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Lorentz Invariance in Chiral Kinetic Theory

Jing-Yuan Chen,[1] Dam T. Son,[1] Mikhail A. Stephanov,[2,1] Ho-Ung Yee,[2,3] and Yi Yin[2]

[1]*Kadanoff Center for Theoretical Physics, University of Chicago, Chicago, Illinois 60637, USA*

[2]*Physics Department, University of Illinois at Chicago, Chicago, Illinois 60607, USA*

[3]*RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973-5000, USA*

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We show that Lorentz invariance is realized nontrivially in the classical action of a massless spin-$\frac{1}{2}$ particle with definite helicity. We find that the ordinary Lorentz transformation is modified by a shift orthogonal to the boost vector and the particle momentum. The shift ensures angular momentum conservation in particle collisions and implies a nonlocality of the collision term in the Lorentz-invariant kinetic theory due to side jumps. We show that 2/3 of the chiral-vortical effect for a uniformly rotating particle distribution can be attributed to the magnetic moment coupling required by the Lorentz invariance. We also show how the classical action can be obtained by taking the classical limit of the path integral for a Weyl particle.

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Introduction.—The parity-odd response of a chiral medium and its deep relationship to topology and quantum anomalies have attracted significant theoretical interest. Two such phenomena, the chiral magnetic and chiral vortical effects (CME and CVE), which is the appearance of nonzero current in a magnetic field or when the system is in rotation, have been considered some time ago in astrophysical context [1, 2]. More recently, the interest in such phenomena was rekindled by developments in various subfields of physics. It was observed that charge-dependent correlations can be used to detect the CME in heavy-ion collisions [3]. Independently, the chiral vortical effect has been found in a calculation using gauge/gravity duality [4, 5], and a general argument based on second law of thermodynamics was put forward in Ref. [6] to demonstrate the generality of this result. The recent experimental discovery of "3D graphene" [7, 8] brings closer the possibility of realizing the materials with non-trivial chiral properties, such as Weyl semimetals [9].

One promising approach to explore anomaly-related phenomena is the kinetic theory, which can go beyond the regime of thermodynamic equilibrium. This kinetic approach is applicable when the external fields and the interactions between the (quasi-)particles are sufficiently weak, so each particle can be considered as moving along a classical trajectory, punctuated by rare collisions. Between collisions, one has essentially a single-particle problem. The information about the quantum anomaly is encoded in the momentum-space Berry curvature [10]. The classical action for such a motion can be derived either from a single-particle quantum Hamiltonian [11] or, more directly, from field theory [12].

There is, however, a puzzling aspect of the kinetic theory: it does not have a manifest Lorentz symmetry, which it should inherit from the original quantum field theory. The issue was first raised in Ref. [12] by comparing the kinetic theory and field theory results for a Fermi-liquid at zero temperature and later, in Ref.[13], at finite temperature. In this Letter we confirm the suggestion made in Ref. [12] that Lorentz symmetry requires an additional magnetic moment coupling term in the classical action of the particle. Unexpectedly, we also find that the Lorentz transformation laws of the coordinates and momenta contain extra terms associated with particle spin. Another nontrivial consequence of the analysis is a magnetization current contribution to the total current, which is required to reproduce the correct (Lorentz-covariant) magnitude of the CVE.

Classical action..—We shall argue that the motion of a massless right-handed spin-$\frac{1}{2}$ particle in an external electromagnetic field is described, in the classical regime, by the following phase-space action,

$$\mathcal{I} = \int (\boldsymbol{p}+\boldsymbol{A})\cdot d\boldsymbol{x} - (\mathcal{E}+\Phi)dt - \boldsymbol{a_p}\cdot d\boldsymbol{p}, \qquad (1)$$

where $\boldsymbol{a_p}$ is the Berry connection such that

$$\boldsymbol{b} \equiv \boldsymbol{\nabla}\times\boldsymbol{a} = \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2}, \qquad \hat{\boldsymbol{p}} \equiv \frac{\boldsymbol{p}}{|\boldsymbol{p}|}, \qquad (2)$$

