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Generalized Kitaev Models and Slave Genons

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We present a wide class of partially integrable lattice models with two-spin interactions, which generalize the Kitaev honeycomb model. These models have a conserved quantity associated with each plaquette, conserved large loop operators on the torus, and topological degeneracy. We introduce a 'slave-genon' approach, which generalizes the Majorana fermion approach in the Kitaev model. The Hilbert space of our spin model can be embedded into an enlarged Hilbert space of non-Abelian twist defects, referred to as genons. In the enlarged Hilbert space, the spin model is exactly reformulated as a model of non-Abelian genons coupled to a discrete gauge field. We discuss in detail a particular Z_3 generalization, and show that in a certain limit the model is analytically tractable and produces a non-Abelian topological phase with chiral parafermion edge states.

Introduction–The Kitaev honeycomb model [1] is an exactly solvable spin model on the two-dimensional hexagonal lattice, which can realize different exotic topologically ordered phases of matter, along with non-Abelian quasiparticle excitations. Over the past decade, this model has generated remarkable excitement[2]: its solvability has provided a theoretical framework to study the emergence of topological order and non-Abelian anyons from microscopic models, while its simplicity supports the hope for experimental realization, either in Mott insulators with strong spin orbit coupling, such as various Iridate compounds [3, 4], or directly engineered with designer Hamiltonians [5]. In particular, the non-Abelian state in the Kitaev model would open the possibility of topological quantum computation [6].

In this paper, we generalize the Kitaev model to a much larger class of partially integrable spin models with only nearest-neighbor interactions. We show that there is an exact transformation whereby these models can be reformulated in terms of an array of interacting non-Abelian defects coupled to a static discrete gauge field. In order to implement the exact transformation, we introduce a "slave genon" approach, where the local Hilbert space on each site is rewritten in terms of the topological degeneracy of a set of extrinsic non-Abelian twist defects, referred to as genons [7, 8], together with a constraint on their overall fusion channel. This generalizes the Majorana fermion representation of the original Kitaev honeycomb model [1]. While the transformed problem is itself a non-trivial interacting problem, certain results in 1+1 dimensional critical phenomena can then be utilized to solve the model in certain limits.

We will focus on a particular Z_n rotor generalization of the Kitaev model for most of the paper, and discuss more general models in the end of the draft. We introduce a graphical method to perform the slave genon technique, making use of genons in bilayer FQH states [7, 8], with a

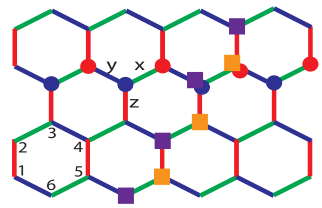

FIG. 1: The links of a honeycomb lattice are labelled x, y, or z, depending on their orientation. Sites on a plaquette are labelled 1,..,6, as shown. Red and blue circles illustrate the path L_1, purple and orange squares illustrate the path L_2, which are used to define the string operators Φ_1 and Φ_2.

$1/n$ Laughlin state in each layer. When $n = 2$, the genons localize Majorana zero modes, thus reproducing Kitaev's construction. More generally they localize parafermion zero modes [7–15]. For $n = 3$, we present preliminary numerical results, and discuss the realization of a non-Abelian Z_3 parafermion phase, which contains the non-Abelian Fibonacci anyon [6] in its excitation spectrum.

Z_n Kitaev model–We consider the following Hamiltonian on the honeycomb lattice with n states per site:

$$H = -\sum_{\langle ij \rangle} J_{s_{ij}} (T_i^{s_{ij}} T_j^{s_{ij}} + H.c.), \tag{1}$$

where $s_{ij} = x, y, z$ depends on the direction of the link ij (Fig. 1). T_i^x and T_i^y are $n \times n$ matrices satisfying the relations: $T_i^x T_i^y = T_i^y T_i^x \omega$, $(T_i^x)^n = (T_i^y)^n = 1$, where $\omega \equiv e^{i2\pi/n}$.We further define: $T_i^z \equiv (T_i^x T_i^y)^\dagger$, implying $T_i^z T_i^x = T_i^x T_i^z \omega$, $T_i^y T_i^z = T_i^z T_i^y \omega$. T_i^s from different sites commute with each other. The original Kitaev model corresponds to $n = 2$.

