Challenge to the a Theorem in Six Dimensions

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A challenge to the a-theorem in six dimensions

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The possibility of a strong a-theorem in six dimensions is examined in multi-flavor ϕ^3 theory. Contrary to the case in two and four dimensions, we find that in perturbation theory the relevant quantity $\tilde{a}$ increases monotonically along flows away from the trivial fixed point. $\tilde{a}$ is a natural extension of the coefficient a of the Euler term in the trace anomaly, and it arises in any even spacetime dimension from an analysis based on Weyl consistency conditions. We also obtain the anomalous dimensions and beta functions of multi-flavor ϕ^3 theory to two loops. Our results suggest that some new intuition about the a-theorem is in order.

INTRODUCTION

The counting of degrees of freedom in quantum field theories (QFTs) is of paramount importance in understanding their structure and phases. In particular, it is often of interest to understand how low-energy, long-range "IR" degrees of freedom might be related to the underlying microscopic "UV" degrees of freedom. For example, in quantum chromodynamics we observe mesons, hadrons, *etc.* at low energies, but believe them to consist of the quarks and gluons of the microscopic theory.

A rather good understanding of QFT degrees of freedom exists in two dimensions. There, a quantity can be defined that undergoes a monotonically decreasing renormalization group (RG) flow from a critical point in the UV to a critical point in the IR. At the critical points the quantity is stationary with respect to variations in scale, and becomes the central charge c of the Virasoro algebra of the corresponding conformal field theory (CFT), which is also the coefficient of the topological term (the Ricci scalar) in the two-dimensional trace anomaly. This is the result of Zamolodchikov [1].

In the four-dimensional case, which is of great interest to particle physicists, results are not so definitive. Cardy suggested [2] that the four-dimensional analog of c is the coefficient of the (topological) Euler term in the four-dimensional trace anomaly, a. In fact, it was shown, using heat kernel methods for field theories on curved backgrounds [3] and Weyl consistency conditions [4], that a perturbative version of Zamolodchikov's result holds [4, 5]. More recently, non-perturbative methods have made headway into a weaker version of the a-theorem, where, instead of establishing a monotonic flow, a relation between the value of a at the critical points is argued for [6], namely that $a_{\text{UV}} > a_{\text{IR}}$.

In this paper we investigate the possibility of an a-theorem in six dimensions. The weak version of the a-theorem in $d = 6$ was studied in [7] using the methods of [6], but no definitive conclusion could be reached. The six-dimensional case is of interest in clarifying the basic structure of QFT in general. Interesting CFTs arise in $d = 6$ by string-theoretic constructions and the low energy dynamics of M5 branes. The spectrum of operators in these theories can be studied without any knowledge of what Lagrangian "describes" them, but not much is known about RG flows to and from these theories. As far as Lagrangian theories are concerned, the ϕ^3 theory is of interest as the unique classically scale-invariant theory in $d = 6$. Its RG running can be easily studied with well-known methods, and expectations stemming from the intuition behind the a-theorem can be put to the test.

In this work we examine the possibility of an a-theorem in six-dimensional multi-flavor ϕ^3 theory. We find that the opposite conclusion of two- and four-dimensional a-theorems may be drawn in six dimensions, at least in perturbation theory. More specifically, we find that the candidate for an a-theorem singled out by the Weyl consistency conditions *increases* monotonically along the renormalization group flow out of the trivial fixed point. To come to our conclusion, we use the methods developed in [4] (see also [8, 9]). This involves constraining the form of the Weyl anomaly utilizing the Abelian nature of the Weyl group; because the Weyl group is related to a change of scale, this imposes constraints on the RG properties of quantities in the anomaly and, in particular, produces a candidate for an a-theorem. In section we explain this method and in section we show that a quantity that becomes a at critical points *increases* monotonically along the renormalization group flow, at least in perturbation theory. We discuss the implications of this result in section .