while the dispersion relation

$$\mathcal{E} \equiv |\boldsymbol{p}| - \frac{\hat{\boldsymbol{p}}\cdot\boldsymbol{B}}{2|\boldsymbol{p}|} \qquad (3)$$

is modified to linear order in the field by the magnetic moment coupling [12, 13] (see also Ref. [14] for a comprehensive review of earlier studies of massive fermions). Although we work in the convenient units $\hbar = c = 1$, it is easy to see, by restoring $\hbar$, that both the Berry connection term in Eq. (1) and the magnetic coupling term in Eq. (3) are of order $\mathcal{O}(\hbar)$. Later in the Letter, we will derive the action (1) from the Weyl Hamiltonian by taking the classical limit of a path integral, but for now we take it as the starting point.

Lorentz invariance.—To zeroth order in $\hbar$ the action, $\int(\boldsymbol{p}+\boldsymbol{A})\cdot d\boldsymbol{x} - (|\boldsymbol{p}|+\Phi)dt$, which is the action of a spinless particle, is invariant with respect to the infinitesimal

Lorentz boost

$$\delta_{\boldsymbol\beta}\boldsymbol{x} = \boldsymbol\beta t; \quad \delta_{\boldsymbol\beta} t = \boldsymbol\beta\cdot\boldsymbol{x}; \quad \delta_{\boldsymbol\beta}\boldsymbol{p} = \boldsymbol\beta|\boldsymbol{p}|;$$
$$\delta_{\boldsymbol\beta}\boldsymbol{B} = \boldsymbol\beta\times\boldsymbol{E}; \quad \delta_{\boldsymbol\beta}\boldsymbol{E} = -\boldsymbol\beta\times\boldsymbol{B}; \qquad (4)$$

The $\mathcal{O}(\hbar)$ terms in (1) are not invariant with respect to this boost , and the action changes by

$$\delta_{\boldsymbol\beta}\mathcal{I} = \int\left[\frac{\boldsymbol\beta\times\hat{\boldsymbol p}}{2|\boldsymbol p|}(\dot{\boldsymbol p} - \boldsymbol E - \hat{\boldsymbol p}\times\boldsymbol B) + \frac{\boldsymbol B\cdot\hat{\boldsymbol p}}{2|\boldsymbol p|}\boldsymbol\beta\cdot(\dot{\boldsymbol x} - \hat{\boldsymbol p})\right]dt. \qquad (5)$$

However, noting that the two expressions in parentheses are the variations of the $\mathcal{O}(\hbar^0)$ part of action with respect to $\boldsymbol x$ and $\boldsymbol p$ respectively, one can find a modified Lorentz transformation for $\boldsymbol x$ and $\boldsymbol p$

$$\delta'_{\boldsymbol\beta}\boldsymbol x = \boldsymbol\beta t + \frac{\boldsymbol\beta\times\hat{\boldsymbol p}}{2|\boldsymbol p|}; \qquad \delta'_{\boldsymbol\beta}\boldsymbol p = \boldsymbol\beta\mathcal{E} + \frac{\boldsymbol\beta\times\hat{\boldsymbol p}}{2|\boldsymbol p|}\times\boldsymbol B; \quad (6)$$

under which the action is invariant up to order $\hbar$ inclusively: $\delta'_{\boldsymbol\beta}\mathcal{I} = \mathcal{O}(\hbar^2)$.

Thus, the action (1) has, in fact, a hidden Lorentz invariance, under which the position and the momentum of the particle transform in a nontrivial manner. We now give a physical interpretation of the modified Lorentz transformations.

Angular momentum and side jump.—We will assume for simplicity that $\boldsymbol E = \boldsymbol B = 0$. Since the Berry connection comes into play when the particle changes its momentum, we consider an elastic scattering of two particles. For simplicity, consider the process in the center of mass frame, and assume zero impact parameter. The angular momentum conservation is trivial in this frame: $\boldsymbol J_{\rm in} = \boldsymbol J_{\rm out} = 0$ with both orbital $\boldsymbol L$ and spin $\boldsymbol S$ contributions vanishing before and after the collision.