The key fact about this model is that there is a conserved operator associated with each plaquette. Define: $W_p \equiv \prod_{\langle ij \rangle \in \hexagon} K_{ij} = (\omega T_1^x T_2^y T_3^z T_4^x T_5^y T_6^z)^\dagger$, where the site labels are shown in Fig. 1. Following Kitaev, we define $K_{jk} = T_j^{s_{jk}} T_k^{s_{jk}}$. It can be verified directly that $[W_p, H] = 0$, so that the spectrum can be decomposed into eigenstates of W_p. Note that $W_p^n = 1$.

In addition to the above conserved plaquette operators, the model (for $n \geq 3$) with periodic boundary conditions also admits conserved, non-commuting, loop operators:

$$\Phi_1 \equiv \prod_{2i-1,2i \in L_1} T_{2i-1}^z T_{2i}^{z\dagger},\ \Phi_2 \equiv \prod_{2i-1,2i\in L_2} T_{2i-1}^y T_{2i}^{y\dagger} \quad (2)$$

where $[\Phi_1, H] = [\Phi_2, H] = 0$, and $\Phi_2\Phi_1 = \Phi_1\Phi_2\omega^2$. The loops L_1 and L_2 are shown in Fig. 1, and describe non-contractible paths around the hexagonal lattice in the two directions. Since these operators are conserved, eigenstates must form a representation of their algebra. This rigorously implies a ground state degeneracy on the torus that is a multiple of n (or $n/2$) for n odd (even).

Just as in the original Kitaev model, the generalized model can be defined on any *planar* trivalent graph. A key difference between the $n \geq 3$ and the $n = 2$ cases is that for $n \geq 3$, the three operators $T_i^{x,y,z}$ on each site must be ordered with the same chirality. In other words, the direction $x \to y \to z \to x$ must be either all counter-clockwise or all clockwise on all sites. This requirement also means that the model can only be defined on planar graphs. Physically, this is because the large loops Φ_1, Φ_2 defined above can be considered as Wilson loops of a particle with statistical angle $\frac{2\pi}{n}$. For $n > 2$ this particle is an Abelian anyon, which can only be defined in two-dimensions, while for $n = 2$ it is a fermion. Multi-site terms can be added to the Hamiltonian without affecting the conservation laws, as long as they are products of bond terms K_{ij} and/or $K_{ij}^\dagger$. In the supplementary materials[16], we present more details of the computation of commutation relations and conserved quantities by setting up convenient diagrammatic rules.

Anisotropic limit and the Abelian phase – Similar to the original model[1], the anisotropic limit $J_z \gg J_x, J_y$ can be easily solved. In this limit, we first diagonalize the J_z terms in the Hamiltonian. To do this, let us pick a basis of n states on each site, $|a\rangle_i$, which diagonalize T_i^z: $T_i^z|a\rangle_i = \omega^a|a\rangle_i$, for $a = 0, ..., n-1$. Pairs of sites i, j coupled by J_z have their n^2 states split into n degenerate lowest energy states, $|a\rangle_i|n-a\rangle_j$, for $a = 0, .., n-1$. These states are separated by a gap of order J_z relative to the remaining $n^2 - n$ states. For large J_z, we can treat pairs of sites separated by vertical links effectively as a single site, thus obtaining at low energies a square lattice with n states per site. Within the degenerate n-dimensional space on each site, we can define a new set of Z_n rotor operators L_i^x, L_i^y, such that $L_i^x|a\rangle_i|n-a\rangle_j = \omega^a|a\rangle_i|n-a\rangle_j$, and $L_i^y|a\rangle_i|n-a\rangle_j = |a-1\rangle_i|n-a+1\rangle_j$.

Within this low-energy subspace, the remaining J_x, J_y terms can be treated within perturbation theory. The lowest order term that does not change the J_z bond energy is $\frac{J_x^2 J_y^2}{(6J_z)^3} K_{12}K_{23}K_{45}K_{56}$ (with the label of sites defined in Fig. 1). It is straightforward to show that this gives $H_{eff} = \frac{J_x^2 J_y^2}{(6J_z)^3} \sum_{ijkl\in\square} L_i^x L_j^y L_k^x L_l^y$, which is the Z_n toric code Hamiltonian [17–19].

