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Field theory representation of gauge-gravity symmetry-protected topological invariants, group cohomology and beyond

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The challenge of identifying symmetry-protected topological states (SPTs) is due to their lack of symmetry-breaking order parameters and intrinsic topological orders. For this reason, it is impossible to formulate SPTs under Ginzburg-Landau theory or probe SPTs via fractionalized bulk excitations and topology-dependent ground state degeneracy. However, the partition functions from path integrals with various symmetry twists are universal SPT invariants, fully characterizing SPTs. In this work, we use gauge fields to represent those symmetry twists in closed spacetimes of any dimensionality and arbitrary topology. This allows us to express the SPT invariants in terms of continuum field theory. We show that SPT invariants of pure gauge actions describe the SPTs predicted by group cohomology, while the mixed gauge-gravity actions describe the beyond-group-cohomology SPTs. We find new examples of mixed gauge-gravity actions for U(1) SPTs in 4+1D via the gravitational Chern-Simons term. Field theory representations of SPT invariants not only serve as tools for classifying SPTs, but also guide us in designing physical probes for them. In addition, our field theory representations are independently powerful for studying group cohomology within the mathematical context.

Gapped systems without symmetry breaking[1,2] can have intrinsic topological order.[3–5] However, even without symmetry breaking and without topological order, gapped systems can still be nontrivial if there is certain global symmetry protection, known as Symmetry-Protected Topological states (SPTs).[6–9] Their nontrivialness can be found in the gapless/topological boundary modes protected by a global symmetry, which shows gauge or gravitational anomalies.[10–30] More precisely, they are short-range entangled states which can be deformed to a trivial product state by local unitary transformation[31–33] if the deformation breaks the global symmetry. Examples of SPTs are Haldane spin-1 chain protected by spin rotational symmetry[34,35] and the topological insulators[36–38] protected by fermion number conservation and time reversal symmetry.

While some classes of topological orders can be described by topological quantum field theories (TQFT),[39–42] it is less clear *how to systematically construct field theory with a global symmetry to classify or characterize SPTs for any dimension*. This challenge originates from the fact that SPTs is naturally defined on a discretized spatial lattice or on a discretized spacetime path integral by a group cohomology construction[6,43] instead of continuous fields. Group cohomology construction of SPTs also reveals a duality between some SPTs and the Dijkgraaf-Witten topological gauge theory.[43,62]

Some important progresses have been recently made to tackle the above question. For example, there are 2+1D[44] Chern-Simons theory,[45–49] non-linear sigma models,[50,51] and an orbifolding approach implementing modular invariance on 1D edge modes.[25,28] The above approaches have their own benefits, but they may be either limited to certain dimensions, or be limited to some special cases. Thus, the previous works may not fulfill all SPTs predicted from group cohomology classifications.

In this work, we will provide a more systematic way to tackle this problem, by constructing topological response field theory and topological invariants for SPTs (SPT invariants) in any dimension protected by a symmetry group G. The new ingredient of our work suggests a one-to-one correspondence between the continuous semi-classical probe-field partition function and the discretized cocycle of cohomology group, $\mathcal{H}^{d+1}(G, \mathbb{R}/\mathbb{Z})$, predicted to classify $d+1$D SPTs with a symmetry group G.[52] Moreover, our formalism can even attain SPTs beyond group cohomology classifications.[16–18,20–22]

