# Topological Magnetoelectric Effect Decay

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# Topological Magneto-Electric Effect Decay

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We address the influence of realistic disorder and finite doping on the effective magnetic monopole induced near the surface of an ideal topological insulator (TI) by currents that flow in response to a suddenly introduced external electric charge. We show that when the longitudinal conductivity $\sigma_{xx} = g(e^2/h) \neq 0$, the apparent position of a magnetic monopole initially retreats from the TI surface at speed $v_M = \alpha c g$, where $\alpha$ is the fine structure constant and $c$ is the speed of light. For the particular case of TI surface states described by a massive Dirac model, we further find that the temperature $T = 0$ Hall currents vanish when the external potential is screened.

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*Introduction–* When a time-reversal-symmetry breaking perturbation opens a gap in the surface state spectrum of a three-dimensional topological insulator (TI)[1, 2], surface Hall currents and orbital magnetism are induced by electrical perturbations. This magneto-electric coupling effect can be attractively described[3] by adding a $E \cdot B$ term to the electromagnetic Lagrangian. The duality of the resulting *axion electrodynamics* model[4] leads to a curious *topological magneto-electric* effect[5, 11, 12] in which an electric charge placed above the TI surface induces Hall currents and associated orbital magnetization that appears to emanate from a magnetic monopole below the surface.

Experimental demonstration of the topological magneto-electric effect relies on the achievement of a (half-quantized) quantum anomalous Hall effect on a TI surface. There has been steady progress in this direction [6], culminating in the recent experiments on Cr-doped (Bi,Sb)$_2$Te$_3$ [7]. However, all experimental work so far has found considerable longitudinal conductivity, even in Ref. [7]. Longitudinal transport is in fact an omnipresent experimental reality that is not captured by the axion electrodynamics model.

In this paper we study the modifications of the surface magneto-electric response brought about by a finite TI surface state longitudinal conductivity, $\sigma_{xx} = g(e^2/h)$. The two main issues that have to be addressed in this context are $(i)$ the dynamics of magneto-electric effect during the onset of screening on a topological surface, and $(ii)$ the description of the equilibrium surface currents which remain after screening has been fully established. The importance of screening for the description of the image magnetic monopole was first emphasized in Ref. [8]. However, transient phenomena were not considered at all, and the previous description of magnetic response to a general potential was incomplete as we explain below. We find[9] that when the external charge is placed more than a screening length $\lambda$ from the surface, the monopole moves away with velocity $v_M = 2\pi\sigma_{xx} = \alpha c g$, $\alpha$ being the fine structure constant. In the long-time limit the screened external potential becomes static. In this case we find that the orbital magnetization response depends on details of the surface state electronic structure, and that it vanishes identically in the particular case of a two-dimensional massive Dirac model with temperature $T = 0$ and a Fermi level position outside the gap.

*Macroscopic Theory–* We assume that the TI surface has a well defined surface Hall conductivity and diffusion constant; this assumption can fail for very well developed quantum Hall effects. We introduce an external charge $Qe$ located a distance $d$ from the TI surface; since we wish to treat this object as a source of macroscopic inhomogeneity rather than as a contribution to the disorder potential we imagine that $Q \gg 1$ and that $d$ is longer than microscopic lengths. Currents flow in the TI surface in response to the electric fields from the external charge and the screening charges that accumulate in the TI surface layer. Working in two-dimensional momentum space and assuming that the total electric field changes sufficiently slowly with time, we use the continuity equation to conclude that

$$\frac{\partial n_q^{2D}}{\partial t} = -2\pi\sigma_{xx} q(Qe^{-qd} + n_q^{2D}) - D_F q^2 n_q^{2D}. \qquad (1)$$

where $d$ is the distance from the surface to the external charge $Qe$, $q$ is the two-dimensional wave vector, and $n_q^{2D}$ is the Fourier transform of the induced surface state density. The longitudinal conductivity, $\sigma_{xx}$, is related to the diffusion coefficient via the Einstein relation $\sigma_{xx} = (\partial n/\partial \mu) e^2 D_F$.