Let us now perform a Lorentz boost along the the direction of motion of one of the incoming particles. Then the total angular momentum of incoming particles is still zero $\boldsymbol J_{\rm in} = 0$. However, the spins of the outgoing particles no longer cancel each other, since their momenta are not collinear in the new frame. That means that the orbital momentum of the outgoing pair should be nonzero, which would be impossible if the particle trajectories were going through a single collision point.

However, the modified Lorentz transformation in Eq. (6) shifts the trajectory in the direction perpendicular to the boost and the particle momentum: $\Delta\boldsymbol x = \boldsymbol\beta\times\hat{\boldsymbol p}/(2|\boldsymbol p|)$. Since the momenta of the particles, $\boldsymbol p$ and $-\boldsymbol p$, are opposite before the boost, the shifts are also opposite. As a result the two outgoing particles are moving in two parallel planes. It is easy to check that such a shift leads to a contribution to the orbital momentum

$$\boldsymbol L_{\rm out} = \frac{\boldsymbol\beta\times\hat{\boldsymbol p}}{|\boldsymbol p|}\times\boldsymbol p \qquad (7)$$

equal and opposite to the total spin of the outgoing particles

$$\boldsymbol S_{\rm out} = \delta_{\boldsymbol\beta}(\hat{\boldsymbol p}) = \frac{\boldsymbol\beta - \hat{\boldsymbol p}(\boldsymbol\beta\cdot\hat{\boldsymbol p})}{|\boldsymbol p|} = -\boldsymbol L_{\rm out}. \qquad (8)$$

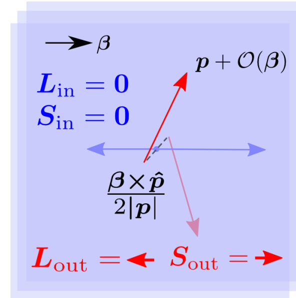

FIG. 1: (color online) Side jump

Therefore, collisions of two particles with spin involves a shift in the position. This is similar to the "side jump" phenomenon in impurity scatterings with spin-orbit interaction [15]. The magnitude of the side jump is frame-dependent and does not depend on the details of the collision. This phenomenon has a classical analog: the center of mass of a spinning extended particle is frame dependent [16]. We expect the side jump to be important for constructing Lorentz invariant chiral kinetic theory with collisions, and that in such a theory the collision kernel must be nonlocal in space and time.

Lorentz algebra.—We now check that the modified Lorentz transformations satisfy the algebra of the Lorentz group. For simplicity, we set the electromagnetic field to zero – similar results hold in the presence of the field. It is well-known that the commutator of the ordinary Lorentz transformations is a rotation. For example, $[\delta_{\boldsymbol\beta_1}, \delta_{\boldsymbol\beta_2}]\boldsymbol x = \boldsymbol\varphi\times\boldsymbol x$, where $\boldsymbol\varphi \equiv \boldsymbol\beta_2\times\boldsymbol\beta_1$. For the modified Lorentz transformation, however,

$$[\delta'_{\boldsymbol\beta_1}, \delta'_{\boldsymbol\beta_2}]\boldsymbol x = \boldsymbol\varphi\times\boldsymbol x + \hat{\boldsymbol p}\,\delta t\,; \quad [\delta'_{\boldsymbol\beta_1}, \delta'_{\boldsymbol\beta_2}]t = \delta t\,, \qquad (9)$$