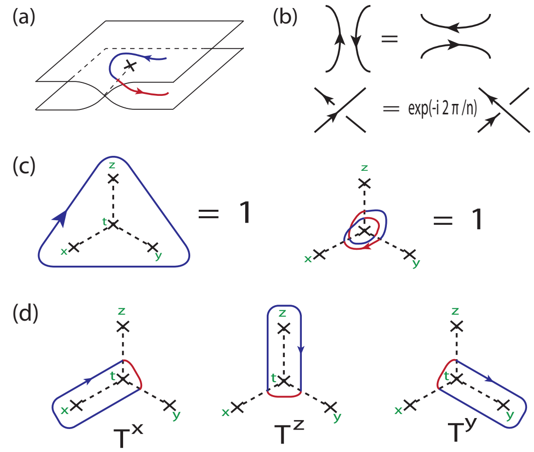

FIG. 2: (a) A genon (twist defect) in a bilayer FQH system, is marked by the X. The branch cut emanating from the genon connects the two layers. (b) The Wilson lines of the Abelian quasiparticles can locally be cut and rejoined. (c) A spin is represented in terms of 4 genons, labelled x, y, z, t. The constraint that the Wilson loop around all 4 genons be trivial reduces the number of states to n. The double loop around each genon can always be set to one. (d) T^x, T^y, T^z correspond to Wilson loop operators around pairs of genons.

Slave Genons – In order to further analyze the model beyond this strongly anistropic limit, we introduce a 'slave genon' approach, which maps the spin model to a model of coupled non-Abelian twist defects [7, 8, 10–13, 20–27], referred to as genons [7, 8], in a topologically ordered state. This generalizes the Majorana fermion representation introduced in the original Kitaev honeycomb model [1], along with well-known slave fermion/boson techniques [28]. A key difference in the $n \geq 3$ Z_n models is that the slave particles must be topological defects in a topologically ordered system, instead of fermions or bosons.

As a formal aid in defining these slave particles, we introduce a Laughlin $1/n$ fractional quantum Hall (FQH) state on the surface shown in Fig. 2 (a). The surface is obtained by introducing a branch cut line in a bilayer system, such that the two layers are exchanged across the branch cut line. A genon is defined as the endpoint of the [7, 8, 20]. Consider 4 genons with the constraint that they fuse to vacuum. As is shown in Fig. 2 (c), this constraint means a Laughlin quasparticle going around the 4 genon cluster obtains no Berry's phase. With this constraint, the disk region with 4 genons is topologically equivalent to a torus with a single layer of $1/n$ state[20], which has n topological ground states. The slave genon approach is defined by mapping the n-state rotor on each site of the honeycomb lattice to such a cluster of 4 genons.

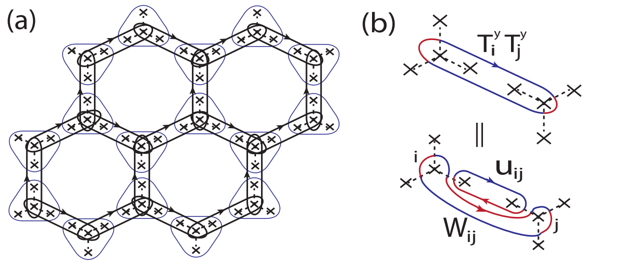

FIG. 3: (a) The interaction terms in the Hamiltonian correspond to the three types of loops. The blue loop around each site represents the local constraint which commute with the Hamiltonian terms. (b) A loop corresponding to the interaction $T_i^y T_j^y$ can be decomposed into two non-overlapping loops, W_{ij} and u_{ij}.