If we assume that the external charge is introduced suddenly at time $t = 0$, then by solving Eq. (1) and using the Poisson equation we find that the total electric potential on the surface is

$$\phi_{tot}(q,t) = 2\pi e Q e^{-qd} \left( \frac{1 + (q\lambda)^{-1} e^{-(D_F q^2 + v_M q)t}}{q + \lambda^{-1}} \right), \qquad (2)$$

where $\lambda^{-1} = 2\pi\sigma_{xx}/D_F$ is the screening wave vector and $\lambda$ the screening length. For $t \to \infty$, longitudinal currents vanish due to the Einstein-relation and $\phi_{tot}$ reduces to the standard result for 2D Thomas-Fermi screening.

This expression for the total electric potential is particularly illuminating in the limit in which the separation $d$ between the external charge and the TI surface is much larger than the screening length $\lambda$:

$$\phi_{tot}(q,t) = \frac{2\pi e Q}{q}\ \exp(-q(d+v_M t)). \tag{3}$$

The potential at time $t$, which controls the instantaneous Hall currents and hence the instantaneous magnetization, is identical to that from an external charge that is located not at vertical position $d$, but at vertical position $d+v_M t$. As shown elsewhere[5], because of the magneto-electric duality of axion electrodynamics, these Hall currents give rise to a magnetization that is identical to that produced by a magnetic monopole located at a distance $d+v_M t$ below the TI surface. Hence we arrive at the conclusion that screening is initially equivalent to the *apparent* monopole position moving away from the TI surface with velocity $v_M$. Currents flow until macroscopic electric fields vanish. The topological magneto-electric effect is therefore purely transient when $d \gg \lambda$.

Since the external potential remains large for $t \to \infty$ at length scales smaller than $\lambda$, there will be a macroscopic orbital magnetic response to the screened potential if the contributions to the transverse current from the screened electric field and from the induced density inhomogeneities do not cancel. Is there an Einstein relation for Hall currents? Below we use a quantum kinetic theory to answer this question microscopically. We conclude that the answer is no in general. Both drift and diffusion type terms do appear. The contribution to the Hall current from density inhomogeneities can be understood as being due to a non-uniform internal magnetic moment [13] density. For the particular case of a two-dimensional massive Dirac equation model for TI surface states, however, we explain below that the drift and diffusion Hall currents *do cancel* when the carrier density is non-zero and $T \to 0$, further limiting the experimental accessibility of the topological magneto electric effect. Quantitative estimates intended to assess its observability are provided in the Supplemental Material.[14]

*Do transverse currents flow in equilibrium?–* After screening is fully established, the electrochemical potential is constant. Transport currents are absent, and current flow on the surface, if any, can only exist due to non-uniform magnetization. In addressing these currents, it is helpful to first consider the simplified problem illustrated in Fig. 1 in which the electrostatic potential depends on only one coordinate and has a jump from $\phi_L$ to $\phi_R$ near an interior point. Currents can flow only in the narrow region where the potential has a gradient. Current flows in the $y$-direction in the transition region only if the values of the magnetization in the uniform regions are different.

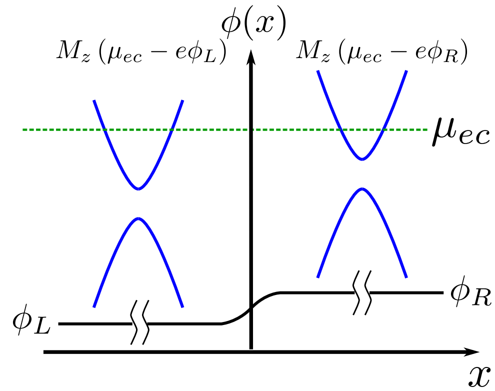


FIG. 1: (color online) Calculation of the 2D massive Dirac model equilibrium magnetization in the presence of an electric potential step, $\phi(x)$. The latter is assumed to have values $\phi_{L,R}$ at $x \to \mp\infty$, respectively. The green dashed line indicates the position of the electrochemical potential, $\mu_{ec}$, constant in the surface. Out-of-plane magnetization values in the macroscopic regions on both sides of the step are given by $M_z(\mu_{ec} - e\phi_{L,R})$, where $M_z(\mu)$ is the $z$-component of the magnetization of a uniform 2D massive Dirac model with chemical potential $\mu$.