where $\delta t = -\boldsymbol\varphi\cdot\hat{\boldsymbol p}/|\boldsymbol p|$. We see that the Lorentz algebra closes up to an additional shift δt and $\delta\boldsymbol x = \delta t\,\hat{\boldsymbol p}$ which, by virtue of the fact that $d\boldsymbol x = dt\,\hat{\boldsymbol p}$ on equations of motion, is an invariance of the action (for a classical trajectory it amounts to time reparametrization). Using this (gauge) freedom, one can accompany boost by such a transformation, i.e., define $\delta''_{\boldsymbol\beta}t = 0$ and $\delta''_{\boldsymbol\beta}\boldsymbol x = \delta'_{\boldsymbol\beta}\boldsymbol x - \hat{\boldsymbol p}\delta_{\boldsymbol\beta}t$, so that the algebra will close: $[\delta''_{\boldsymbol\beta_1}, \delta''_{\boldsymbol\beta_2}]\boldsymbol x = \boldsymbol\varphi\times\boldsymbol x$ and $[\delta''_{\boldsymbol\beta_1}, \delta''_{\boldsymbol\beta_2}]t = 0$. This would correspond (at $\boldsymbol B = 0$) to the representation of the Lorentz algebra found in Ref. [17].

Chiral vortical effect.—Another nontrivial consequence of the magnetic moment coupling is a contribution to the current which turns out to be essential for reproducing correct value of the chiral vortical effect.

The current is determined by variation of the action with respect to external gauge potential $\boldsymbol A$. The resulting single-particle current (in zero field) is given by

$$\boldsymbol J(\boldsymbol x, t) \equiv \left.\frac{\delta\mathcal{I}}{\delta\boldsymbol A(\boldsymbol x, t)}\right|_{\boldsymbol A=0} = \left(\hat{\boldsymbol p} - \frac{\hat{\boldsymbol p}}{2|\boldsymbol p|}\times\boldsymbol\nabla\right)\delta^3(\boldsymbol x - \boldsymbol x'(t)) \qquad (10)$$

where $\boldsymbol{x}'(t)$ is the position of the particle at time t. Consider now an ensemble of particles with a distribution function f. The corresponding current is given by

$$\boldsymbol{J}(\boldsymbol{x},t) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\Big(\hat{\boldsymbol{p}} f - \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}\times\boldsymbol{\nabla} f\Big). \tag{11}$$

The first term is the classical Liouville current, while the second term, which is due to the magnetic moment coupling, is $\mathcal{O}(\hbar)$. It is trivially conserved because it can be written as $\boldsymbol{\nabla}\times\boldsymbol{M}$, where

$$\boldsymbol{M} = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|} f, \tag{12}$$

is the total magnetization (the sum of the magnetic moments). However, this contribution is needed to make the current a Lorentz vector and, as we shall now show, to reproduce the correct magnitude of the CVE.

Consider a distribution f such that there exist a frame in which the distribution is isotropic in momentum. Denoting the energy of particles in this frame ϵ' we can write $f = f(\epsilon')$. Now consider a distribution which, in addition, varies very slowly in space because the velocity $\boldsymbol{u}$ of the frame in which the distribution in momentum is isotropic varies very slowly with space point $\boldsymbol{x}$. Since the distribution function is a Lorentz scalar we can write the distribution in the lab frame as $f = f(\epsilon')$, where $\epsilon' = \epsilon - \boldsymbol{p}\cdot\boldsymbol{u} - \lambda\hat{\boldsymbol{p}}\cdot\boldsymbol{\omega}$ is the energy in the locally comoving frame expressed in terms of the lab energy ϵ and momentum $\boldsymbol{p}$ and the helicity of the particle $\lambda = \frac{1}{2}$. The last term is present if the velocity distribution has vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{u}/2$ since the particle carries intrinsic angular momentum $\lambda\hat{\boldsymbol{p}}$.

