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Universal Bounds on the Time Evolution of Entanglement Entropy

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Using relative entropy, we derive bounds on the time rate of change of geometric entanglement entropy for any relativistic quantum field theory in any dimension. The bounds apply to both mixed and pure states, and may be extended to curved space. We illustrate the bounds in a few examples and comment on potential applications and future extensions.

MOTIVATION AND INTRODUCTION

Recently, entanglement entropy has become an important theoretical tool for probing quantum physics in diverse situations. Of especial interest is the geometric entanglement entropy, S_V, associated with some spatial region, V. To wit, the von Neumann entropy of the reduced density matrix found by tracing out the degrees of freedom associated with the complementary region, $\bar{V}$.[1]

In this paper, we are interested in how causality and locality bound the rate of change of entanglement entropy, $\frac{\mathrm{d}S}{\mathrm{d}t}$, for excited states in a relativistic quantum field theory. As is now well-known, the geometric entanglement entropy is UV divergent in the vacuum, with the leading divergence proportional to the area of ∂V. Because of this UV sensitivity, one might question whether there are any interesting bounds at all; however, entropy differences are frequently finite for reasonable states, and therefore one should expect that $\frac{\mathrm{d}S}{\mathrm{d}t}$ is UV finite for reasonable states. This is also supported by some previous explicit calculations, cf. [2–5].

There are two relevant bodies of research in the literature. Firstly, there are bounds on $\frac{\mathrm{d}S}{\mathrm{d}t}$ for finite-dimensional nonrelativistic quantum mechanical systems. The most relevant to us is the proof of the small incremental entangling (SIE) conjecture in [6], building on work in [7]. The SIE conjecture states that for a four-part system $aABb$ evolving with Hamiltonian of the form $H = H_{aA} + H_{bB} + H_{AB}$,[2] the maximum growth of the entanglement entropy of aA is bounded [7]:

$$\left.\frac{\mathrm{d}S_{aA}}{\mathrm{d}t}\right|_{\max} \leq k\,\|H_{AB}\|\log d \qquad d = \min(d_A, d_B), \tag{1}$$

where k is an order unity constant, $\|H_{AB}\|$ is the operator norm of the interacting Hamiltonian, and the maximum is taken over all states. This can be used to argue that if one state obeys the area law, then all adiabatically connected states do as well [6, 7]; for a lattice system $\log d \simeq A \log d_s$, where d_s is the dimension of each lattice site's Hilbert space and A is the area measured in lattice units.

Unfortunately, it is difficult to directly apply this to quantum field theory, since even in lattice QFT the per-site Hilbert space is infinite dimensional.[3] Moreover, Lorentz invariance would enter into this argument only indirectly in the form of the Hamiltonian. Finally, let us note that there are states for which $\frac{\mathrm{d}S}{\mathrm{d}t}$ *is* UV divergent [9–11], and thus we expect that this probably is not even the right starting point. These examples have divergent stress tensors $T_{\mu\nu}$, and since the accessible phase space grows with energy scale, it is perhaps not surprising that $\frac{\mathrm{d}S}{\mathrm{d}t}$ diverges.

The second body of literature concerns fundamental relativistic bounds on the transmission rate of classical information—[12] gives an extensive discussion. In so far as the von Neumann entropy is the quantum analogue of the classical Shannon entropy, and that many bounds on classical information carry over to analogous bounds on quantum information [13], it is natural to ask whether these bounds have quantum anologues as bounds on $\frac{\mathrm{d}S}{\mathrm{d}t}$. The most important bound for us derives from the Bekenstein bound [14]:

$$H \leq \frac{2\pi E R}{\hbar c}, \tag{2}$$

where H is the thermodynamic entropy, E the energy of some object that can be circumscribed by a radius R ball. While this bound originated from black hole thermodynamics, it is supposed to be valid for any system one can throw into a black hole. If one considers information transmission via material transport, then one finds that [15]

$$\dot{I} \leq \frac{2\pi E}{\hbar}, \tag{3}$$

where $\dot{I}$ is the classical communication rate measured in "nats" per unit time.[4]

[1] We do not worry about issues associated with the ability to decompose the Hilbert space into a tensor product, $\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_{\bar{V}}$. See [1], for a recent discussion.
[2] Systems a and b are called ancilla, since they do not directly interact with each other.
[3] Although see [8] for a regularization scheme that uses a finite dimensional Hilbert space.
[4] A nat is $(\ln 2)^{-1} \approx 1.44$ bits.