The spin operators $T_i^{x,y,z}$ are mapped to Wilson loop operators, defined as the unitary rotation of topological ground states induced by adiabatic propagation of charge $1/n$ Laughlin quasiparticles along a non-contractible loop. $T_i^{x,y,z}$ corresponds to the three non-contractible loops shown in Fig. 2 (d). During topological deformations of the Wilson loops, we also require that a double loop around a genon is contractible, as is illustrated in 2 (c). Physically this removes the ambiguity that a genon may trap a Laughlin quasiparticle. We emphasize that the genons and associated FQH state are *entirely auxiliary* degrees of freedom – the spin model is not required to have a FQH state physically. When $n = 2$, it can be shown that the genons localize Majorana fermions, so this approach is equivalent to the Majorana representation of the original Kitaev model. For general n, it can be shown that the genons localize Z_n parafermion zero modes. Thinking in terms of the genons described above admits useful graphical representations of the operators.

Therefore in this representation, the spin model is mapped to a two-dimensional array of genons, with couplings given by Wilson loop operators. The two-site terms K_{ij} in H correspond to Wilson loops surrounding 4 genons, as shown in Fig. 3. Importantly, H commutes with the local constraint at each site, since the Wilson loop corresponding to the local constraint commutes with that of K_{ij}, as is illustrated in Fig. 3 (a). On each site, the constraint can be expressed in the spin operators $T_i^{x,y,z}$ as $T_i^x T_i^y T_i^z = 1$, which projects the n^2 states of 4 genons[20] to n states of the physical spin.

From the pictorial representation, we readily infer that the Hamiltonian can be rewritten as:

$$H = -\sum_{\langle ij\rangle} J_{s_{ij}} u_{ij} W_{ij} + H.c., \tag{3}$$

where W_{ij} and u_{ij} are the loop operators corresponding to the operation of moving charge $1/n$ Laughlin quasiparticles around the loops shown in Fig. 3b. Note that u_{ij} only appears in the Hamiltonian in the term $T_i^{s_{ij}} T_j^{s_{ij}}$. From Fig. 3, we deduce that $[u_{ij}, W_{ij}] = 0$, $[u_{ij}, u_{kl}] = 0$ and therefore $[u_{ij}, H] = 0$. We can hence replace the u_{ij} by c-numbers, associated with different superselection sectors. W_{ij} can be considered as a two-dimensional "parafermion hoping" term, while the eigenvalues of u_{jk} can be considered as a Z_3 gauge field coupled to the parafermions [9]. The precise meaning of the parafermion coupling will be discussed in next paragraph. By deforming the loops u_{ij} and using the constraints shown in Fig. 2c, it is straightfoward to show that the conserved plaquette operators, W_p, correspond to the Z_n "gauge flux" through a plaquette: $W_p = \prod_{\langle ij\rangle \in \hexagon} u_{ij} = u_{12}u_{23}u_{34}u_{45}u_{56}u_{61}$.

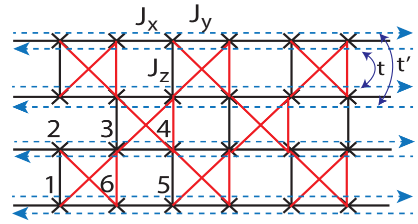

FIG. 4: Hamiltonian (3) describes a hexagonal array of coulpled genons, or Z_n parafermion zero modes. For $J_x = J_y$, and in the absence of interchain interactions, each chain is at criticality, which in the $n = 3$ case is described by a Z_3 parafermion CFT. Interchain coupling terms can be added to gap out counterpropagating parafermion modes from each chain, leading to a gapped topologically ordered state with a chiral Z_3 parafermion edge mode. Red bonds correspond to the next neighbor interactions (see eq. (4) in text).

To understand more explicitly the meaning of coupled parafermion zero modes, we first consider the Hamiltonian for a single chain, with u_{ij} uniformly set to 1: $H_{1D} = -\sum_i (J_x W_{2i-1,2i} + J_y W_{2i,2i+1} + H.c.)$, with $W_{i-1,i}W_{i,i+1} = W_{i,i+1}W_{i-1,i}\omega$. This Hamiltonian is equivalent to the transverse field Z_n Potts model. Following the results in the Potts model [9, 29, 30], a pair of parafermion operators α_{Li}, α_{Ri} can be introduced, which satisfies the algebra $\alpha_{R/Li}\alpha_{R/Lj} = \alpha_{R/Lj}\alpha_{R/Li}e^{\pm i2\pi sgn(j-i)/n}$. In terms of spin operators of the Kitaev model, we have $\alpha_{Ri} = T_{1y}^\dagger K_{12}K_{23}...K_{i-1,i}$, $\alpha_{Li} = T_{1y}^\dagger K_{12}^\dagger K_{23} K_{34}^\dagger ... K_{i-1,i}^{s_i}$, with $s_i = -(-1)^i$. H_{1D} can be rewritten in terms of a "parafermion chain" by setting $W_{i,i+1} \propto \alpha_{Ri}^\dagger \alpha_{R,i+1}$. The 2D Hamiltonian (3) can then be reinterpreted as an array of coupled 1D parafermion chains [9, 15, 16, 31–33].

