Collinear Limit of Scattering Amplitudes at Strong Coupling

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On the collinear limit of scattering amplitudes at strong coupling

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In this letter we consider the collinear limit of gluon scattering amplitudes in planar $\mathcal{N}=4$ SYM theory at strong coupling. We argue that in this limit scattering amplitudes map into correlators of twist fields in the two dimensional non-linear $O(6)$ sigma model, similar to those appearing in recent studies of entanglement entropy. We provide evidence for this assertion by combining the intuition springing from the string worldsheet picture and the predictions coming from the OPE series. One of the main implications of these considerations is that scattering amplitudes receive equally important contributions at strong coupling from both the minimal string area and its fluctuations in the sphere.

INTRODUCTION

In planar $\mathcal{N}=4$ Super-Yang-Mills theory, scattering amplitudes and null polygonal Wilson loops are one and the same [1, 2] at any value of the coupling $\lambda = g_{YM}^2 N$. Through the prism of the AdS/CFT correspondence, a scattering amplitude can then be viewed as a path integral over the open string configurations that end on a light-like polygon at the boundary of $AdS_5 \times S^5$. At strong coupling, this partition function is dominated by its saddle point which in turn is given by a minimal string area in AdS_5. For the n-gluon amplitude [1]

$$\mathcal{W}_n = e^{-\frac{\sqrt{\lambda}}{2\pi} A_n + \dots} \tag{1}$$

where $\mathcal{W}_n$ is the renormalized amplitude introduced in [3] and A_n is the corresponding subtracted area of the classical string ending on the n-gon [4]. Thanks to the integrability of the classical worldsheet theory, the problem of computing this area can be reduced to solving a set of Thermodynamic Bethe Ansatz equations with A_n being identified with a free energy of sort, known as the critical Yang-Yang functional [3, 6–8]. Except for that, little is known about scattering amplitudes at strong coupling, that is about the ellipsis in (1) – in contrast with the flood of results at weak coupling [9].

Building upon earlier work [8], we proposed in [3] an alternative method for computing the open string partition function, at any value of the coupling. In this so-called *pentagon approach* a generic polygon is broken down into a sequence of pentagon transitions $\mathcal{P}$ between one flux tube to the next. This decomposition takes the form [3, 10]

$$\mathcal{W}_n = \langle 0|\mathcal{P} e^{-H\tau_{n-5} + iP\sigma_{n-5} + iJ\phi_{n-5}} \mathcal{P} \dots \\ \dots \mathcal{P} e^{-H\tau_1 + iP\sigma_1 + iJ\phi_1} \mathcal{P}|0\rangle \,. \tag{2}$$

Here, τ_i, σ_i and ϕ_i are a base of conformal cross ratios that are conjugate to the energy, momentum and angular momentum of the state that propagates in the i'th flux tube, see [3] for more details. This representation is particularly suitable to the analysis of the multi-collinear limit which corresponds to the regime of large τ_i.

Based on this approach as well as on world-sheet considerations, we shall see that at strong coupling the *collinear limit* is governed by the string dynamics in the five sphere. More precisely, we will show that in this limit the entire partition function reduces to a correlator of twist operators in the $O(6)$ sigma model, similar to those encountered in the study of entanglement entropy [11–13].

A surprising consequence of this identification and of the strongly coupled dynamics of the $O(6)$ sigma model is an additional exponentially large contribution to $\mathcal{W}_n$ of the same order as the classical area A_n. As we will explain, accommodating for the sphere indeed corrects the minimal area prescription such that

$$\log \mathcal{W}_n = -\frac{\sqrt{\lambda}}{2\pi} A_n + \frac{\sqrt{\lambda}}{48} \frac{(n-4)(n-5)}{n} + o(\sqrt{\lambda}) \,, \tag{3}$$

to leading order at strong coupling. More excitingly, the dynamics of the $O(6)$ sigma model also allows us to start unveiling the α' corrections. For the six gluons amplitude, for instance, we shall find that

$$\mathcal{W}_6 = f_6 \, \lambda^{-\frac{7}{288}} e^{\frac{\sqrt{\lambda}}{144} - \frac{\sqrt{\lambda}}{2\pi} A_6} (1 + O(1/\sqrt{\lambda})) \tag{4}$$

where the λ independent prefactor f_6 is a yet to be determined function of the 3 hexagon cross-ratios. Computing this function for generic kinematics is beyond the scope of the present paper. However, based on the $O(6)$ analysis alone we will predict that in the collinear limit $\tau \gg 1$

$$f_6(\tau, \sigma, \phi) \simeq \frac{1.04}{(\sigma^2 + \tau^2)^{1/72}} \,, \tag{5}$$

with the critical exponent in this power-law being related to dimensions of the twist fields mentioned before.