Unfortunately, the precise range of applicability and validity of the Bekenstein bound is obscured by ambiguities in defining all three related quantities: H, E, and R. The original argument for the bound has also been challenged [16, 17]; see [18, 19] for recent defenses of the bound.

Fortunately, positivity of relative entropy provides an apodictic quantum analogue of the Bekenstein bound, which is not plagued by the same ambiguities [20, 21].[5] Since the original Bekenstein bound immediately led to a bound on the transmission of classical information, one should guess that the new refined version should imply a bound on $\frac{\mathrm{d}S}{\mathrm{d}t}$. In fact, the calculation is not as straightforward as the classical case, because we must carefully formulate bounds that subtract off contributions from the vacuum. Instead of using the positivity of relative entropy, we primarily use the monotonicity property.

DERIVATION

Causal Domains

We begin our derivation by first noting that we are working with a relativistic QFT in d-dimensional Minkowski space. We are interested in the entanglement entropy of a region V as a function of t. Let us consider evaluating $\frac{\mathrm{d}S}{\mathrm{d}t}$ as usual in a limiting procedure via

$$\frac{\mathrm{d}S_V(t)}{\mathrm{d}t} = \lim_{\delta t\to 0} \frac{S_V(t+\delta t) - S_V(t)}{\delta t}. \tag{4}$$

Since entropy differences are finite for reasonable states, we expect this to be finite for "nice" states. We spend most of our effort manipulating the entropy difference in the numerator.

Let Σ_t denote the spatial slice at time t, so that $V \subset \Sigma_t$. In this language, $V' \subset \Sigma_{t+\delta t}$ is the time translation of V and $\frac{\mathrm{d}S}{\mathrm{d}t} = \lim_{\delta t\to 0}(S_{V'} - S_V)/\delta t$. The causal domain of V, $\mathcal{D}(V) = \mathcal{D}^+(V) \cup \mathcal{D}^-(V)$, is given by the set of events for which either the past or future lightcone intersects Σ_t as a subset of V. The entanglement entropy S_V more correctly is a function of $\mathcal{D}(V)$, since changes in the slicing Σ_t that keep ∂V fixed effect unitary transformations on the density matrix and leave S_V invariant.

Thus, we can deform the spatial slices inside ∂V or outside ∂V at the two times without changing the answer. It seems convenient to deform the two slices as shown in Figure 1. We decompose the slices into an invariant spatial region B, followed by two (in the limit) null regions C and D, and another invariant spatial region E. The total state on $BCDE$ and $BC'D'E$ will be pure if the total system is in a pure state. This slightly singular evolution[6] gives us two states related by a unitary transformation that acts only on the the CD space:

$$|\psi_{BC'D'E}\rangle = U_{CD}\,|\psi_{BCDE}\rangle\,. \tag{5}$$

The original density matrices for V, ρ_V and $\rho_{V'}$, are related by unitary transformations to

$$\rho_V = U_1 \rho_{BC} U_1^\dagger \qquad \rho_{V'} = U_2 \rho_{BC'} U_2^\dagger. \tag{6}$$

Hence, the entropy is the same. Note that the transformation from ρ_{BC} to $\rho_{BC'}$ looks like a quantum operation that depends on the state of C. The regions B and E seem to play the role of ancilla, although keep in mind that we are going to be taking the limit as $\delta t \to 0$ and these regions all depend on δt.