These values can be found by solving the *thermodynamic* problem in those regions, which is insensitive to boundary effects. If the common electrochemical potential on the surface is $\mu_{ec}$, the problem then reduces to the calculation of magnetization of 2D massive Dirac fermions at chemical potentials $\mu_{L,R} = \mu_{ec} - e\phi_{L,R}$.

If a non-zero magnetization exists in the *absence* of an external magnetic field in a uniform sample of area $A$ and with chemical potential $\mu$ we can find it using the thermodynamic expression:

$$M_z = -\frac{1}{A}\left(\frac{\partial \Omega}{\partial B_z}\right)_{T,\mu} = -\frac{1}{A}\lim_{B_z \to 0}\frac{\Omega(B_z) - \Omega(-B_z)}{2B_z}, \tag{4}$$

where $\Omega$ is the thermodynamic potential of the system, which can be calculated knowing the spectrum of Landau levels on the surface: [15]

$$\begin{aligned} \varepsilon_n &= \mathrm{sgn}(n)\sqrt{\frac{2\hbar^2 v^2}{\ell^2}|n| + \Delta^2}, \quad n \neq 0, \\ \varepsilon_0 &= -\mathrm{sgn}(B_z)|\Delta|, \quad n = 0. \end{aligned} \tag{5}$$

In Eq. (5) $v$ is the Dirac velocity, $\Delta$ is the mass parameter, $n$ is the Landau level number, and $\ell = \sqrt{\hbar c/|e|B_z}$ is the magnetic length. Since only the position of the zeroth Landau level depends on the *sign* of the magnetic field, only this level contributes to $M_z$. If the chemical potential is in either the conduction or valence band, the temperature is zero, and disorder is neglected, no change in the thermodynamic potential occurs. It follows that the magnetization does not depend on the value of the chemical potential as long as it is outside of the gap, and therefore that no current flows in the region of varying electric potential. We now derive this result microscopically and discuss conditions under which the cancelation is incomplete.

*Microscopic Theory*–When time-reversal symmetry is broken, the surface states of a 3D strong topological insulators are approximately described[1, 2, 16] by a 2D massive Dirac Hamiltonian:

$$H_{\rm MD} = \int d^2\mathbf{r}\,\Psi^\dagger\left(\mathbf{B_p}\boldsymbol{\sigma} + e\phi_{ext} + U_{\rm dis}\right)\Psi. \tag{6}$$

Here $\mathbf{B_p} = (vp_x, vp_y, \Delta)$ is a $\mathbf{p}$-dependent effective Zeeman field which acts on electron spins. The mass term $\Delta$ breaks time-reversal symmetry and arises[7] from proximity exchange coupling to an insulating ferromagnet. $U_{\rm dis}$ describes an atomic scale disorder potential which we take to be created by short-range impurities with concentration $n_{imp}$: $U_{\rm dis} = \sum_i u\delta(\mathbf{r} - \mathbf{r}_i)$, with $\mathbf{r}_i$ denoting the positions of impurities.

We calculate the currents flowing in equilibrium using a quasiclassical kinetic equation for the electron density matrix $\hat{f}$:

$$\partial_t \hat{f}_{\mathbf{p}} + \frac{1}{2}\left\{\partial_{\mathbf{p}}(\mathbf{B_p}\cdot\boldsymbol{\sigma}), \partial_{\mathbf{r}}\hat{f}_{\mathbf{p}}\right\} + \frac{i}{\hbar}[\mathbf{B_p}\cdot\boldsymbol{\sigma}, \hat{f}_{\mathbf{p}}] + e\mathbf{E}_{\rm tot}\cdot\partial_{\mathbf{p}}\hat{f}^{\rm eq}_{\mathbf{p}} = \hat{I}_{st}. \tag{7}$$

In the above equation $[A, B]$ and $\{A, B\}$ stand respectively for the commutator and anticommutator of $A$ and $B$, $\mathbf{E}_{tot}$ is the screened electric field, and $\hat{I}_{st}$ is the collision integral.[17] We allow for the possibility of an imperfect quantum Hall effect by considering the case in which carriers are present in at least one of the bands due either to doping or to finite temperature.