For systems that realize topological orders, we can adiabatically deform the ground state $|\Psi_{g.s.}(g)\rangle$ of parameters g via:

$$\langle \Psi_{g.s.}(g+\delta g)|\Psi_{g.s.}(g)\rangle \simeq \dots \mathbf{Z}_0 \dots \tag{1}$$

to detect the volume-independent universal piece of partition function, $\mathbf{Z}_0$, which reveals non-Abelian geometric phase of ground states.[5,30,53–58] For systems that realize SPTs, however, their fixed-point partition functions $\mathbf{Z}_0$ always equal to 1 due to its unique ground state on any closed topology. We cannot distinguish SPTs via $\mathbf{Z}_0$. However, due to the existence of a global symmetry, we can use $\mathbf{Z}_0$ with the *symmetry twist*[59–61] to probe the SPTs. To define the symmetry twist, we note that the Hamiltonian $H = \sum_x H_x$ is invariant under the global symmetry transformation $U = \prod_{\text{all sites}} U_x$, namely $H = UHU^{-1}$. If we perform the symmetry transformation $U' = \prod_{x\in\partial R} U_x$ only near the boundary of a region R (say on one side of ∂R), the local term H_x of H will be modified: $H_x \to H'_x|_{x \text{ near } \partial R}$. Such a change along a codimension-1 surface is called a symmetry twist, see Fig.1(a)(d), which modifies $\mathbf{Z}_0$ to $\mathbf{Z}_0$(sym.twist). Just like the geometric phases of the degenerate ground states characterize topological orders,[30] we believe that $\mathbf{Z}_0$(sym.twist), on different spacetime manifolds and for different symmetry twists, fully characterizes SPTs.[59,60]

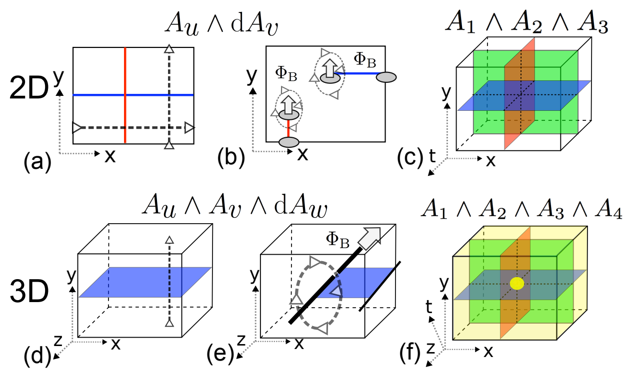

FIG. 1. On a spacetime manifold, the 1-form probe-field A can be implemented on a codimension-1 symmetry-twist[59,60] (with flat $dA = 0$) modifying the Hamiltonian H, but the global symmetry G is preserved as a whole. The symmetry-twist is analogous to a branch cut, going along the arrow - - -▷ would obtain an Aharonov-Bohm phase e^{ig} with $g \in G$ by crossing the branch cut (Fig.(a) for 2D, Fig.(d) for 3D). However if the symmetry twist ends, its ends are *monodromy defects* with $dA \neq 0$, effectively with a gauge flux insertion. Monodromy defects in Fig.(b) of 2D act like 0D point particles carrying flux,[26,59,62,64,65] in Fig.(e) of 3D act like 1D line strings carrying flux.[66–69] The non-flat monodromy defects with $dA \neq 0$ are essential to realize $\int A_u \, dA_v$ and $\int A_u A_v \, dA_w$ for 2D and 3D, while the flat connections ($dA = 0$) are enough to realize the *top* Type $\int A_1 A_2 \dots A_{d+1}$ whose partition function on a spacetime $\mathbb{T}^{d+1}$ torus with $(d+1)$ codimension-1 sheets intersection (shown in Fig.(c),(f) in 2+1D, 3+1D) renders a nontrivial element for Eq.(2).

The symmetry twist is similar to gauging the on-site symmetry[62,63] except that the symmetry twist is non-dynamical. We can use the gauge connection 1-form A to describe the corresponding symmetry twists, with probe-fields A coupling to the matter fields of the system. So we can write[52]

$$\mathbf{Z}_0(\text{sym.twist}) = e^{i\mathbf{S}_0(\text{sym.twist})} = e^{i\mathbf{S}_0(A)}. \qquad (2)$$

Here $\mathbf{S}_0(A)$ is the SPT invariant that we search for. Eq.(2) is a partition function of classical probe fields, or a topological response theory, obtained by integrating out the matter fields of SPTs path integral. Below we would like to construct possible forms of $\mathbf{S}_0(A)$ based on the following principles:[52] (1) $\mathbf{S}_0(A)$ is independent of spacetime metrics (*i.e.* topological), (2) $\mathbf{S}_0(A)$ is gauge invariant (for both large and small gauge transformations), and (3) "Almost flat" connection for probe fields.