Finally, we will also see that another face of the strong coupling dynamics of the $O(6)$ sigma model is the breakdown of the string α' expansion for extremely stretched

Wilson loops. Namely, we shall observe that for exponentially large cross-ratios $\log\tau \gg \sqrt{\lambda} \gg 1$ the open string partition function is fully non-perturbative and governed by the exponentially small dynamical scale of the model. In brief, the emergence of this new scale is the main reason for the richness of the collinear limit at strong coupling. Studying all the various collinear behaviours and their cross-over is the main subject of this paper.

PENTAGONS AS TWIST OPERATORS

In the collinear limit $\tau_i \gg 1$ the lightest states dominate in (2). At strong coupling, these are the string excitations in the sphere [14, 15], dual to the gauge theory scalars, see e.g. figure 2 in [16]. Their dynamics is governed by the $O(6)$ non-linear sigma model and, in particular, their mass is non-perturbatively generated and exponentially small at strong coupling [15]

$$m = \frac{2^{1/4}}{\Gamma(5/4)}\,\lambda^{1/8}e^{-\sqrt{\lambda}/4}(1+O(1/\sqrt{\lambda}))\,. \tag{6}$$

All the other string excitations, i.e., both the AdS and the fermionic modes, have masses of order 1 at strong coupling [14, 15] and hence decouple in the collinear limit.

This leads us to interpret the strong coupling collinear limit of (2) as a correlator in the $O(6)$ model

$$\mathcal{W}_n \simeq \langle 0|\phi_{\pentagon}(w_{n-4})\dots\phi_{\pentagon}(w_1)|0\rangle_{\mathrm{O}(6)} \tag{7}$$

where $w_i - w_{i-1} = (\sigma_i, \tau_i)$ and $\phi_{\pentagon}(w)$ are operators whose matrix elements coincide with the pentagon transitions

$$\langle\theta_1',\dots|\phi_{\pentagon}(0)|\theta_1,\dots\rangle^{j_1,\dots}_{i_1,\dots} = P(\theta_1,\dots|\theta_1',\dots)^{j_1,\dots}_{i_1,\dots}\,. \tag{8}$$

Here, θ_j are the usual hyperbolic rapidities parametrizing the scalars' relativistic dispersion relation while the indices refer to the $O(6)$ polarizations of the incoming and outgoing multi-scalar states.

The clue about what the operator $\phi_{\pentagon}(w)$ is comes from the observation that one needs to perform 5 so-called mirror rotations (equivalently Euclidean boost) $\theta \to \theta + 5i\pi/2$ to go around the pentagon [3]. This should be contrasted with the more standard monodromy for conventional local operators which involves 4 such transformations only. This hints that the effect of the operator $\phi_{\pentagon}(w)$ is to generate a conical excess angle $\frac{1}{4}\times 2\pi$ around w. Such fields are not entirely new and belong to a broad class of operators known in the CFT literature as *twist operators* [17]. Most directly relevant for our discussion is their appearance in the context of entanglement entropy [11, 12]. There, such operators were introduced to study QFTs on k-sheeted Riemann surfaces with branch points being viewed as twist operators with excess angle $\varphi = 2\pi(k-1)$ in the replica theory. Our case is somewhat special in that it requires a "fractional number of sheets" since $k = 5/4$ for a pentagon, see figure 1. Once can indeed verify that the pentagon transitions in the right-hand side of (8) satisfy the axioms for the form factors of twist operators as spelled in [12] with $k = 5/4$.

