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Minimum Energy-Surface Required by Quantum Memory Devices

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We address the question what physical resources are required and sufficient to store classical information. While there is no lower bound on the required energy or space to store information, we find that there is a nonzero lower bound for the product $P = \langle E \rangle \langle r^2 \rangle$ of these two resources. Specifically, we prove that any physical system of mass m and d degrees of freedom that stores S bits of information will have lower bound on the product P that is proportional to $d^2/m(\exp(S/d) - 1)^2$. This result is obtained in a non-relativistic, quantum mechanical setting and it is independent from earlier thermodynamical results such as the Bekenstein bound on the entropy of black holes.

I. INTRODUCTION.

Although information may seem abstract and elusive, it has made a number of surprisingly concrete connections to physics. Originally prompted by Maxwell’s demon, the link of information to thermodynamics has been made lucid by the works of Bennett and Landauer (see [1] for complete coverage). Physicists and computer scientists alike have come to recognize that information does not exist as an independent entity but is encoded in physical devices, and, therefore, “Information is physical” has become a new mantra [2]. Moreover, the technological trend from the last few decades continually has been to compress more information in less space [3]. Due to Moore’s law we sense that we are near the end of the classical regime and entering into the quantum regime where exciting possibilities and challenges abound. Thus, to understand the ultimate limits to information storage based on the quantum laws of physics may be of intellectual as well as practical significance. In this paper we seek a general and robust result concerning information storage using non-relativistic quantum theory. Abstracting away the details of hardware, we compute the absolute minimal amount of physical resources required to store information, and the result further illustrates its physical nature.

Typically, information is stored by preparing a device in a number of different stationary states corresponding to different messages. The device is effectively described by a mixed state of a certain entropy, and looking at a number of simple devices such as particle in a box, harmonic oscillator, and the hydrogen atom, we notice an interesting interplay among energy, surface area, mass, and entropy (see Table I). Keeping the constituent particles’ masses the same, one could save on energy or size

but never both. For instance, the particle-in-a-box model requires the same amount of space regardless of the number of states used in the encoding but the energy increases with the number of states. The hydrogen atom is an example of the opposite extreme. There is an infinite number of states in a finite energy window, roughly equal to 13.6 eV, but the spatial size of the orbitals increases without bound. However, if we could change the mass, then both energy and space can be reduced arbitrarily. Since a number of variables could affect the cost, it is necessary to hone in on a precise notion of the cost of information storage. We will do so in the next section and determine the optimal device based on a number of clearly stated plausible hypotheses. Then, we proceed to compute the minimum cost and prove a lemma which is needed in the calculation. Our main result is a Heisenberg-like uncertainty principle governing energy, space, and entropy. We conclude with a comparison to rough estimates existing in the literature as well as some thoughts upon its implications for future technological advances.

II. DEFINITION OF COST

Consider storing information in a device described by the Hamiltonian

$$H_{\text{dev}} = -\frac{\hbar^2}{2m} \sum_{i=1}^d \frac{d^2}{dx_i^2} + V(\vec{x}), \quad (1)$$

by preparing it in a stationary (energy eigenstate) Ψ_α with probability p_α . The amount of information thus encoded is

$$S = -\sum_{\alpha} p_{\alpha} \log p_{\alpha} \quad (2)$$

and the device will be in the state

$$\rho_{\text{dev}} = \sum_{\alpha} p_{\alpha} |\Psi_{\alpha}\rangle \langle \Psi_{\alpha}|. \quad (3)$$

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We consider the average energy and size associated with such a mixed state the cost in physical resources required

System	Energy	Surface Area	Product
Particle of mass m in a 1d box of width L	$\frac{\hbar^2 \pi^2}{m L^2} \langle n^2 \rangle$	L^2	$\frac{\hbar^2}{m} \langle n^2 \rangle$
Harmonic oscillator of mass m and natural frequency ω	$\hbar \omega \langle n \rangle$	$\frac{\hbar}{m \omega} \langle n \rangle$	$\frac{\hbar^2}{m} \langle n \rangle^2$
Electronic states of Hydrogen atom	$\frac{e^4 m}{\hbar^2} \langle \frac{1}{n^2} \rangle$	$\frac{\hbar^4}{m^2 e^4} \langle n^4 \rangle$	$\frac{\hbar^2}{m} \langle n^4 \rangle \langle 1/n^2 \rangle$