Let us start with a simple example of SPTs with a single global U(1) symmetry. We can probe the system by coupling the charge fields to an external probe 1-form field A (with a U(1) gauge symmetry), and integrate out the matter fields. In 1+1D, we can write down a partition function by dimensional counting: $\mathbf{Z}_0(\text{sym.twist}) = \exp[\,i\,\frac{\theta}{2\pi}\int F]$ with $F \equiv dA$, this is the only term allowed by U(1) gauge symmetry $U^\dagger(A - i d)U \simeq A + df$ with $U = e^{if}$. More generally, for an even $(d+1)$D spacetime, $\mathbf{Z}_0(\text{sym.twist}) = \exp[\,i\,\frac{\theta}{(\frac{d+1}{2})!(2\pi)^{\frac{d+1}{2}}}\int F \wedge F \wedge \dots]$. Note that θ in such an action has no level-quantization (θ can be an arbitrary real number). Thus this theory does *not* really correspond to any nontrivial class, because any θ is smoothly connected to $\theta = 0$ which represents a trivial SPTs.

In an odd dimensional spacetime, such as 2+1D, we have Chern-Simons coupling for the probe field action $\mathbf{Z}_0(\text{sym.twist}) = \exp[\,i\,\frac{k}{4\pi}\int A \wedge dA]$. More generally, for an odd $(d+1)$D, $\mathbf{Z}_0(\text{sym.twist}) = \exp[\,i\,\frac{2\pi k}{(\frac{d+2}{2})!(2\pi)^{(d+2)/2}}\int A \wedge F \wedge \dots]$, which is known to have level-quantization $k = 2p$ with $p \in \mathbb{Z}$ for bosons, since U(1) is compact. We see that only *quantized* topological terms correspond to non-trivial SPTs, the allowed responses $\mathbf{S}_0(A)$ reproduces the group cohomology description of the U(1) SPTs: an even dimensional spacetime has no nontrivial class, while an odd dimension has a $\mathbb{Z}$ class.

Next we consider SPTs with $\prod_u Z_{N_u}$-symmetry. Previously the evaluation of U(1) field on a closed loop (Wilson-loop) $\oint A_u$ can be arbitrary values, whether the loop is contractable or not, since U(1) has continuous value. For finite Abelian group symmetry $G = \prod_u Z_{N_u}$ SPTs, (1) the large gauge transformation δA_u is identified by 2π (this also applies to U(1) SPTs). (2) probe fields have discrete Z_N gauge symmetry,

$$\oint \delta A_u = 0 \pmod{2\pi}, \qquad \oint A_u = \frac{2\pi n_u}{N_u} \pmod{2\pi}. \quad (3)$$

For a non-contractable loop (such as a S^1 circle of a torus), n_u can be a quantized integer which thus allows large gauge transformation. For a contractable loop, due to the fact that small loop has small $\oint A_u$ but n_u is discrete, $\oint A_u = 0$ and $n_u = 0$, which imply the curvature $dA = 0$, thus A is *flat* connection locally.

For 1+1D, the only quantized topological term is: $\mathbf{Z}_0(\text{sym.twist}) = \exp[\,i\,k_{\text{II}}\int A_1 A_2]$. Here and below we omit the wedge product $\wedge$ between gauge fields as a conventional notation. Such a term is *gauge invariant* under transformation if we impose flat connection $dA_1 = dA_2 = 0$, since $\delta(A_1A_2) = (\delta A_1)A_2 + A_1(\delta A_2) = (df_1)A_2 + A_1(df_2) = -f_1(dA_2) - (dA_1)f_2 = 0$. Here we have abandoned the surface term by considering a 1+1D closed bulk spacetime $\mathcal{M}^2$ without boundaries. The level quantization of k_{II} and its group structure can be derived from two rules: *large gauge transformation* and *flux identification*.