The above picture can also be understood more directly from the worldsheet analysis. From our previous discussion it follows that the partition function (2) receives, in the collinear limit, its dominant contribution from the sphere. This means that we can write $\mathcal{W}_n \propto \int \mathcal{D}X e^{-\delta S_{NG}}\,\delta(X^2-1)$ where

$$\delta S_{NG} = \frac{\sqrt{\lambda}}{4\pi}\int d^2z\,\sqrt{g}g^{\alpha\beta}\partial_\alpha X\cdot\partial_\beta X \tag{9}$$

is the expansion of the Nambu-Goto action to quadratic order in the sphere embedding coordinates X and $g_{\alpha\beta}$ is the induced metric of the classical minimal surface in AdS. We thus face the problem of computing the partition function of the $O(6)$ sigma model on the minimal surface. From the low-energy viewpoint, this surface looks everywhere flat, except for a few points where the curvature is concentrated. Indeed the induced metric in the collinear limit is approximately

$$ds^2 \simeq [P(z)\bar{P}(\bar{z})]^{1/4}dzd\bar{z} \tag{10}$$

where $P(z) = \prod_{j=1}^{n-4}(z-z_j)$ is the auxiliary polynomial entering the Pohlmeyer description of the minimal surface [6]. In agreement with the pentagon picture, we see that there are $n-4$ marked points around which we have a conical excess of $2\pi\times\frac{1}{4}$. Following [11], the partition function in this geometry can be recast as a correlator of $n-4$ twist operators as (7).

OPE AS FORM FACTOR EXPANSION

As elaborated above, at strong coupling, the collinear limit is governed by the dynamics of the $O(6)$ sigma model whose physics is strongly coupled. As such, at the moment, the only available tool for studying this regime in a controllable way is the pentagon approach [3]. In this section we will focus on the simplest possible case, the hexagon $\mathcal{W} = \mathcal{W}_6$.

Given the relativistic invariance of the O(6) sigma model, the Wilson loop can only depend (in the collinear limit) on the dimensionless Lorentz invariant distance

$$z \equiv m\sqrt{\sigma^2+\tau^2}\,. \tag{11}$$

For any value of z, the correlator in (7) then reads

$$\mathcal{W} = \sum_{n\text{ even}}\int\frac{\prod_i d\theta_i}{n!\,(2\pi)^n}|P(0|\theta_1,\dots,\theta_n)_{i_1,\dots,i_n}|^2\,e^{-z\sum\limits_{k=1}^{n}\cosh\theta_k}\,. \tag{12}$$

This is the familiar form factor expansion, which simply follows from inserting the resolution of the identity between consecutive operators in (7). Alternatively, from the Wilson loop point of view, this sum stands for the truncation of the full OPE series to the scalar subsector in the strong coupling limit.

As illustrated in [16], the transitions can be factored out into a dynamical factor and a so-called matrix part taking care of the matrix structure of these objects,

$$|P(0|\theta_1,\ldots,\theta_n)_{i_1,\ldots,i_n}|^2 = \Pi_{\rm dyn}\times\Pi_{\rm mat}\,. \qquad (13)$$

Working out these contributions (most notably the matrix part) in a systematic fashion is a fascinating problem which we will report elsewhere. The main conjecture arising from this analysis is that $\Pi_{\rm mat}$ is a simple group theoretic factor whose derivation will be reported elsewhere. It is a rational function of the rapidities θ_j with a very simple integral representation involving $2n$ auxiliary rapidities as

$$\Pi_{\rm mat} = \frac{1}{n!(\frac{n}{2}!)^2}\int\limits_{-\infty}^{+\infty}\frac{dw_1^1\ldots dw^1_{\frac{n}{2}}dw_1^2\ldots dw_n^2 dw_1^3\ldots dw^3_{\frac{n}{2}}}{(2\pi)^{2n}} \times\frac{\prod\limits_{i<j} g(w_i^1-w_j^1)\prod\limits_{i<j} g(w_i^2-w_j^2)\prod\limits_{i<j} g(w_i^3-w_j^3)}{\prod\limits_{i,j} f(\frac{2}{\pi}\theta_i-w_j^2)\prod\limits_{i,j} f(w_i^1-w_j^2)\prod\limits_{i,j} f(w_i^3-w_j^2)} \qquad (14)$$

with $f(x)=x^2+1/4$ and $g(x)=x^2(x^2+1)$. For any fixed number of particles, n, the integrals over the auxiliary roots can be straightforwardly evaluated by residues. Finally, the dynamical part takes the factorized form

$$\Pi_{\rm dyn} = \mu^n\prod_{i<j}F(\theta_i-\theta_j)\,, \qquad \mu=\frac{2\Gamma(\frac{3}{4})}{\sqrt{\pi}\Gamma(\frac{1}{4})}\,, \qquad (15)$$

with

$$F(\theta)=\frac{8\,\theta\tanh\left(\frac{\theta}{2}\right)\Gamma\left(\frac{3}{4}+\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{3}{4}-\frac{i\theta}{2\pi}\right)}{\pi\,\Gamma\left(\frac{1}{4}+\frac{i\theta}{2\pi}\right)\Gamma\left(\frac{1}{4}-\frac{i\theta}{2\pi}\right)}\,. \qquad (16)$$