Formally, we can define the above regions as follows:

$$\begin{aligned} &C = \partial\mathcal{D}^+(V) \cap \overline{\mathcal{D}^-(V')},\ C' = \partial\mathcal{D}^-(V') \cap \overline{\mathcal{D}^+(V)} \\ &D = \partial\mathcal{D}^+(\bar V) \cap \overline{\mathcal{D}^-(\bar V')},\ D' = \partial\mathcal{D}^-(\bar V') \cap \overline{\mathcal{D}^+(\bar V)} \\ &B \subset \mathcal{D}^-(V') \cap \mathcal{D}^+(V),\ \partial B = \partial\mathcal{D}^-(V') \cap \partial\mathcal{D}^+(V). \end{aligned} \tag{7}$$

The various regions are illustrated in Figure 1 for the half space. We now need to bound $S_{BC'} - S_{BC}$ in the limit of small δt. Note these definitions suggest a clear generalization to curved background metrics.

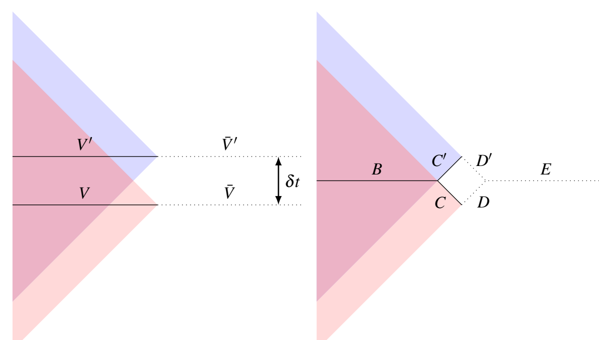

FIG. 1. Entanglement entropy is invariant under different spatial slicings that preserve the causal domain. Thus, we may deform the evolution from S_V to $S_{V'}$ into evolution from S_{BC} to $SC_{BC'}$ as shown above for the half space. This isolates all of the interesting dynamics into a δt-size "diamond".

Relative Entropy

Recently, it was pointed out that the relative entropy furnishes a more precise version of the Bekenstein bound

[5] In [22] another related inequality, which we do not find useful here, was called a "Bekenstein bound".

[6] See the recent paper [23], for some interesting results and subtleties related to null surfaces.

[21]. Recall that the relative entropy is a measure of the distinguishability of a density matrix ρ from a density matrix σ given by[7] [13]

$$S(\rho\|\sigma) = \operatorname{tr}\rho\log\rho - \operatorname{tr}\rho\log\sigma. \tag{8}$$

Note the asymmetry between ρ and σ.

The relative entropy satisfies two inequalities [13] that are important for our purposes. First, Klein's inequality: $S(\rho\|\sigma) \geq 0$ with equality if and only if $\rho = \sigma$. Second, the relative entropy monotonically decreases under partial tracing: $S(\rho_{\alpha\beta}\|\sigma_{\alpha\beta}) \geq S(\rho_\alpha\|\sigma_\alpha)$. Heuristically, decreasing the number of degrees of freedom one can access decreases distinguishability.

As noted in [21], if we write $\sigma = Ne^{-K}$,[8] then the relative entropy can be cleverly rewritten as

$$S(\rho\|\sigma) = \Delta\langle K\rangle - \Delta S, \tag{9}$$

where Δ indicates the difference of the quantity when evaluated in state ρ from state σ. We will always take σ to be the reduced density matrix one gets from the vacuum, for which K is the modular Hamiltonian. Then the nonnegativity of the relative entropy implies an upper bound on the regulated (vacuum-subtracted) entropy ΔS,

$$\Delta S \leq \Delta\langle K\rangle. \tag{10}$$

In the cases where we understand the modular Hamiltonian K, this bears a remarkable similarity to the original Bekenstein bound [21].

