac{G^2M^2\pi}{b^2} + \frac{8bu^\eta + 9 + 48\log\frac{b}{2r_o}}{\pi}\frac{G^2\hbar M}{b^3}. \end{aligned}$$

The first two terms give the correct classical values, including the first post-Newtonian correction, expressed in term of the gauge-invariant impact parameter b (see for instance [18]). The last term is a quantum gravity effect of the order $G^2\hbar M/b^3 = \ell_P^2 r_S/(2b^3)$ which involves the product of the Planck length and the Schwarzschild radius of the massive object divided by the cube of the impact parameter.

The quantum effect depends on the spin of massless particle scattering on the massive target. Of course, this dependence does not necessarily violate the equivalence principle, in that the logarithmic quantum corrections correspond to non-local effects in coordinate space. Because of the long-distance propagation of massless photons and gravitons in loops, such quantum effects are not localized, and the difference can be interpreted as a tidal correction in that the massless particle can no longer be described as a point particle. There is no requirement from the equivalence principle that such non-local effects be independent of the spin of the massless particle. Nevertheless, we see that particles no longer travel on geodesics and that different particles bend differently. This is certainly in contrast to classical applications of the equivalence principle.

Let us compare the bending angle of a photon with that of a massless scalar by the sun. The only difference given the above treatment will be given by the bubble effect

$$\theta_\gamma - \theta_\varphi = \frac{8(bu^\gamma - bu^\varphi)}{\pi}\frac{G^2\hbar M}{b^3}. \qquad (13)$$

and is far too small to be seen experimentally [19]. However, it is interesting that quantum effects do predict such a difference, modifying one of the key features of classical general relativity. Moreover, this is another demonstration that effective field techniques can make well-defined predictions within quantum gravity. We have focused on the bending of light in the vicinity of a massive object. One can envision other situations wherein the effective field theoretic framework might be very useful to analyze and understand effects in quantum gravity. We find such a prospect indeed to be very exciting!

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* Electronic address: bjbohr@nbi.dk
† Electronic address: donoghue@physics.umass.edu
‡ Electronic address: holstein@physics.umass.edu
§ Electronic address: ludovic.plante@cea.fr
¶ Electronic address: pierre.vanhove@cea.fr

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