WEYL CONSISTENCY CONDITIONS

In general, the classical symmetries of a theory may be broken for its renormalized Green functions. The form of this "anomaly" is constrained by the algebra of the symmetry group: for an infinitesimal transformation generated by Δ^a acting on the generating functional of renormalized Green functions Γ, we have

$$\left[\Delta^a, \Delta^b\right] \Gamma = i f^{abc} \Delta^c \Gamma, \tag{1}$$

where f^{abc} are the structure constants of the symmetry group. These are the so-called Wess–Zumino consistency conditions [10].

It is useful to study a QFT on a curved background with spacetime-dependent couplings so that the metric $\gamma_{\mu\nu}(x)$ and couplings $g^I(x)$ act as sources for the stress-energy tensor and the operators (labelled by I) in the Lagrangian, respectively. We only consider the case of dimensionless couplings, so that in perturbation theory all the interaction terms in the Lagrangian are nearly marginal. We introduce their infinitesimal local Weyl transformations as

$$\begin{aligned} \Delta_\sigma \gamma^{\mu\nu}(x) &= 2\sigma(x)\gamma^{\mu\nu}(x)\,, \\ \Delta_\sigma g^I(x) &= \sigma(x)\beta^I(x)\,, \end{aligned} \tag{2}$$

where $\beta^I(x)$ is the beta function of the associated coupling and depends on x only through $g^I(x)$. The group of Weyl transformations is Abelian and has only a single generator. Thus, Eq. (1) becomes

$$\left[\Delta_\sigma, \Delta_{\sigma'}\right] \Gamma = 0, \tag{3}$$

where it is understood that $\Gamma = \Gamma[\gamma^{\mu\nu}, g^I]$, indicating the dependence on the metric and couplings as background fields. If the flat-background theory is a CFT, (3) has been solved in [11–13].

The response of Γ to Weyl rescaling produces the Weyl anomaly

$$\begin{aligned} \Delta_\sigma \Gamma[\gamma^{\mu\nu}, g^I] = \int d^d x \sqrt{\gamma}\, \sigma \sum_i \big(a_i A_i[\gamma^{\mu\nu}] \\ + b_i B_i[\gamma^{\mu\nu}, g^I] + c_i C_i[g^I]\big)\,, \end{aligned} \tag{4}$$

where d is the dimension of spacetime (presumed even here), and i is a counting index. The form of Eq. (4) is fixed by general diffeomorphism invariance and power counting. A_i, B_i and C_i are functions of the metric and couplings, and by dimensional analysis must include d spacetime derivatives. The A_i do not contain any derivatives on couplings and are therefore of $d/2$-th order in curvature, the C_i are functions of d derivatives on the couplings, and, finally, the B_i are functions of both curvature and derivatives of the couplings. The coefficients a_i, b_i and c_i are all functions of the couplings only. In particular, the A_i contain the Euler term in d dimensions with coefficient $(-1)^{d/2} a$, so that at fixed points $a > 0$.

Now, the consistency conditions from Eq. (3) impose integrability relations on the terms in Eq. (4). The relation of interest involves the coefficient of the Euler term in Eq. (4) and coefficients of terms in the B_i involving $H_{\mu\nu}$, a generalization of the Einstein tensor to d dimensions found by Lovelock [14]. In even dimensions, it was shown that an integrability relation exists [15] involving a such that [16]

$$\partial_I \tilde{a} = \frac{1}{d}(\chi_{IJ} + \partial_I w_J - \partial_J w_I)\beta^J, \tag{5}$$

which can be brought to the form

$$\frac{d\tilde{a}}{d\log\mu} = \frac{1}{d}\chi_{IJ}\beta^I\beta^J, \tag{6}$$

where μ is the renormalization scale. Here χ_{IJ} and w_I are tensors in the space of couplings and they appear in the coefficients of the B_i terms $\partial_\mu g^I \partial_\nu g^J H^{\mu\nu}$ and $\nabla_\mu \partial_\nu g^I H^{\mu\nu}$ in Eq. (4), where $\tilde{a}$ is a scalar in the space of couplings [17]. Both quantities may be related to correlation functions of the stress-energy tensor, its trace, and the operators in the QFT. Since $\beta^I = 0$ at the critical points, $\tilde{a}$ is stationary with respect to variations of scale there. In fact

$$\tilde{a} = a + w_I\beta^I + \sum_p a_p, \tag{7}$$

where a_p are some of the a_i's in (4) that vanish at criticality. Hence, at critical points, $\tilde{a} = a$. Moreover, Eq. (6) that $\tilde{a}$ satisfies is very similar to that found for the analogous quantity in two dimensions in [1]. This suggests $\tilde{a}$ as the analog of Zamolodchikov's monotonically-decreasing function in two dimensions.