Bounds

We can now use the monotonicity property of relative entropy for the regions defined above. First note that the monotonicity condition $S(\rho_{BCD}\|\sigma_{BCD}) \geq S(\rho_B\|\sigma_B)$ implies

$$\Delta S_{BCD} - \Delta S_B \leq \Delta\langle K_{BCD}\rangle - \Delta\langle K_B\rangle. \tag{11}$$

We also have an equivalent bound for the complementary regions:

$$\Delta\langle K_E\rangle - \Delta\langle K_{CDE}\rangle \leq \Delta S_E - \Delta S_{CDE} \tag{12}$$

We want to relate the LHS of the first inequality to the RHS of the second. If the total state of the QFT is pure, then the two quantities are equal since $S = \bar{S}$ for a pure state; but if the total state is mixed, for instance thermal, then we have to work a little harder. First note that strong subadditivity (SSA) of entanglement implies

$$S_E - S_{CDE} \leq S_{BCD} - S_B. \tag{13}$$

Unfortunately, SSA does not directly apply to the regulated entanglement entropy. In this case, however, purity of the vacuum implies

$$S_E^{\text{vac}} - S_{CDE}^{\text{vac}} = S_{BCD}^{\text{vac}} - S_B^{\text{vac}}, \tag{14}$$

and therefore

$$\Delta S_E - \Delta S_{CDE} \leq \Delta S_{BCD} - \Delta S_B. \tag{15}$$

This allows us to write

$$\begin{aligned}\Delta\langle K_E\rangle - \Delta\langle K_{CDE}\rangle &\leq \Delta S_E - \Delta S_{CDE} \\ &\leq \Delta S_{BCD} - \Delta S_B \leq \Delta\langle K_{BCD}\rangle - \Delta\langle K_B\rangle\end{aligned} \tag{16}$$

Interestingly, this inequality holds as long as either ρ or σ come from a pure state. Dividing by δt and taking δt to zero, this becomes an upper and lower bound on the normal derivative of the regulated entanglement entropy in terms of normal derivatives of modular hamiltonians.

Monotonicity implies

$$S(\rho_B\|\sigma_B) \leq S(\rho_{BC}\|\sigma_{BC}) \leq S(\rho_{BCD}\|\sigma_{BCD}) \tag{17}$$

together with

$$S(\rho_B\|\sigma_B) \leq S(\rho_{BC'}\|\sigma_{BC'}) \leq S(\rho_{BCD}\|\sigma_{BCD}), \tag{18}$$

as well as the equivalent relations for the complementary region. Judicious use of the inequalities allows us to write

$$S(\rho_{BC'}\|\sigma_{BC'}) - S(\rho_{BC}\|\sigma_{BC}) \geq S(\rho_B\|\sigma_B) - S(\rho_{CDE}\|\sigma_{CDE}) - \Delta\langle K_{BC}\rangle + \Delta\langle K_{DE}\rangle \tag{19a}$$

$$S(\rho_{BC'}\|\sigma_{BC'}) - S(\rho_{BC}\|\sigma_{BC}) \geq S(\rho_E\|\sigma_E) - S(\rho_{BCD}\|\sigma_{BCD}) + \Delta\langle K_{BC'}\rangle - \Delta\langle K_{D'E}\rangle. \tag{19b}$$

[7] We use $S(\cdot\|\cdot)$ instead of $S(\cdot|\cdot)$ to distinguish the relative entropy from the conditional entropy.

[8] The normalization N can be fixed by demanding σ have unit trace.

While the two inequalities hold separately, it is convenient to add them to find

$$\begin{aligned}2[S(\rho_{BC'}\|\sigma_{BC'}) - S(\rho_{BC}\|\sigma_{BC})] \geq\\ \Delta\langle K_{BC'}\rangle - \Delta\langle K_{D'E}\rangle - \Delta\langle K_{BC}\rangle + \Delta\langle K_{DE}\rangle\\ + \Delta\langle K_{B}\rangle - \Delta\langle K_{CDE}\rangle - \Delta\langle K_{BCD}\rangle + \Delta\langle K_{E}\rangle\\ + [\Delta S_{BCD} - \Delta S_E + \Delta S_{CDE} - \Delta S_B].\end{aligned} \quad (20)$$