The result (12) is a novel exact result for scattering amplitudes. It holds at strong coupling and in the collinear limit with the Lorentz invariant distance (11) held fixed, but otherwise arbitrary.

LONG AND SHORT DISTANCE ANALYSIS

Two very interesting regimes one might want to analyze in greater detail are the IR regime $z\gg1$ and the UV regime $z\ll1$. The former is dominated by the vacuum and is straightforwardly extracted from (12),

$$\mathcal{W}=1+O(e^{-2z})\,. \qquad (17)$$

The first deviation is controlled by the 2-particle integral analyzed in [16]. The trivialization of the Wilson loop in this limit is in perfect agreement with the expected behaviour of scattering amplitudes in the collinear limit. We note that it is achieved for τ much greater than the Compton wavelength $1/m$ of the lightest excitations.

As usual with such expansions, it is much more challenging to analyze the UV regime $z\ll1$. The point is that the higher-particle terms in the sum (12) are no longer suppressed at small z. Instead, they typically explode and the full series (12) must be resummed. The expectation – which we confirmed numerically on few examples – is that the n-particle contribution should follow the same trend and diverge as $\log(1/z)^{n/2}$ at small z. Clearly, without further information, it is challenging to predict what the true z dependence will be upon re-summing all contributions in (12). Fortunately, the twist-field interpretation introduced before sheds light on this issue and provides us with a physical picture for what the result should be, as we now explain.

The hexagonal Wilson loop is computed by a correlator of two twist operators in the $O(6)$ sigma model. In the short distance limit, these two operators are fused according to their OPE. Given that each operator has the effect of producing a conical excess of $\pi/4$, a pair of close by pentagons should act as an effective 'hexagon' operator producing a conical excess of $\pi/2$. In sum,

$$\phi_{\pentagon}(\sigma,\tau)\phi_{\pentagon}(0,0)\sim\frac{\log(1/z)^B}{z^A}\,\phi_{\hexagon}(0,0)\,, \qquad (18)$$

where $A=2\Delta_{\pentagon}-\Delta_{\hexagon}=2\Delta_{5/4}-\Delta_{3/2}$ with Δ_k the dimension of the twist field. The latter dimension has been known for a long time [18] and reads

$$\Delta_k=\frac{c}{12}\left(k-\frac{1}{k}\right)\,, \qquad (19)$$

where c is the central charge. In our case $c=5$ since the short distance CFT is that of 5 free massless bosons. All together, this leads to the prediction $A=1/36$.

The critical exponent B might look less familiar at first sight, as it is absent from the OPE of primaries in standard CFTs. It controls however a celebrated logarithmic enhancement which comes about because we are dealing with an asymptotically free theory and because our operators receive anomalous dimensions. (This is very well known from QCD and $B=-(2\gamma_{\pentagon}-\gamma_{\hexagon})/(2\beta_0)$ when expressed in terms of one-loop anomalous dimension and beta function coefficients, see e.g. [19].) Unfortunately, to our knowledge, these anomalous dimensions are not yet available from direct QFT computations. Still, it is possible to argue for a possible relation between them and the free energy of the $O(6)$ sigma model. We defer the details of the argument to the [20] and quote here the main conjecture $B=-3A/2$.