While the consistency conditions impose this integrability relation, a strong version of the a-theorem must establish that the "metric" χ_{IJ} is positive-definite, which then proves that $d\tilde{a}/d\log\mu > 0$. To compute χ_{IJ}, other methods must be used.

RESULTS FROM THE EFFECTIVE POTENTIAL

To compute χ_{IJ} in six dimensions, we work with the conformally-coupled scalar field theory [18] on a curved background with Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu\phi_i\partial_\nu\phi^i\gamma^{\mu\nu} + \tfrac{1}{5}R\phi_i\phi^i) + \tfrac{1}{3!}g_{ijk}\phi^i\phi^j\phi^k, \tag{8}$$

with the fields, spacetime metric, and couplings all implicitly functions of spacetime. The generic coupling constants g^I are here specifically g_{ijk} with the label $I = (ijk)$. At the classical level the term $\partial_\mu g^I \partial_\nu g^J H^{\mu\nu}$, where the

Lovelock tensor in $d = 6$ is

$$
\begin{aligned}
H_{\mu\nu} = {} & (R^2 - 4R_{\kappa\lambda}R^{\kappa\lambda} + R_{\kappa\lambda\rho\sigma}R^{\kappa\lambda\rho\sigma})\gamma_{\mu\nu} \\
& - 4RR_{\mu\nu} + 8R_{\mu\kappa}R^{\kappa}{}_{\nu} + 8R^{\kappa\lambda}R_{\kappa\mu\lambda\nu} \\
& - 4R_{\kappa\lambda\rho\mu}R^{\kappa\lambda\rho}{}_{\nu},
\end{aligned}
$$

clearly does not show up, so $\chi_{IJ} = 0$ at the classical level. To find the first (quantum) contributions to χ_{IJ}, we can compute the effective potential in a curved background with the loop expansion to two loop order or, equivalently, second order in $\hbar$ [19].

The six-dimensional two-loop effective potential can be computed using heat kernel methods in dimensional regularization [3, 20, 21]. This is done in position space, and it involves the computation of the two-loop graph and the associated graph with the counterterm insertion in Fig. 1. These two graphs generate the full two-loop effective potential. Such computations have been explained

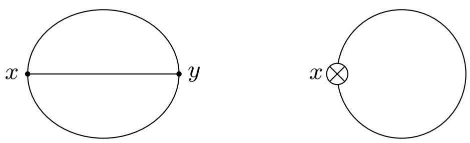
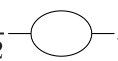
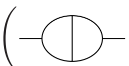
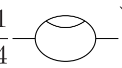


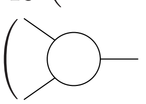
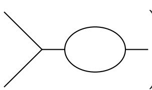
FIG. 1. The diagrams that need to be considered at the two-loop level.

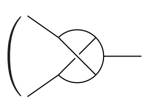
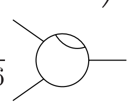
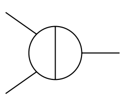
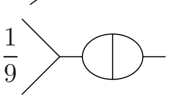
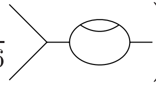
in great detail in [3]. The case of $d = 6$ single-flavor ϕ^3 theory with x-independent coupling has been worked out in [20, 21], and we find agreement with these papers in cases checked.