The single chain system with $n = 3$ is particularly interesting. When $J_x = J_y$, the model is at a self-dual critical point of the 1D Z_3 Potts model, which is described by a Z_3 parafermion conformal field theory (CFT) with central charge $c = 4/5$[34]. At small but finite J_z, the system can be viewed as coupled parafermion chains, as is illustrated in Fig. 4. It is known that a "chiral" coupling between 1D gapless chains can realize a chiral 2D topologically ordered state[15, 33, 35, 36], if the right-moving (left-moving) states of a chain are only coupled

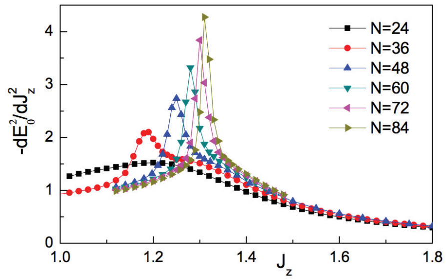

FIG. 5: Second derivative of the ground state energy density as a function of J_z, computed from DMRG with 3 chains with cylindrical boundary conditions, for $J_x = J_y = (3 - J_z)/2$.

to the left-moving (right-moving) states of the chain below (above) by a relevant coupling. In our $n = 3$ system, such a coupling would result in a non-Abelian topological state with chiral Z_3 parafermion edge states. This is similar to the proposal of [15] although the latter is not a local spin model and therefore realizes a different topological order which we elaborate on below. In $n > 2$ models, the J_z coupling breaks time-reversal symmetry, so that it is possible for the system with some proper J_z to be in the same non-Abelian phase as the ideal system with only chiral coupling.

Numerical Results – To further understand the $n = 3$ system, we have performed preliminary numerical analyses. For the single chain with $J_x = J_y$, our DMRG results [37, 38] for the entanglement entropy shows that the chain is indeed described by a CFT with central charge $c = 4/5$. When $J_z \gg J_x, J_y$, we have verified through exact diagonalization of system sizes up to 12 sites that the system is gapped, with a 9-fold ground state degeneracy. As J_z is lowered relative to J_x, J_y, we expect a phase transition from the Abelian phase to the isotropic phase. Fig. 5 shows DMRG results for the second derivative of the ground state energy density, $-d^2E_0/dJ_z^2$, which indeed shows evidence of a sharp phase transition. We note that the first derivative dE_0/dJ_z appears smooth across this transition, allowing us to rule out the possibility of a level crossing between the nearly degenerate ground states. These results confirm non-trivial features of the Z_3 Kitaev model, while they do not fully establish the nature of the isotropic phase. More complete numerical study of the isotropic phase will be left for future works.

Multi-site terms and the controlled limit– In the original Kitaev model[1], a three site term drives the model into the non-Abelian Ising phase. For $n = 3$, the model is not fully solvable, but it is possible to consider a modification of the Hamiltonian (1) that makes the model analytically tractable, allowing us to demonstrate the appearance of a non-Abelian phase in this limit. As is pointed out in Ref. [15, 39], there is a known correspondence between the lattice parafermion operators and continuous fields in the Z_3 Potts model CFT. Using this correspondence, one can see that the parafermion coupling of the form $-\lambda \sum_{j,m} \left(\alpha^\dagger_{R,2j,m} + \alpha^\dagger_{R,2j+1,m}\right)(\alpha_{L,2j,m+1} + \alpha_{L,2j+1,m+1}) + h.c.$ between two neighboring chains labelled by m and $m+1$ induces the chiral coupling between the right movers of the m-th chain and the left movers of the $m+1$-th chain. Since this is a direct application of Ref. [15, 39]'s result, we will leave more detailed derivation of this term for the supplementary materials[16].