All in all, once inserted into the correlator (7) the OPE (18) generates the short distance behaviour

$$\mathcal{W}(z)=\frac{C}{z^{1/36}\log(1/z)^{1/24}}+\ldots \qquad (20)$$

where C is a constant which reflects the freedom in adopting different normalizations for the twist fields. For the problem at hand, the physical normalization is set by the collinear limit. Namely, it is unambiguously fixed by the long distance asymptotics (17). Because this condition is imposed in the IR, where the non-perturbative physics dominates, it is challenging, if not impossible, to fix C from the CFT directly. What we can do, however, is to fix our constant C numerically, through the exact series representation (12) truncated at some large number of particles. Dealing with the multi-dimensional integrals in (12) is numerically challenging. One way to do it is by Monte-Carlo, along the lines of [21] which analysed a similar (yet simpler) form factor sum related to a correlator in the 2d Ising model. In figure 2 we represent the numerical evaluation of the OPE series for increasingly small values of z. Once we subtract the leading and subleading logarithmic behavior from these numerics, $\log \mathcal{W}$ does approach a constant value. In this way we read $\log C \simeq -0.01$. It would be interesting to improve the numerics and get C with higher precision. Even better, it would be great if we could compute it analytically from the OPE sum (12).

CROSS-OVER AND CLASSICAL ENHANCEMENT

We are now in position to explain the prediction (4), (5) for the α' expansion of the six-gluon amplitude. Essentially what we want to show is that the short-distance $O(6)$ result (20) is enough to fix the prefactor dressing the minimal area prediction (1) in the collinear limit. In this limit the classical area A_6 falls off exponentially [8]

$$A_6 = O(e^{-\sqrt{2}\tau})\,, \qquad \tau \gg 1\,, \tag{21}$$

and similarly for the n-gluon area A_n in the multi-collinear limit $\tau_i \gg 1$. This behaviour is most clearly understood by recalling that the AdS_5 modes, which control the physics of the minimal surface, *all* have masses of order $O(1)$. (The lightest ones have mass $\sqrt{2}$ [14, 15]). Therefore, whatever survives in the collinear limit is necessarily captured by the prefactor dressing the minimal area prediction (1).

That the aforementioned prefactor is non-trivial in this limit directly follows from our previous analysis. The main point is that regardless of how big τ is, from the string α' expansion point-of-view, we always end up in the short-distance regime $z \ll 1$ of the $O(6)$ model. Indeed, for fixed τ and very large λ, the dimensionless distance z given by (11) is very small. In other words, $z \ll 1$ is the cross over domain between the non-perturbative regime $z \sim 1$ analyzed in this paper and the perturbative regime of the string worldsheet theory.

This being said, it is straightforward to convert the short-distance result (20) into the prediction (4), (5). It literally amounts to matching the latter against the former using the expressions (11) and (6) for the distance z and the mass gap m.

What is perhaps the most surprising outcome of all this analysis is the semi-classical enhancement stemming from the dynamics in the sphere. Namely, we see that the contribution from the sphere is visible already at the leading order in the $\sqrt{\lambda}$ expansion. Technically, this is a consequence of the fact that the twist fields carry scaling dimensions. Namely, our correlators are all dimensionless by construction and thence all distances come multiplied by m. In the short distance limit the overall dependence on the mass of the correlators can then be directly read off the OPE of the twist fields. In the case of n-gluon scattering, we would have $n-4$ pentagons that fuse together into an object with excess angle $\varphi = 2\pi \times \frac{n-4}{4}$. Keeping track of the mass dependence only we would then write

$$\underbrace{\phi_{\pentagon} \dots \phi_{\pentagon}}_{n-4} \sim m^{-(n-4)\Delta(\frac{5}{4})+\Delta(\frac{n}{4})}\phi_\varphi \tag{22}$$

which immediately leads to (3).

As a final remark, let us add that the O(6) model can also be used to predict the pre-factor dressing the strong coupling result (3) in the multi-collinear limit for any n-gon. To leading order at strong coupling, it should relate to the correlation function $\langle \phi_{\pentagon}(w_1)\dots\phi_{\pentagon}(w_{n-4})\phi_\varphi(\infty)\rangle_{\text{CFT}}$ in the free theory whose computation should lead to a beautiful mathematical problem in classical Liouville theory [22].

CONCLUSIONS

In this paper we start unveiling the structure of scattering amplitudes at strong coupling in planar $\mathcal{N}=4$ SYM theory beyond the minimal area paradigm. In particular, we have seen how strong coupling dynamics might challenge our intuition about scattering amplitudes, or their dual description in terms of Wilson loops, already in such a seemingly simple regime as the collinear limit. The rich behaviour we observed directly reflected the strong IR effects on the dual world-sheet which come about because the colour flux tube of the theory is infinite and its spectrum effectively gapless at strong coupling. These features will survive beyond the planar limit and are common to some other strongly coupled flux tubes, see e.g. [24].