The last term in brackets we can drop since it is positive definite from (16). Using the same techniques, we can find an upper bound as well:

$$\begin{aligned}2[S(\rho_{BC'}\|\sigma_{BC'}) - S(\rho_{BC}\|\sigma_{BC})] \leq\\ \Delta\langle K_{BC'}\rangle - \Delta\langle K_{D'E}\rangle - \Delta\langle K_{BC}\rangle + \Delta\langle K_{DE}\rangle\\ - \Delta\langle K_{B}\rangle + \Delta\langle K_{CDE}\rangle + \Delta\langle K_{BCD}\rangle - \Delta\langle K_{E}\rangle.\end{aligned} \quad (21)$$

Let us define the time and normal derivatives as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}\Delta S}{\mathrm{d}t} = \lim_{\delta t\to 0}\frac{S_{BC'} - S_{BC}}{\delta t} \quad (22a)$$

$$\frac{\mathrm{d}\Delta S}{\mathrm{d}x_\perp} = \lim_{\delta t\to 0}\frac{\Delta S_{BCD} - \Delta S_B}{\delta t} \quad (22b)$$

$$\frac{\mathrm{d}\Delta \bar{S}}{\mathrm{d}x_\perp} = -\lim_{\delta t\to 0}\frac{\Delta S_{CDE} - \Delta S_E}{\delta t}, \quad (22c)$$

with the obvious parallel definitions for Ks. Note that time translation invariance of the vacuum means that the vacuum subtraction drops out from the time derivatives; this is *not* true for the normal derivative since the vacuum entanglement is not invariant under increases in the region size. Also, note the $x_\perp$ is oriented outward from V.

The normal derivative defined in (22) deserves some explication. For an arbitrary region at constant t with a smooth boundary $\mathrm{d}x_\perp$ is a normal shift of the boundary. To be precise, if we consider some F which is a functional of the entangling surface parametrized by $x^\mu(s_a)$, with $\mu = 0,\dots,d-1$, $a = 1,\dots d-2$ then we have

$$\frac{\mathrm{d}F}{\mathrm{d}x_\perp} \equiv \int \mathrm{d}^{d-2}s\,\frac{\delta F}{\delta x^\mu(s)}\frac{\omega^{\mu\nu}\hat{t}_\nu}{|\omega|}, \quad (23)$$

with $\hat{t}^\mu$ being the unit time vector, and $\omega_{\mu\nu} = \epsilon_{\mu\nu\rho_1\dots\rho_{d-2}}\frac{\partial x^{\rho_1}}{\partial s^1}\cdots\frac{\partial x^{\rho_{d-2}}}{\partial s^{d-2}}$. With this notation, Equation (16) becomes

$$\frac{\mathrm{d}\Delta\langle\bar{K}\rangle}{\mathrm{d}x_\perp} \leq \frac{\mathrm{d}\Delta\bar{S}}{\mathrm{d}x_\perp} \leq \frac{\mathrm{d}\Delta S}{\mathrm{d}x_\perp} \leq \frac{\mathrm{d}\Delta\langle K\rangle}{\mathrm{d}x_\perp}, \quad (24)$$

and the two inequalities (20) and (21) become upper and lower bounds on $\frac{\mathrm{d}S}{\mathrm{d}t}$:

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Delta\langle K\rangle + \Delta\langle\bar{K}\rangle\Big) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x_\perp}\Big(\Delta\langle K\rangle - \Delta\langle\bar{K}\rangle\Big)\\ \frac{\mathrm{d}S}{\mathrm{d}t} &\leq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Delta\langle K\rangle + \Delta\langle\bar{K}\rangle\Big) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x_\perp}\Big(\Delta\langle K\rangle - \Delta\langle\bar{K}\rangle\Big),\end{aligned} \quad (25)$$

where note that $\frac{\mathrm{d}(K-\bar{K})}{\mathrm{d}x_\perp}$ is nonnegative from (24). In fact, there are many other bounds one can derive from monotonicity; however, in practice these seem to be the tightest bounds on $\frac{\mathrm{d}S}{\mathrm{d}t}$. These bounds hold universally for any unitary Lorentz invariant theory, but for the cases in which the modular Hamiltonian is local and known, we may write them directly in terms of the stress tensor and simplify them further. We shall now consider the two better known cases in turn.