The shift $-\lambda\hat{\boldsymbol{p}}\cdot\boldsymbol{\omega}$ arises naturally when f is a local equilibrium solution of Boltzmann equation. The detailed balance dictates that, for fermions, $\ln[f/(1-f)]$ is a linear function of the conserved quantities ϵ, $\boldsymbol{p}$ and angular momentum $\boldsymbol{j}$, i.e., $-\beta(\epsilon - \boldsymbol{p}\cdot\boldsymbol{u}_0 - \boldsymbol{j}\cdot\boldsymbol{\alpha})$ with some constants β, $\boldsymbol{u}_0$ and $\boldsymbol{\alpha}$. Inserting $\boldsymbol{j} = \boldsymbol{x}\times\boldsymbol{p} + \lambda\hat{\boldsymbol{p}}$ gives $-\beta(\epsilon - \boldsymbol{p}\cdot\boldsymbol{u} - \lambda\hat{\boldsymbol{p}}\cdot\boldsymbol{\alpha})$ with $\boldsymbol{u}\equiv\boldsymbol{u}_0 + \boldsymbol{\alpha}\times\boldsymbol{x}$. This means the equilibrium distribution could be inertially moving as well as rotating and that $\boldsymbol{\alpha} = \boldsymbol{\omega}$.

Substituting the distribution $f(\epsilon - \boldsymbol{p}\cdot\boldsymbol{u} - \lambda\hat{\boldsymbol{p}}\cdot\boldsymbol{\omega})$ into Eq. (11) and Taylor expanding to linear order in $\boldsymbol{u}$ and $\boldsymbol{\omega}$ one finds that magnetization current contributes 2/3 of the total current:

$$\begin{aligned}\boldsymbol{J} &= -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial f}{\partial\epsilon}\,\frac{1}{2}\left[\hat{\boldsymbol{p}}(\hat{\boldsymbol{p}}\cdot\boldsymbol{\omega}) - \hat{\boldsymbol{p}}\times\boldsymbol{\nabla}(\hat{\boldsymbol{p}}\cdot\boldsymbol{u})\right]\\ &= -\frac{\boldsymbol{\omega}}{2}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\partial f}{\partial\epsilon}\left[\frac{1}{3}+\frac{2}{3}\right].\end{aligned} \tag{13}$$

where we used the isotropy of f to replace $\hat{p}^i\hat{p}^j$ by $\delta^{ij}/3$ under the integral. Now using $\epsilon = |\boldsymbol{p}|$, taking the integral over angular directions of $\boldsymbol{p}$ and then integrating by parts, we find for the current:

$$\boldsymbol{J} = \frac{\boldsymbol{\omega}}{4\pi^2}\int_0^\infty d\epsilon\, 2\epsilon f, \tag{14}$$

which agrees with the expression for the CVE obtained from the CME by the substitution $\boldsymbol{B}\to 2\epsilon\boldsymbol{\omega}$ (for isotropic distributions) [11]. Such an agreement between the results in different frames is another manifestation of the Lorentz covariance of the current in Eq. (11).

Classical action from path integral.—We now show that the action (1), including the magnetic moment coupling, can be derived systematically from the path integral. This derivation is different but complementary to the previously developed wave-packet approach for massive fermions (see Ref. [14] for a review). Path-integral derivation was introduced for the free case in Ref. [11], however, the coupling to electromagnetic field was incorrectly assumed to be minimal. Here we show that a careful analysis reveals the presence of the magnetic moment coupling dictated by Lorentz invariance.

We start from a path-integral representation of an amplitude for the Weyl Hamiltonian in an external field

$$\mathcal{H} = \boldsymbol{\sigma}\cdot(\boldsymbol{p} - \boldsymbol{A}(\boldsymbol{x},t)) + \Phi(\boldsymbol{x},t) \tag{15}$$

where $\boldsymbol{x}$ and $\boldsymbol{p}$ are canonically conjugate operators of position and momentum: $[x^i, p^j] = i\delta^{ij}$. Inserting sums over complete sets of momentum and coordinate eigenstates, a transition amplitude can be rewritten as a matrix element of the path-ordered products of 2×2 matrices $e^{-i\mathcal{H}\Delta t}$, where $\boldsymbol{x}(t)$ and $\boldsymbol{p}(t)$ are now classical c-number variables of path integration. As in Ref. [11], we diagonalize each of these matrices along the path using $\boldsymbol{p}$-dependent matrix $V_{\boldsymbol{p}}$ satisfying $V_{\boldsymbol{p}}^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{p}V_{\boldsymbol{p}} = \sigma_3|\boldsymbol{p}|$.