The invariance of $\mathbf{Z}_0$ under the allowed *large gauge transformation* via Eq.(3) implies that the volume-integration of $\int \delta(A_1A_2)$ must be invariant mod 2π, namely $\frac{(2\pi)^2 k_{\text{II}}}{N_1} = \frac{(2\pi)^2 k_{\text{II}}}{N_2} = 0 \pmod{2\pi}$. This rule implies the *level-quantization*.

On the other hand, when the Z_{N_1} flux from A_1, Z_{N_2} flux from A_2 are inserted as n_1, n_2 multiple units of $2\pi/N_1$, $2\pi/N_2$, we have $k_{\text{II}}\int A_1A_2 = k_{\text{II}}\frac{(2\pi)^2}{N_1N_2}n_1n_2$. We see that k_{II} and $k'_{\text{II}} = k_{\text{II}} + \frac{N_1N_2}{2\pi}$ give rise to the same partition function $\mathbf{Z}_0$. Thus they must be identified $(2\pi)k_{\rm II} \simeq (2\pi)k_{\rm II} + N_1N_2$, as the rule of *flux identification*. These two rules impose

$$\mathbf{Z}_0(\text{sym.twist}) = \exp[\ \mathrm{i}\ p_{\rm II}\frac{N_1N_2}{(2\pi)N_{12}}\int_{\mathcal{M}^2} A_1A_2], \quad (4)$$

with $k_{\rm II} = p_{\rm II}\frac{N_1N_2}{(2\pi)N_{12}}$, $p_{\rm II} \in \mathbb{Z}_{N_{12}}$. We abbreviate the greatest common divisor (gcd) $N_{12\ldots u} \equiv \gcd(N_1, N_2, \ldots, N_u)$. Amazingly we have independently recovered the formal group cohomology classification predicted as $\mathcal{H}^2(\prod_u Z_{N_u}, \mathbb{R}/\mathbb{Z}) = \prod_{u<v}\mathbb{Z}_{N_{uv}}$.

For 2+1D, we can propose a naive $\mathbf{Z}_0$(sym.twist) by dimensional counting, $\exp[\ \mathrm{i}\ k_{\rm III}\int A_1A_2A_3]$, which is gauge invariant under the flat connection condition. By the large gauge transformation and the flux identification, we find that the level $k_{\rm III}$ is quantized,[52] thus

$$\mathbf{Z}_0(\text{sym.twist}) = \exp[\ \mathrm{i}\ p_{\rm III}\frac{N_1N_2N_3}{(2\pi)^2N_{123}}\int_{\mathcal{M}^3} A_1A_2A_3] (5)$$

named as Type III SPTs with a quantized level $p_{\rm III} \in \mathbb{Z}_{N_{123}}$. The terminology "Type" is introduced and used in Ref.70 and 68. As shown in Fig.1, the geometric way to understand the 1-form probe field can be regarded as (the Poincare-dual of) codimension-1 sheet assigning a group element $g \in G$ by crossing the sheet as a branch cut. These sheets can be regarded as the *symmetry twists*[59,60] in the SPT Hamiltonian formulation. When three sheets (yt, xt, xy planes in Fig.1(c)) with nontrivial elements $g_j \in Z_{N_j}$ intersect at a single point of a spacetime $\mathbb{T}^3$ torus, it produces a nontrivial topological invariant in Eq.(2) for Type III SPTs.