The distribution function can be decomposed into scalar and vector pieces, $\hat{f}_{\mathbf{p}} = n_{\mathbf{p}} + \boldsymbol{\sigma}\cdot\mathbf{f_p}$, and $\mathbf{f_p}$ further separated into contributions parallel and perpendicular to $\mathbf{B_p}$, $f^{\parallel}_{\mathbf{p}}\mathbf{b_p}$ and $\mathbf{f}^{\perp}_{\mathbf{p}}$. ($\mathbf{b_p}$ is a unit vector in the direction of $\mathbf{B_p}$.) In this parameterization $(n_{\mathbf{p}}, f^{\parallel}_{\mathbf{p}}) = (f^c_{\mathbf{p}} \pm f^v_{\mathbf{p}})/2$, where $f^v_{\mathbf{p}}$ and $f^c_{\mathbf{p}}$ are valence and conduction band occupation numbers, while $\mathbf{f}^{\perp}_{\mathbf{p}}$ describes interband coherence. The kinetic equation for the full density matrix can be separated into a set of four equations for these components.

The model's intraband response is entirely standard [18], except that scattering on the Fermi surface is influenced by the inner product of the momentum-dependent conduction band states. For the conduction band, which we assumed to have a Fermi surface, we find that

$$\frac{\partial f^c_{\mathbf{p}}}{\partial t} + \mathbf{v_p}\nabla f^c_{\mathbf{p}} + e\mathbf{E}_{tot}\mathbf{v_p}\frac{\partial n_F(B_{\mathbf{p}} - \mu)}{\partial B_{\mathbf{p}}} = -\frac{\pi n_{imp}u^2}{\hbar}\int\frac{d^2p'}{(2\pi\hbar)^2}\delta(B_{\mathbf{p}} - B_{\mathbf{p}'})(1 + \mathbf{b_p}\mathbf{b_{p'}})(f^c_{\mathbf{p}} - f^c_{\mathbf{p}'}), \tag{8}$$

where $\mathbf{v_p} = v^2\mathbf{p}/B_{\mathbf{p}}$ is the band velocity appropriate for the conduction band of Hamiltonian (6), and $n_F(\epsilon)$ is the Fermi-Dirac distribution function. It follows that the longitudinal conductivity, $\sigma_{xx}$ is related to the diffusion coefficient via the Einstein relation $\sigma_{xx} = \nu_F e^2 D_F$, where $\nu_F = B_{\mathbf{p}_F}/2\pi v^2\hbar^2$ is the density of states at the Fermi level. This leads to the usual expression for the Thomas-Fermi screening length, $\lambda_{TF}$, define by $\lambda_{TF}^{-1} = 2\pi\sigma_{xx}/D_F = 2\pi\nu_F e^2$. The absence of a longitudinal current in equilibrium, assumed in the macroscopic theory, then follows from the cancelation between the second (diffusion) and third (drift) terms of the left-hand-side of Eq. (8), when $f^c$ is replaced by its equilibrium Fermi function value. Here the diffusion coefficient $D_F = \mathrm{v}^2_{\mathbf{p}}\tau_{tr}/2$ with

$$\tau_{tr}^{-1} = \frac{n_{imp}u^2}{4v^2\hbar^3}\frac{v^2p_F^2 + 4\Delta^2}{\sqrt{v^2p_F^2 + \Delta^2}}. \tag{9}$$

As suggested above, the naive guess that one just has to multiply the screened electric field with the intrinsic Hall conductivity to find the current definitely fails in the case of the 2D massive Dirac model. Below we show that this happens because gradients in the density of carriers, all of which generally carry intrinsic magnetic moments[13, 19], also yield an azimuthal current.