In the classical regime, we can neglect off-diagonal elements of the propagator matrix and consider only the contribution given by the diagonal matrix elements between positive-energy eigenvectors of $\sigma_3|\boldsymbol{p}|$ which we denote as $[\ldots]_{++}$. The key ingredient for the magnetic moment coupling is found in the matrix element $\left[V_{\boldsymbol{p}}^\dagger e^{i\boldsymbol{\sigma}\cdot\boldsymbol{A}\Delta t}V_{\boldsymbol{p}-\Delta\boldsymbol{p}}\right]_{++}$ which we can evaluate using Gordon identity to linear order in $\Delta\boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p}'$:

$$\begin{aligned}\left[V_{\boldsymbol{p}}^\dagger e^{i\boldsymbol{\sigma}\cdot\boldsymbol{A}\Delta t}V_{\boldsymbol{p}-\Delta\boldsymbol{p}}\right]_{++} = u_{\boldsymbol{p}}^\dagger u_{\boldsymbol{p}'}\exp&\left[i\,\frac{\hat{\boldsymbol{p}}+\hat{\boldsymbol{p}}'}{2}\cdot\boldsymbol{A}\Delta t\right.\\ &\left.+\ \frac{\Delta\boldsymbol{p}\times\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|}\cdot\boldsymbol{A}\Delta t\right] + \mathcal{O}(\Delta\boldsymbol{p}^2,\Delta t^2).\end{aligned} \tag{16}$$

where $u_{\boldsymbol{p}}$ is the positive energy eigenvector—the solution of the Weyl equation: $\boldsymbol{\sigma}\cdot\boldsymbol{p}\,u_{\boldsymbol{p}} = |\boldsymbol{p}|u_{\boldsymbol{p}}$. The first term in the square brackets combines with neighboring factors $e^{-i|\boldsymbol{p}|\Delta t}$ in the path-ordered product to replace $|\boldsymbol{p}|$ with $|\boldsymbol{p}-\boldsymbol{A}|\approx\boldsymbol{p} - \hat{\boldsymbol{p}}\cdot\boldsymbol{A} + \mathcal{O}(\boldsymbol{A}^2/|\boldsymbol{p}|)$ [20].

Naively, we could neglect the last term in the square brackets in Eq. (16) because it contains an additional factor $\Delta\boldsymbol{p}$. However, $\boldsymbol{p}$ and $\boldsymbol{p}'$ are independent integration variables and the difference $\Delta\boldsymbol{p}$ is not small in general. Rather, it is the factor $\prod \exp(i\boldsymbol{p}\cdot\Delta\boldsymbol{x}) = \prod \exp(-i\boldsymbol{x}\cdot\Delta\boldsymbol{p})$ which, upon integration over $\boldsymbol{x}$, makes rapidly oscillating contributions at large $\Delta\boldsymbol{p}$ cancel out. If $\Delta\boldsymbol{p}$ multiplies a function of $\boldsymbol{x}$ the result of integration is the same as if we replaced $\Delta\boldsymbol{p}$ with $-i\partial/\partial\boldsymbol{x}$ as in this example:

$$\int dx\, e^{-ix\Delta p}\Delta p F(x) = -i\int dx\, e^{-ix\Delta p}\,\frac{dF(x)}{dx}\,. \qquad (17)$$

This relation is the path-integral representation of the canonical commutation relation between x and p (similar to the commutation relation between coordinate and velocity discussed in Ref. [18]). Thus we cannot consider $\Delta\boldsymbol{p}$ as small in the second term in Eq. (16) if $\boldsymbol{A}$ depends on $\boldsymbol{x}$. Replacing $\Delta\boldsymbol{p}$ with $-i\partial/\partial\boldsymbol{x}$ we find that this term contributes $i\hat{\boldsymbol{p}}\cdot\boldsymbol{B}/(2|\boldsymbol{p}|)\Delta t$ to the phase, representing the interaction energy of the particle's magnetic moment.