There are also other types of partition functions, which require to use the insert flux $\mathrm{d}A \neq 0$ only at the *monodromy defect* (i.e. at the end of branch cut, see Fig.1(b)) to probe them:[11,47–49,70,71]

$$\mathbf{Z}_0(\text{sym.twist}) = \exp[\ \mathrm{i}\ \frac{p}{2\pi}\int_{\mathcal{M}^3} A_u\,\mathrm{d}A_v], \quad (6)$$

where u, v can be either the same or different gauge fields. They are Type I, II actions: $p_{\rm I,1}\int A_1\,\mathrm{d}A_1$, $p_{\rm II,12}\int A_1\,\mathrm{d}A_2$, etc. In order to have $\mathrm{e}^{\,\mathrm{i}\,\frac{p_{\rm II}}{2\pi}\int_{\mathcal{M}^3}A_1\,\mathrm{d}A_2}$ invariant under the large gauge transformation, $p_{\rm II}$ must be integer. In order to have $\mathrm{e}^{\,\mathrm{i}\,\frac{p_{\rm I}}{2\pi}\int_{\mathcal{M}^3}A_1\,\mathrm{d}A_1}$ well-defined, we separate $A_1 = \bar{A}_1 + A_1^F$ to the non-flat part A_1 and the flat part A_1^F. Its partition function becomes $\mathrm{e}^{\,\mathrm{i}\,\frac{p_{\rm I}}{2\pi}\int_{\mathcal{M}^3}A_1^F\,\mathrm{d}\bar{A}_1}$.[52] The invariance under the large gauge transformation of A_1^F requires $p_{\rm I}$ to be quantized as integers. We can further derive their level classification via Eq.(3) and two more conditions:

$$\oiint \mathrm{d}A_v = 0 \pmod{2\pi}, \quad \oiint \delta\,\mathrm{d}A_v = 0. \quad (7)$$

The first means that the net sum of all monodromy-defect fluxes on the spacetime manifold must have integer units of 2π. Physically, a 2π flux configuration is trivial for a discrete symmetry group Z_{N_v}. Therefore two SPT invariants differ by a 2π flux configuration on their monodromy-defect should be regarded as the same one. The second condition means that the variation of the total flux is zero. From the above two conditions for flux identification, we find the SPT invariant Eq.(6) describes the Z_{N_1} SPTs $p_{\rm I} \in \mathbb{Z}_{N_1} = \mathcal{H}^3(Z_{N_1}, \mathbb{R}/\mathbb{Z})$ and the $Z_{N_1}\times Z_{N_2}$ SPTs $p_{\rm II} \in \mathbb{Z}_{N_{12}} \subset \mathcal{H}^3(Z_{N_1}\times Z_{N_2}, \mathbb{R}/\mathbb{Z})$.[52]

For 3+1D, we derive the *top* Type IV partition function that is independent of spacetime metrics:

$$\mathbf{Z}_0(\text{sym.twist}) = \exp[\mathrm{i}\frac{p_{\rm IV}N_1N_2N_3N_4}{(2\pi)^3N_{1234}}\int_{\mathcal{M}^4} A_1A_2A_3A_4], \ (8)$$

where $\mathrm{d}A_i = 0$ to ensure gauge invariance. The large gauge transformation δA_i of Eq.(3), and flux identification recover $p_{\rm IV} \in \mathbb{Z}_{N_{1234}} \subset \mathcal{H}^4(\prod_{i=1}^4 Z_{N_i}, \mathbb{R}/\mathbb{Z})$. Here the 3D SPT invariant is analogous to 2D, when the four codimension-1 sheets (yzt, xzt, yzt, xyz-branes in Fig.1(f)) with flat A_j of nontrivial element $g_j \in Z_{N_j}$ intersect at a single point on spacetime $\mathbb{T}^4$ torus, it renders a nontrivial partition function for the Type IV SPTs.