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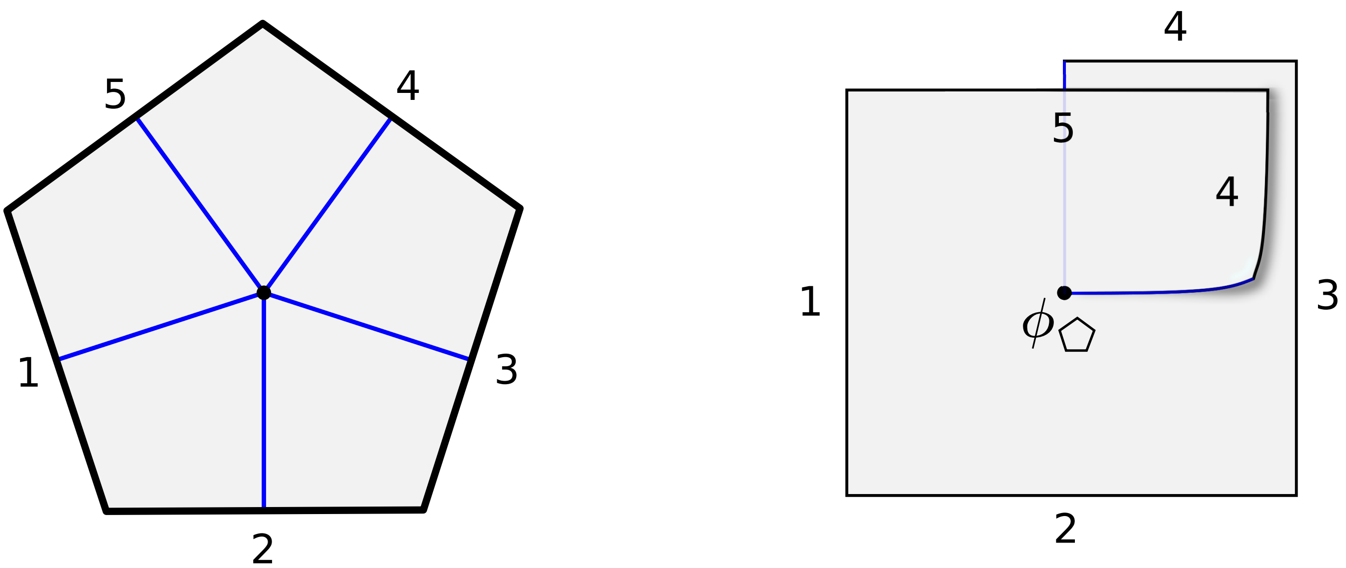
FIGURES

FIG. 1. (a) The world-sheet of the string ending on a pentagon can be viewed as made out of five quadrants. (b) Equivalently, we can engineer these five quadrants starting from the square by inserting the twist operator $\phi_{\pentagon}$.

FIG. 2. Plot of $\log \mathcal{W}$ truncated to $n_{\max}$ particles for z's as small as 10^{-6} and as large as $1/250$. The 2-particle approximation corresponding to $n_{\max} = 2$ (i.e., the upper line) already yields a reasonable estimate of the exact result; this is not unusual for such form factor representations, see e.g. [12].



(**a**) (**b**)

Figure 1 26Nov2014

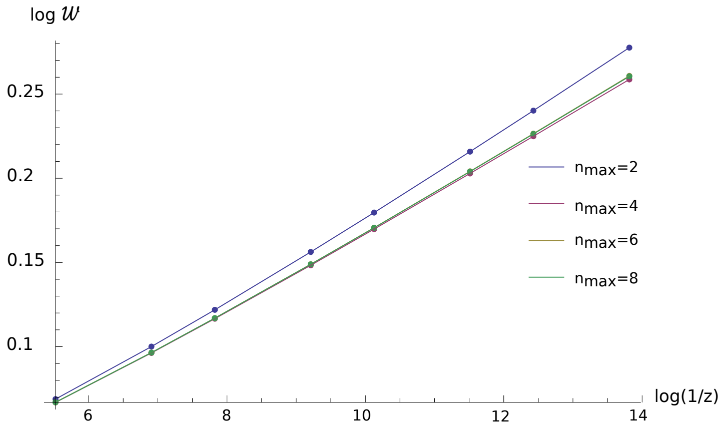


Figure 2 26Nov2014