Using the Wilson loop representation, the chiral coupling between parafermions reviewed above can be achieved in a local spin Hamiltonian:

$$H' = H - J_z \sum_{\hexagon} \mathcal{O}_{\hexagon}, \tag{4}$$

with $\mathcal{O}_{\hexagon} = (T_1^z T_1^y T_2^z T_6^y + T_3^x T_2^x T_2^z T_1^z T_1^y T_6^y + T_3^x T_2^x T_2^z T_1^z + H.c.)$, and H given by Eq. (1). Therefore, the above Hamiltonian, with $J_x = J_y \gg J_z > 0$, realizes a gapped 2D topologically ordered state, with a robust chiral Z_3 parafermion CFT propagating along its edge. The topological order can then be read off from the field content of the Z_3 parafermion CFT [34, 40], which has 6 topologically distinct quasiparticles. In contrast, the system proposed by [15] has 2 distinct quasiparticles.

The coupling between parafermion chains in our model involves only single parafermion operators from different chains. This is not possible with the usual transverse field Potts model, but is possible with the approach described here. The slave genon transformation thus provides a way to design general interactions in 2D lattices of parafermions in terms of *local* interactions of a 2D spin model. A similar method can perhaps be employed more generally, which may enable a spin model realization of recently studied anyon lattice models [41, 42].

Further generalizations – The model described here can be further generalized. For example, one can consider genons in a generic Abelian FQH state. Quasiparticles in each layer are labeled by integer vectors $\vec{l}$, with the fractional mutual statistics $\theta_{ll'} = 2\pi \vec{l}^T K^{-1} \vec{l'}$ and self statistics $\theta_l = \pi \vec{l}^T K^{-1} \vec{l}$ determined by an integer valued K matrix[28]. 4 genons with the local constraint in Fig. 2 now correspond to a spin with $|K|$ states[8]. The spin operators $T_i^{x,y,z}$ generalize to Wilson loop operators $T_{\vec{l}}^{x,y,z}$ of a quasiparticle $\vec{l}$ around the same loops as those in Fig. 2 (d). These operators satisfy the algebra $T_{\vec{l}}^x T_{\vec{l'}}^y = T_{\vec{l'}}^y T_{\vec{l}}^x e^{i2\pi \vec{l}^T K^{-1} \vec{l'}}$, $T_{\vec{l}}^s T_{\vec{l'}}^s = T_{\vec{l}+\vec{l'}}^s$, and $T_{K\vec{n}} = 1$ for all $\vec{n} \in \mathbb{Z}^N$. Therefore, we can consider the more general Kitaev-type Hamiltonian: $H = \sum_{\vec{l} \in \mathbb{Z}^N} \sum_{\langle ij \rangle} J_{\vec{l};s_{ij}} T_{\vec{l}}^{s_{ij}} T_{\vec{l}}^{s_{ij}} + H.c.$ This model can be analyzed similarly to the Z_n generalization presented earlier. In particular, there are conserved quantities associated with each plaquette and conserved large loop operators on torus geometry, and one can consider an exact transformation to a lattice model of interacting genons or, alternatively, generalized parafermion zero modes[14].

Acknowledgement. The Z_n generalization of the Kitaev model has also been studied independently in unpublished works of P. Fendley and C. L. Kane (*c.f.* discussions in [9]). This work was presented in June of 2013 on a workshop[43]. As this manuscript was being completed, we were made aware that a related but different generalization of Kitaev model has been studied by A. Vaezi[44]. We thank A. Ludwig and A. Kitaev for discussions. HCJ was supported by the Templeton Fund. RT has been supported by the European Research Council through ERC-StG-TOPOLECTRICS-336012. XLQ is supported by David & Lucile Packard foundation. We also acknowledge computing support from the Center for Scientific Computing at the CNSI and MRL: NSF MRSEC (DMR-1121053) and NSF CNS-0960316.

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