ILLUSTRATION

The Half-Space

In our first example the region V is the half-space $x_1 < X$. It is a quite general result [24, 25] that for this geometry the modular hamiltonian corresponding to the vacuum state necessarily becomes the boost charge. That is,

$$\Delta K = 2\pi\int_{x_1<X}\mathrm{d}^{d-1}x\,(X - x_1)\,T^{00}(t,\vec{x}), \quad (26a)$$

$$\Delta \bar{K} = 2\pi\int_{x_1>X}\mathrm{d}^{d-1}x\,(x_1 - X)\,T^{00}(t,\vec{x}). \quad (26b)$$

A simple computation yields

$$\begin{aligned}\frac{\mathrm{d}\langle K\rangle}{\mathrm{d}t} &= -2\pi P^1 & \frac{\mathrm{d}\langle\bar{K}\rangle}{\mathrm{d}t} &= 2\pi\bar{P}^1\\ \frac{\mathrm{d}\Delta\langle K\rangle}{\mathrm{d}X} &= 2\pi P^0 & \frac{\mathrm{d}\Delta\langle\bar{K}\rangle}{\mathrm{d}X} &= -2\pi\bar{P}^0\end{aligned} \quad (27)$$

with P^1 the momentum along x_1 of the half-space, P^0 its energy, along with $\bar{P}^0$ and $\bar{P}^1$ the energy and momentum of its complement. Defining the total energy $E_T = P^0 + \bar{P}^0$ and total momentum $P^1_T = P^1 + \bar{P}^1$, the bounds become

$$-2\pi P^1 + \pi(P^1_T - E_T) \leq \frac{\mathrm{d}S}{\mathrm{d}t} \leq -2\pi P^1 + \pi(P^1_T + E_T) \quad (28)$$

Note the qualitative similarity with the classical bound (3).

The Ball

Now consider the case where the region V is the ball of radius R centered at the origin. If we are dealing with a *conformal* field theory, we may use a conformal mapping from the Rindler wedge onto the causal development of the ball to obtain the modular hamiltonian [26, 27],

$$\begin{aligned}\Delta K &= \pi\left(R\,P^0 - K^0/R\right)\\ &= \pi\int_{r<R}\mathrm{d}^{d-1}x\,\frac{R^2-r^2}{R}\,T^{00}(t,\vec{x}) \qquad (29)\\ \Delta\bar{K} &= -\pi\left(R\,\bar{P}^0 - \bar{K}^0/R\right)\\ &= \pi\int_{r>R}\mathrm{d}^{d-1}x\,\frac{r^2-R^2}{R}\,T^{00}(t,\vec{x}). \qquad (30)\end{aligned}$$

Notice that for $r\simeq R$ we recover the result of the previous section. As before we can use conservation of the stress-tensor to obtain

$$\begin{aligned}\frac{\mathrm{d}\langle K\rangle}{\mathrm{d}t} &= -\frac{2\pi}{R}D, \qquad \frac{\mathrm{d}\langle\bar{K}\rangle}{\mathrm{d}t} = \frac{2\pi}{R}\bar{D}\\ \frac{\mathrm{d}\Delta\langle K\rangle}{\mathrm{d}R} &= \pi P^0 + \pi K^0/R^2, \qquad (31)\\ \frac{\mathrm{d}\Delta\langle\bar{K}\rangle}{\mathrm{d}R} &= -\pi\bar{P}^0 - \pi\bar{K}^0/R^2,\end{aligned}$$

with D the dilatation charge. In this way the bounds become

$$\begin{aligned}&-\frac{2\pi}{R}D + \frac{\pi}{2}\left(\frac{2D_T}{R} - E_T - \frac{K_T^0}{R}\right)\\ &\qquad\le\frac{\mathrm{d}S}{\mathrm{d}t}\le -\frac{2\pi}{R}D + \frac{\pi}{2}\left(\frac{2D_T}{R} + E_T + \frac{K_T^0}{R^2}\right) \qquad (32)\end{aligned}$$