Finally, the factor $u^\dagger_{\boldsymbol{p}}u_{\boldsymbol{p}'} = \exp(-i\boldsymbol{a}_{\boldsymbol{p}}\cdot\Delta\boldsymbol{p})$ is the Berry phase. If we express it using the physical (gauge-invariant) momentum $\boldsymbol{P} = \boldsymbol{p} - \boldsymbol{A}$, we can, to linear order in $\boldsymbol{A}$, write

$$\begin{aligned}\langle\ldots u^\dagger_{\boldsymbol{p}}u_{\boldsymbol{p}'}\ldots\rangle &= \langle\ldots(1 - i\boldsymbol{a}_{\boldsymbol{p}}\cdot\Delta\boldsymbol{p})\ldots\rangle\\ &= \langle\ldots(1 - i(\boldsymbol{a}_{\boldsymbol{P}}\cdot\Delta\boldsymbol{P} + \Delta\boldsymbol{p}\times\boldsymbol{b}\cdot\boldsymbol{A})\ldots\rangle\\ &= \langle\ldots(1 + \boldsymbol{b}\cdot\boldsymbol{B} - i\boldsymbol{a}_{\boldsymbol{P}}\cdot\Delta\boldsymbol{P})\ldots\rangle\\ &= \langle\ldots(1 + \boldsymbol{b}\cdot\boldsymbol{B})e^{-i\boldsymbol{a}_{\boldsymbol{P}}\cdot\Delta\boldsymbol{P}}\ldots\rangle \qquad (18)\end{aligned}$$

where $\langle\ldots\ldots\rangle$ denote remaining factors and limits in the path integral and in the third line we replaced $\Delta\boldsymbol{p}$ with $-i\partial/\partial\boldsymbol{x}$ as before. We find that if we change variables to physical momentum $\boldsymbol{P}$, the factor $u^\dagger_{\boldsymbol{p}}u_{\boldsymbol{p}'}$, expanded to order $\Delta\boldsymbol{P}$, and under path integration, cannot be treated as a pure phase. The magnitude factor $(1 + \boldsymbol{b}\cdot\boldsymbol{B})$ in Eq. (18) combined with the path-integral measure $d\boldsymbol{x}\,d\boldsymbol{P}$ gives the correct conserved (up to the anomaly [10, 11]) Liouville measure for a Weyl particle.

Conclusions.—We have shown that the theory of a single particle with spin-1/2 and definite helicity can be made Lorentz-invariant if one includes one term in the action that corresponds to the interaction between the particle's magnetic moment with the magnetic field. The magnitude of the magnetic moment is completely determined by Lorentz invariance. We have also shown that the Lorentz transformations of the particle's coordinates and momentum components are nontrivial, and that they are related to the side jumps in scattering processes.

Although our action has Lorentz symmetry, it is not written in a manifestly Lorentz invariant manner. We are currently developing a manifestly Lorentz-invariant formulation, which will be reported elsewhere. It would also be interesting to generalize this analysis to higher dimensions and non-abelian anomalies [19].

From the equation of motion of a single particle one can go to the kinetic description in terms of a Boltzmann equation. We expect that the side jumps required by Lorentz invariance are necessary for the collision term in the Boltzmann equation to be consistent with angular momentum conservation. Understanding how to write down a correct kinetic theory of chiral particles, including their interactions, will provide a link, so far missing, between quantum field theory and hydrodynamics with anomalies and would allow, in particular, treatments of processes far from equilibrium in theories with anomalies.

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[20] Gauge invariance means that tracking $\mathcal{O}(\boldsymbol{A}^2)$ terms would require us to consider contributions to the action at order $\boldsymbol{B}^2$, beyond the $\mathcal{O}(\hbar)$ order we are working within. Note that we can always choose $\boldsymbol{A}$ to vanish in a given space-time point and, because $\boldsymbol{B}$ is small compared to the relevant scale $\boldsymbol{p}^2$, $\boldsymbol{A}$ will remain small compared to $|\boldsymbol{p}|$ in a patch much larger than de Broglie wavelength $1/|\boldsymbol{p}|$ around such a point.