Another response is for Type III 3+1D SPTs:

$$\mathbf{Z}_0(\text{sym.twist}) = \exp[\mathrm{i}\int_{\mathcal{M}^4}\frac{p_{\rm III}N_1N_2}{(2\pi)^2N_{12}}A_1A_2\,\mathrm{d}A_3], \quad (9)$$

which is gauge invariant only if $\mathrm{d}A_1 = \mathrm{d}A_2 = 0$. Based on Eq.(3),(7), the invariance under the large gauge transformations requires $p_{\rm III} \in \mathbb{Z}_{N_{123}}$. Eq.(9) describes Type III SPTs: $p_{\rm III} \in \mathbb{Z}_{N_{123}} \subset \mathcal{H}^4(\prod_{i=1}^3 Z_{N_i}, \mathbb{R}/\mathbb{Z})$.[52]

Yet another response is for Type II 3+1D SPTs:[72,73]

$$\mathbf{Z}_0(\text{sym.twist}) = \exp[\mathrm{i}\int_{\mathcal{M}^4}\frac{p_{\rm II}N_1N_2}{(2\pi)^2N_{12}}A_1A_2\,\mathrm{d}A_2]. \quad (10)$$

The above is gauge invariant only if we choose A_1 and A_2 such that $\mathrm{d}A_1 = \mathrm{d}A_2\mathrm{d}A_2 = 0$. We denote $A_2 = \bar{A}_2 + A_2^F$ where $\bar{A}_2\mathrm{d}\bar{A}_2 = 0$, $\mathrm{d}A_2^F = 0$, $\oint \bar{A}_2 = 0 \bmod 2\pi/N_2$, and $\oint A_2^F = 0 \bmod 2\pi/N_2$. Note that in general $\mathrm{d}\bar{A}_2 \neq 0$, and Eq.(10) becomes $\mathrm{e}^{\mathrm{i}\int_{\mathcal{M}^4}\frac{p_{\rm II}N_1N_2}{(2\pi)^2N_{12}}A_1A_2^F\,\mathrm{d}\bar{A}_2}$. The invariance under the large gauge transformations of A_1 and A_2^F and flux identification requires $p_{\rm II} \in \mathbb{Z}_{N_{12}} = \mathcal{H}^4(\prod_{i=1}^2 Z_{N_i}, \mathbb{R}/\mathbb{Z})$ of Type II SPTs.[52] For Eq.(9),(10), we have assumed the monodromy *line defect* at $\mathrm{d}A \neq 0$ is *gapped*;[66,68] for *gapless* defects, one will need to introduce extra anomalous *gapless* boundary theories.

Now we systematically study the physical probes of SPTs.[52] The SPT invariants can help us to design physical probes for their SPTs. Let us consider: $\mathbf{Z}_0(\text{sym.twist}) = \exp[\mathrm{i}p\frac{\prod_{j=1}^{d+1}N_j}{(2\pi)^dN_{123\ldots(d+1)}}\int A_1A_2\ldots A_{d+1}]$, a generic *top type* $\prod_{j=1}^{d+1}Z_{N_j}$ SPT invariant in $(d+1)$D, and its observables.

If we design the space to have a topology $(S^1)^d$, and add the unit symmetry twist of the $Z_{N_1}, Z_{N_2}, \ldots, Z_{N_d}$ to the S^1 in d directions respectively: $\oint_{S^1} A_j = 2\pi/N_j$. The SPT invariant implies that such a configuration will carry a $Z_{N_{d+1}}$ *induced charge* $p\frac{N_{d+1}}{N_{123\ldots(d+1)}}$.