We can imagine increasing the radius of the sphere and simultaneously translating it to obtain the half-space. In this limit we have

$$D\to R\,P^1, \qquad K_T^0\to R^2E_T, \qquad (33)$$

and we recover the bounds (28).

CONCLUSION

We have derived bounds on the entangling rate valid for any unitary Lorentz invariant quantum field theory in any dimension. We shall not show it here, but we have checked they are satisfied in all cases where we were able to easily test them. The bound can be thought of as a quantum version of the structurally similar bound on classical information in (3).

As it stands, our bounds hold even for theories without a local stress-tensor—such as defect or boundary CFTs. This goes some way to explaining why our bounds involve global charges even when the modular hamiltonian has a local expression. But it also suggests these bounds can be made stronger. For instance, considering a distant perturbation that increases the total energy, we expect that $\frac{\mathrm{d}S}{\mathrm{d}t}$ should vanish until signals from the perturbing event could possibly reach the region, at least for local field theories. Unfortunately, our bound does not seem to account for this aspect of causality, since independently of distance these perturbations still affect global charges such as total energy.

One idea for improving the bounds is to focus our attention on the dynamics inside the small causal diamond at the boundary of the causal developments of two Cauchy slices separated by a small δt, such as the diamond bounded by regions C ,C' ,D, and D' in Figure 1. In this small region, we are probing the UV dynamics of the theory. Assuming a free UV fixed point, then it seems that to leading order the process can only be a "swap gate": $D\to C'$ and $C\to D'$. Using this should be enough to derive a stronger, local bound. And yet, we should offer a word of caution: simple dimensional analysis seems to preclude a linear and local bound, at least for the half-space geometry, unless we are willing to introduce some cutoff dependence; although something non-linear such as

$$\left|\frac{\mathrm{d}S}{\mathrm{d}t}\right|^2\le\frac{\mathrm{d}^2K}{\mathrm{d}t\,\mathrm{d}x}\simeq\int_{\partial V}T^{00}. \qquad (34)$$

is perfectly fine. In fact a bound on classical information very similar to the above appears in the literature; see [12], and references therein. Let us also note, one may derive a bound that maximizes over the Hilbert space like that in (1) by using Bousso's covariant entropy bound [28] as a cutoff; however, such a bound, being in Planck units, would have limited utility.

Another possibility is to consider bounds on the second time derivative. If we consider the relative entropy between states at times t and $t+\delta t$ it is easy to derive

$$\frac{\mathrm{d}S}{\mathrm{d}t}=\frac{\mathrm{d}\langle K\rangle}{\mathrm{d}t}, \qquad \frac{\mathrm{d}^2S}{\mathrm{d}t^2}\le\frac{\mathrm{d}^2\langle K\rangle}{\mathrm{d}t^2} \qquad (35)$$

Unfortunately, K here is the modular hamiltonian for the system in the state at time t, which is inaccessible in general. We have also tried an approach in the same lines as those in this paper, by considering three closely spaced Cauchy slices, and using monotonicity of relative entropy. However, and quite generally, we were not able to find any such bound.

One obvious extension of our work here is to examine what the bounds imply for holographic entanglement entropy [29–31]. The bound, or a suitable extension of it for curved space, may have implications for black hole evaporation and the recent black hole entanglement crisis [32–36]. As in the discussion of [6, 7], we may also use the bound to tell us about entanglement in the vacuum of adiabatically connected theories.

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