Since the response we seek to evaluate includes the time-reversal-symmetry broken system's anomalous Hall effect, side-jump and skew scattering contributions[20] are present. By generalizing the treatment of Refs. [21, 22], developed for uniform electric fields, to the case of a non-uniform static field, one can show that the aforementioned Fermi-surface contributions to the current vanish in equilibrium. Therefore, side-jump and skew scattering contributions need not be considered and the entire Hall response comes from the intrinsic contribution.[23]

The intrinsic contribution to the current is obtained from the equation for $\mathbf{f}^{\perp}_{\mathbf{p}}$. Importantly we can simply drop the contribution to the collision integral for $\mathbf{f}^{\perp}_{\mathbf{p}}$ coming from $f^{\parallel}_{\mathbf{p}}$ which contributes to side-jump processes only. [27] Further, for a sufficiently clean surface, such that $B_{\mathbf{p}}\tau_{tr}/\hbar \gg 1$, we can also neglect the collisional relaxation of $\mathbf{f}^{\perp}_{\mathbf{p}}$ as compared to the precession term, coming from the commutator on the left hand side of Eq. (7). The general expression for the static limit of $\mathbf{f}^{\perp}_{\mathbf{p}}$ is thus obtained simply by isolating the inter-band coherence terms on the left hand side Eq. (7). We obtain

$$2B_{\mathbf{p}}\mathbf{f}^{\perp}_{\mathbf{p}} = \hbar\left((\nabla n_{\mathbf{p}}\partial_{\mathbf{p}})\mathbf{B}_{\mathbf{p}}\right)\times\mathbf{b}_{\mathbf{p}} + \hbar\left((e\mathbf{E}_{tot}\partial_{\mathbf{p}})\mathbf{f}_{\mathbf{p}}\right)\times\mathbf{b}_{\mathbf{p}}. \tag{10}$$

The second term on the right hand side of Eq. (10) leads to the standard intrinsic contribution to the Hall conductivity. The first term is the response to the equilibrium density inhomogeneities. Restricting ourselves to linear response, and substituting equilibrium values for the density matrix gives $\nabla n_{\mathbf{p}} = -e\mathbf{E}_{tot}(\mathrm{d}n_F(B_{\mathbf{p}}-\mu)/\mathrm{d}B_{\mathbf{p}} - \mathrm{d}n_F(-B_{\mathbf{p}}-\mu)/\mathrm{d}B_{\mathbf{p}})/2$ and $\mathbf{f}_{\mathbf{p}} = \mathbf{b}_{\mathbf{p}}(n_F(B_{\mathbf{p}}-\mu) - n_F(-B_{\mathbf{p}}-\mu))/2$. Substituting these expressions in Eq. (10) and taking the local direction of the electric field to be along the $\hat{x}$ axis we obtain:

$$\mathbf{f}^{\perp}_{\mathbf{p}} = -\frac{\hbar}{4}eE_{tot}B_{\mathbf{p}}(\partial_{p_x}\mathbf{b}_{\mathbf{p}})\times\mathbf{b}_{\mathbf{p}}\sum_{\nu=\pm}\nu\frac{\partial}{\partial B_{\mathbf{p}}}\left(\frac{n_F(\nu B_{\mathbf{p}}-\mu)}{B_{\mathbf{p}}}\right). \tag{11}$$

This expression yields the usual intrinsic anomalous Hall conductivity when the derivative acts on the $B_{\mathbf{p}}^{-1}$ factor only. When the derivative acts on both factors we obtain

$$\frac{j_y}{E_x} = \frac{\hbar e^2}{2}\int\frac{d^2p}{(2\pi\hbar)^2}\mathbf{b}_{\mathbf{p}}\cdot(\partial_{p_x}\mathbf{b}_{\mathbf{p}})\times(\partial_{p_y}\mathbf{b}_{\mathbf{p}})\sum_{\nu=\pm}\nu B^2_{\mathbf{p}}\frac{\partial}{\partial B_{\mathbf{p}}}\left(\frac{n_F(\nu B_{\mathbf{p}}-\mu)}{B_{\mathbf{p}}}\right). \tag{12}$$

Note that the equilibrium value of $j_y/E_x$ is *not* the Hall conductivity. The ratio instead describes equilibrium currents that flow along equipotential lines of the screened external potential and generate a contribution to the orbital magnetization. The corresponding magnetic flux is calculated for a specific measurement procedure in the Supplemental Material.[14]