We can also apply dimensional reduction to probe SPTs. We can design the dD space as $(S^1)^{d-1}\times I$, and add the unit Z_{N_j} symmetry twists along the j-th S^1 circles for $j = 3, \dots, d+1$. This induces a 1+1D $Z_{N_1} \times Z_{N_2}$ SPT invariant $\exp[\,\mathrm{i}\, p \frac{N_{12}}{N_{123\dots(d+1)}} \frac{N_1 N_2}{2\pi N_{12}} \int A_1 A_2]$ on the 1D spatial interval I. The 0D boundary of the reduced 1+1D SPTs has *degenerate zero energy modes* that form a projective representation of $Z_{N_1} \times Z_{N_2}$ symmetry.[26] For example, dimensional reducing 3+1D SPTs Eq.(8) to this 1+1D SPTs, if we break the Z_{N_3} symmetry on the Z_{N_4} monodromy defect line, gapless excitations on the defect line will be gapped. A Z_{N_3} symmetry-breaking domain wall on the gapped monodromy defect line will carry degenerate zero modes that form a projective representation of $Z_{N_1} \times Z_{N_2}$ symmetry.

For Eq.(8), we design the 3D space as $S^1 \times M^2$, and add the unit Z_{N_4} symmetry twists along the S^1 circle. Then Eq.(8) reduces to the 2+1D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ SPT invariant $\exp[\,\mathrm{i}\, p_{\text{IV}} \frac{N_{123}}{N_{1234}} \frac{N_1 N_2 N_3}{2\pi N_{123}} \int A_1 A_2 A_3]$ labeled by $p_{\text{IV}} \frac{N_{123}}{N_{1234}} \in \mathbb{Z}_{N_{123}} \subset \mathcal{H}^3(Z_{N_1} \times Z_{N_2} \times Z_{N_3}, \mathbb{R}/\mathbb{Z})$. Namely, the Z_{N_4} monodromy line defect carries *gapless excitations* identical to the edge modes of the 2+1D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ SPTs if the symmetry is not broken.[59]

Now let us consider *lower type* SPTs, take 3+1D $\int A_1 A_2 \mathrm{d}A_3$ of Eq.(9) as an example.[52] There are at least two ways to design physical probes. First, we can design the 3D space as $M^2 \times I$, where M^2 is punctured with N_3 *identical* monodromy defects each carrying n_3 unit Z_{N_3} flux, namely $\oiint \mathrm{d}A_3 = 2\pi n_3$ of Eq.(7). Eq.(9) reduces to $\exp[\,\mathrm{i}\, p_{\text{III}} n_3 \frac{N_1 N_2}{(2\pi) N_{12}} \int A_1 A_2]$, which again describes a 1+1D $Z_{N_1} \times Z_{N_2}$ SPTs, labeled by $p_{\text{III}} n_3$ of Eq.(4) in $\mathcal{H}^2(Z_{N_1} \times Z_{N_2}, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_{N_{12}}$. This again has 0D boundary-degenerate-zero-modes.

Second, we can design the 3D space as $S^1 \times M^2$ and add a symmetry twist of Z_{N_1} along the S^1: $\oint_{S^1} A_1 = 2\pi n_1 / N_1$, then the SPT invariant Eq.(9) reduces to $\exp[\,\mathrm{i}\, \frac{p_{\text{III}}\, n_1 N_2}{(2\pi) N_{12}} \int A_2 \mathrm{d}A_3]$, a 2+1D $Z_{N_2} \times Z_{N_3}$ SPTs labeled by $\frac{p_{\text{III}}\, n_1 N_2}{N_{12}}$ of Eq.(6).

These $\int A \mathrm{d}A$ types in Eq.(6), can be detected by the nontrivial *braiding statistics of monodromy defects*, such as the particle/string defects in 2D/3D.[48,62,66–69] Moreover, a Z_{N_1} monodromy defect line carries gapless excitations identical to the edge of the 2+1D $Z_{N_2} \times Z_{N_3}$ SPTs. If the gapless excitations are gapped by Z_{N_2}-symmetry-breaking, its domain wall will induce fractional quantum numbers of Z_{N_3} charge,[26,74] similar to Jackiw-Rebbi[75] or Goldstone-Wilczek[76] effect.