From our computation we determine the one- and two-loop anomalous dimensions of the elementary fields ϕ_i and the beta functions for the couplings g_{ijk}:

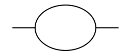
$$\gamma^{(1)} = \frac{1}{64\pi^3}\frac{1}{12}$$  , (9)

$$\gamma^{(2)} = \frac{1}{(64\pi^3)^2}\frac{1}{18}\Big($$  $$-\frac{11}{24}$$  $$\Big),$$ (10)

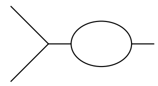
$$\beta^{(1)} = -\frac{1}{64\pi^3}\Big($$  $$-\frac{1}{12}$$  $$\Big),$$ (11)

$$\beta^{(2)} = -\frac{1}{(64\pi^3)^2}\frac{1}{2}\Big($$  $$-\frac{7}{36}$$  $$+\frac{1}{2}$$  $$-\frac{1}{9}$$  $$+\frac{11}{216}$$  $$\Big).$$ (12)

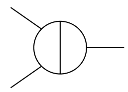
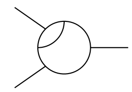
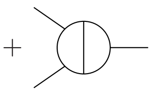
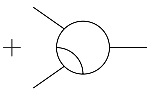
To our knowledge the multi-component two-loop results (10) and (12) have not appeared for general coupling g_{ijk} before in the literature, although they may be extracted from Ref. [22]. Here we have used diagrammatic notation to indicate the corresponding contraction of the couplings, *e.g.*,

 $$= g_{ikl}g_{jkl},$$ (13)

and permutations of the free indices in the wavefunction-renormalization corrections to the beta function are understood. For example,

 $$= g_{ijl}g_{lmn}g_{kmn} + \text{permutations}.$$ (14)

Eq. (11) generalizes the single field result of [23] (see also [20, 21, 24, 25]) to the multi-field case, and agrees with the results of [22, 26, 27] [28]. The first contribution to (12) is non-planar. For the seemingly asymmetric vertex corrections in (12) (the second and third terms) a symmetrization is understood; for example,

 $\sim$  $+$  $+$ 

where "∼" means "the left-hand side stands for the right-hand side."

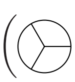
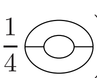
Our main result is the two-loop expression for the "metric" in theory space:

$$\chi^{(2)}_{IJ} = -\frac{1}{(64\pi^3)^2}\frac{1}{3240}\delta_{IJ}\,. \tag{15}$$

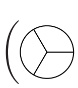
With this result and the one-loop beta function (11) we can use the consistency condition (5) to compute $\tilde{a}$ at three loops. For this we also need $w^{(2)}_I$, which we can obtain from the same heat-kernel computation [29]:

$$w^{(2)}_I = -\frac{1}{(64\pi^3)^2}\frac{1}{6480}g_I. \tag{16}$$

We find [30]

$$\tilde{a}^{(3)} = \frac{1}{(64\pi^3)^3}\frac{1}{77\,760}\Big($$  $$-\frac{1}{4}$$  $$\Big).$$ (17)

The three-loop contribution to the coefficient of the Euler term a can also be computed using the relation between $\tilde{a}$ and a of the form (7) found in [15]. We find

$$a^{(3)} = \frac{1}{(64\pi^3)^3}\frac{7}{388\,800}\Big($$  $$-\frac{1}{4}$$  $$\Big).$$ (18)

Clearly, both $\tilde{a}$ and a increase in the flow out of the trivial fixed point.

One may wonder if the results in (15) and (17) depend on the renormalization scheme we used to compute the two-loop effective potential. Actually, Eq. (5) (and thus Eq. (6)) is invariant under the choice of renormalization scheme. The individual terms are, however, scheme-dependent. The corresponding arbitrariness is of the form $\delta\tilde{a} = z_{IJ}\beta^I\beta^J$ and $\delta\chi_{IJ} = \beta^K\partial_K z_{IJ} + z_{KJ}\partial_I\beta^K + z_{IK}\partial_J\beta^K$, where z_{IJ} is an arbitrary regular symmetric function of the couplings. Since the arbitrariness in $\tilde{a}$ vanishes (quadratically) when fixed points are approached, it cannot change the nature of the flow in the vicinity of fixed points.

DISCUSSION

Using the result of our computation, Eq. (15), in the evolution equation (6), or equivalently, the explicit form of $\tilde{a}$ in (17), it is apparent that in perturbation theory the quantity $\tilde{a}$ in Eq. (6) actually *increases* as one decreases the renormalization scale. This is contrary to intuition developed in $d = 2, 4$, where $\tilde{a}$ seems to count the degrees of freedom in a QFT.