The right-hand-side of this expression vanishes for the 2D massive Dirac equation model for temperature $T \to 0$. In that case Eq. (12) reduces to

$$\frac{j_y}{E_x} = \frac{e^2}{4\pi\hbar}(n_F(-\Delta-\mu) - n_F(\Delta-\mu)), \tag{13}$$

which obviously vanishes at $T \to 0$ for any $\mu > |\Delta|$. Perfect cancelation occurs between the homogenous system anomalous Hall response and the current due to the curl of the internal quasiparticle magnetization density. The same cancellation occurs for generalized Dirac models Eq. (6) with $|\mathbf{p}|$-dependent velocities and constant $\Delta$, as long as the $\mathbf{p}$-integrals are convergent. This precise cancelation is however dependent on our neglect of collisional relaxation in the equation for $\mathbf{f}^{\perp}_{\mathbf{p}}$, which would lead to $\hbar/\Delta\tau_{tr}$ corrections. The cancelation is also imperfect at finite temperature; substantial current signal can be recovered, as illustrated in Fig. 2. The azimuthal current vanishes not only for $T \to 0$ but also for $T \to \infty$ and is therefore a non-monotonic function of temperature.

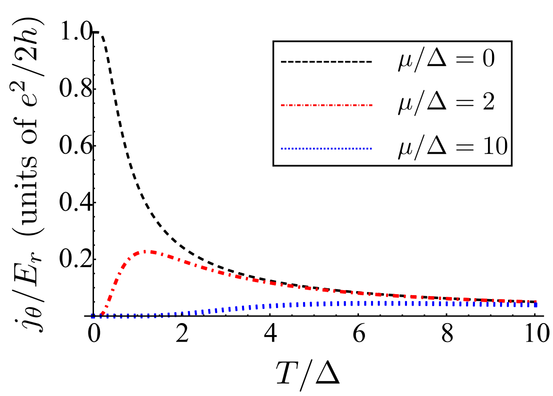


FIG. 2: (Color online) The dependence of the ratio of the azimuthal current, $j_\theta$, to the radial electric field, $E_r$, on temperature for different values of the chemical potential.

*Discussion–* We find that the apparent monopole position moves away from the TI surface with a velocity $v_M = \alpha c g$. Since graphene based two-dimensional electron systems, which are similar to TI surface states, can[28] have $g$ values $\sim 10^{-7}$ or smaller when time-reversal symmetry is broken by an external magnetic field, there is a reasonable hope that it will be possible to obtain TI samples in which $v_M$ is small enough to enable observations in which $\sigma_{xx}$ plays no role and the axion electrodynamics model is directly applicable. In the Supplemental Material [14] we describe a procedure to estimate the required upper limit for $g$ for a specific experimental procedure based on a SQUID measurement of the magnetic flux created by an external charge placed near a topological surface.

After the external charge screening process has been completed, we find that the azimuthal current response has two contributions, one proportional to the Hall conductivity and treated previously by Zang and Nagaosa,[8] and one proportional to an external potential induced change in the internal orbital magnetization[19] of the surface states. For the particular case of a massive Dirac model the two contributions cancel exactly in the clean $T = 0$ limit when the carrier density is finite. This observation constitutes an important experimental prediction: in the presence of a finite doping, the magnetic flux signal is expected to be a non-monotonic function of temperature. This should be contrasted with a naive expectation that above signal should simply diminish with increasing temperature. A specific calculation of the magnetic flux from the surface as a function of temperature, illustrating its non-monotonicity, is presented in the Supplemental Material.[14]

We obtain the aforementioned result using a quasiclassical kinetic equation approach, which may not be reliable near band edges due to both quantum and non-linear screening effects, but nevertheless starkly demonstrates the distinction between azimuthal current and Hall conductivity responses. In general the magnetic flux induced by an electron charge near a time-reversal symmetry broken TI surface is dependent on disorder and on the $|\mathbf{p}|$-dependence of the exchange potential $\Delta_{\mathbf{p}}$, and not simply on surface's Hall conductivity.

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