It is straightforward to apply the above results to SPTs with $\mathrm{U}(1)^m$ symmetry. Again, we find only trivial classes for even $(d+1)$D. For odd $(d+1)$D, we can define the lower type action: $\mathbf{Z}_0(\text{sym.twist}) = \exp[\,\mathrm{i}\, \frac{2\pi k}{(\frac{d+2}{2})!(2\pi)^{(d+2)/2}} \int A_u \wedge F_v \wedge \dots]$. Meanwhile we emphasize that the *top* type action with $k \int A_1 A_2 \dots A_{d+1}$ form will be trivial for $\mathrm{U}(1)^m$ case since its coefficient k is no longer well-defined, at $N \to \infty$ of $(Z_N)^m$ SPTs states. For physically relevant $2+1$D, $k \in 2\mathbb{Z}$ for bosonic SPTs. Thus, we will have a $\mathbb{Z}^m \times \mathbb{Z}^{m(m-1)/2}$ classification for $\mathrm{U}(1)^m$ symmetry.[52]

We have discussed the allowed action $\mathbf{S}_0(\text{sym.twist})$ that is described by pure gauge fields A_j. We find that its allowed SPTs coincide with group cohomology results. For a curved spacetime, we have more general topological responses that contain both gauge fields for symmetry twists and gravitational connections Γ for spacetime geometry. Such *mixed gauge-gravity topological responses* will attain SPTs beyond group cohomology. The possibility was recently discussed in Ref.17 and 18. Here we will propose some additional new examples for SPTs with U(1) symmetry.

In 4+1D, the following SPT response exists,

$$\begin{aligned}\mathbf{Z}_0(\text{sym.twist}) &= \exp[\mathrm{i}\frac{k}{3}\int_{\mathcal{M}^5} F \wedge \mathrm{CS}_3(\Gamma)] \\ &= \exp[\mathrm{i}\frac{k}{3}\int_{\mathcal{N}^6} F \wedge \mathrm{p}_1],\ k \in \mathbb{Z} \qquad (11)\end{aligned}$$

where $\mathrm{CS}_3(\Gamma)$ is the gravitations Chern-Simons 3-form and $\mathrm{d}(\mathrm{CS}_3) = \mathrm{p}_1$ is the first Pontryagin class. This SPT response is a Wess-Zumino-Witten form with a surface $\partial \mathcal{N}^6 = \mathcal{M}^5$. This renders an extra $\mathbb{Z}$-class of 4+1D U(1) SPTs beyond group cohomology. They have the following physical property: If we choose the 4D space to be $S^2 \times M^2$ and put a U(1) monopole at the center of S^2: $\int_{S^2} F = 2\pi$, in the large M^2 limit, the effective 2+1D theory on M^2 space is k copies of E_8 bosonic quantum Hall states. A U(1) monopole in 4D space is a 1D loop. By cutting M^2 into two separated manifolds, each with a 1D-loop boundary, we see U(1) monopole and anti-monopole as these two 1D-loops, each loop carries k copies of E_8 bosonic quantum Hall edge modes.[77] Their gravitational response can be detected by thermal transport with a thermal Hall conductance,[78] $\kappa_{xy} = 8k \frac{\pi^2 k_B^2}{3h} T$.

To conclude, the recently-found SPTs, described by group cohomology, have SPT invariants in terms of *pure gauge actions* (whose boundaries have *pure gauge anomalies*[11,13–15,26]). We have derived the formal group cohomology results from an easily-accessible field theory set-up. For beyond-group-cohomology SPT invariants, while ours of bulk-onsite-unitary symmetry are *mixed gauge-gravity actions*, those of other symmetries (e.g. anti-unitary-symmetry time-reversal $\mathbb{Z}_2^T$) may be *pure gravity actions*.[18] SPT invariants can also be obtained via cobordism theory,[17–19] or via *gauge-gravity actions* whose boundaries realizing *gauge-gravitational anomalies*. We have incorporated this idea into a field theoretic framework, which should be applicable for both bosonic and fermionic SPTs and for more exotic states awaiting future explorations.

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