This result should be taken with two comments in mind. Firstly, that the result is a perturbative one, and we cannot say anything about non-perturbative regimes of six-dimensional QFTs. And secondly, that there are no known perturbative critical points other than the single, trivial one at $g_{ijk} = 0$, so in this context renormalization group flows do not connect pairs of critical points [31]. However, it is still true that, with Eq. (6) identical in $d = 2, 4$, and 6 dimensions, the strong version of the a-theorem holds perturbatively in $d = 2, 4$ but not in $d = 6$ [32].

We do not know the reason for this difference. One possibility may be the unstable nature of the theory we are considering. After all, a cubic potential is unbounded from below. However, the state with $\langle\phi_i(x)\rangle = 0$ is perturbatively stable and our computations are valid only in the perturbative regime. Moreover, the analogous case in four dimensions, the inverted quartic potential, is also unstable, but does satisfy a perturbative a-theorem (since the metric in theory space, χ_{IJ}, is perturbatively positive in four dimensions, independently of the sign of the quartic couplings). Another possibility is that a flow between critical points is required for an a-theorem to hold, but the only perturbatively-accessible critical point in the class of theories in Eq. (8) is the Gaussian fixed point at $g_{ijk} = 0$. But, again comparing to known cases, a perturbative strong a-theorem holds for scalar theories in four dimensions, in spite of only having a Gaussian fixed point at the origin of coupling-constant space.

a-theorems can be used to restrict proposed dynamics of strongly interacting models [2]. If our result that $\tilde{a}$ increases in flows towards the IR holds even non-perturbatively, one could envision using it to restrict putative dynamics of strongly interacting QFTs in $d = 6$. In this sense, the existence of an "anti-a-theorem" may be just as useful as a normal one. It is therefore of interest to investigate renormalization group flows in the vicinity of non-Lagrangian critical QFTs that have been formulated through studies of low energy dynamics of M5 branes. Of course, another avenue of research is the establishment of the theorem non-perturbatively in the presence of a flow between fixed points.

Finally, let us note that there may be quantities that reduce to a at fixed points that are not of the form of $\tilde{a}$ (up to the ambiguity $z_{IJ}\beta^I\beta^J$), but that do undergo monotonically-decreasing RG flow towards the IR. This possibility was explored in [33].

For our computations we relied heavily on *Mathematica* and the package `xAct`. We would like to thank Tom Appelquist, George Fleming, Ken Intriligator, Hugh Osborn, David Poland, and Jaewon Song for valuable discussions. We also thank Ken Intriligator and Hugh Osborn for comments on the manuscript.

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[16] These χ_{IJ} and w_I in $d = 6$ were denoted $\mathcal{H}^1_{IJ}$ and $\mathcal{H}^1_I$ respectively in [15].
[17] For more details the reader is referred to [15].

[18] We do not study fermions or vectors, which do not have interacting dynamics with classical scale invariance at the perturbative level in six dimensions.

[19] There is no quantum correction to χ_{IJ} at one loop order. We work in units where $\hbar = 1$ so that the loop-counting scheme is easier to use.

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[28] We thank J. A. Gracey and one of the referees for bringing these references to our attention.

[29] The fact that $w_I^{(2)} \sim g_I$ implies that $\partial_I w_J^{(2)} - \partial_J w_I^{(2)} = 0$, and also, by (5), that the flow is gradient. In multicomponent systems in four dimensions the gradient-property of the flow at low loop orders was considered long ago in [34].

[30] There is also a contribution to $\tilde{a}$ at zero coupling with $\tilde{a} = \frac{1}{64\pi^3}\frac{1}{9072}$ in the conventions of [15], which we use here.

[31] This does not mean that they do not exist. Non-trivial, perturbative flows between a UV and IR critical point have been studied in $6-\epsilon$ dimensions in the $O(N)$ model recently, as in [35]. It is an open question as to whether or not such results could be extended to six dimensions.

[32] Note that, as suggested in [7], the weak version of the a-theorem may still be true in $d = 6$ for flows at weak coupling connecting two critical points.

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