TABLE I: A system is prepared in different stationary states, labelled by n , according to a probability distribution whose Shannon entropy is the desired amount of encoded information. Quoted are order-of-magnitude estimates of the average energy and surface area, denoted by $\langle \cdot \rangle$, for a number of simple systems [4]. Averages such as $\langle n \rangle$, $\langle n^2 \rangle$, and $\langle n^4 \rangle$ grow with increasing entropy. Therefore, although energy and surface area show different behaviours for different systems, their products invariably grow with entropy.

by the device. More specifically, let $\text{Tr } \rho_{\text{dev}} r^2$ be the measure of spatial cost where

$$r^2 := \sum_{i=1}^d x_i^2. \quad (4)$$

To prevent arbitrary shifts due to displacements of the coordinate origin, we require for all $i \in \{1, \dots, d\}$ that $\text{Tr } \rho_{\text{dev}} x_i = 0$. Moreover, let $\text{Tr } \rho_{\text{dev}} H_{\text{dev}}$ be a measure of the cost in energy, and similarly, to prevent meaningless shifts in the Hamiltonian, we require the ground state (g.s.) energy to be zero,

$$\langle H_{\text{dev}} \rangle_{\text{g.s.}} = 0 \quad (5)$$

The cost of information storage specific to using H_{dev} is the joint product of energy and space

$$C_{\text{dev}} := \min_{\rho_{\text{dev}}} \text{Tr } \rho_{\text{dev}} H_{\text{dev}} \text{Tr } \rho_{\text{dev}} r^2, \quad (6)$$

subject to the constraint of Equation (2). Since the cost in Equation (6) is specific to a device, to find the cost of information storage we need to minimize over all devices having the same number of degrees of freedom and particles' mass as follows:

$$P := \min_{V(\vec{x})} C_{\text{dev}} \quad (7)$$

over potentials that satisfy Equation (5).

We choose to express the intuitive notion of cost in energy and space by considering linear power in energy and quadratic in length; see Equation (6). One could potentially capture the same intuition through other powers of the average H and r such as $\langle H^{s_1} \rangle \langle r^{s_2} \rangle$. Putting the exponents inside or outside of the average may change the

end result slightly but should convey the same physics. Since the only dimensionful parameters in our problem are $\hbar$ and m , if the cost is measured in units of $J^{s_1} l^{s_2}$, where J and l stand for Joules and meters, then the answer must be expressed in powers t_1 of $\hbar$ and t_2 of m such that

$$(J \cdot s)^{t_1} kg^{t_2} = J^{s_1} l^{s_2}, \quad (8)$$

where s and kg stand for seconds and kilograms. Solving this equation leads to these constraints on the exponents: $t_1 = 2s_1$, $t_2 = -s_1$, and $s_2 = 2s_1$. The first two equations guarantee that other powers of energy and space admit a solution and therefore are also valid, but, as implied by the last equation, they all are powers of the basic combination of linear in energy and quadratic in space.

III. OPTIMAL POTENTIAL

For the purpose of computing a lower bound of P , we can relax the condition that the mixed state be diagonal in the eigenstates of H_{dev} and consider instead the following functional of $V(\vec{x})$

$$C[V(\vec{x})] := \min_{\rho} \{ \text{Tr } \rho H_{\text{dev}} \text{Tr } \rho r^2 \}, \quad (9)$$

minimized over all density matrices satisfying the entropy condition

$$S = -\text{Tr } \rho \log \rho. \quad (10)$$

Obviously, for a given $V(\vec{x})$,

$$C_{\text{dev}} \geq C[V(\vec{x})] \quad (11)$$

Hence,

$$P \geq \min_{V(\vec{x})} C[V(\vec{x})] \quad (12)$$

with $V(\vec{x})$ subject to the same constraint of Equation (5). Let us start by observing that $C[V(\vec{x})]$ is rotationally invariant. Under a coordinate rotation $\vec{y} = R\vec{x}$, with $R \in SO(d)$, we have

$$C[V(R\vec{x})] \quad (13)$$

$$= \min_{\rho} \left\{ \text{Tr } \rho \left(-\frac{\hbar^2}{2m} \sum_{i=1}^d \frac{d^2}{dx_i^2} + V(R\vec{x}) \right) \text{Tr } \rho r^2 \right\} \quad (14)$$

$$= \min_{\rho} \left\{ \text{Tr } \rho \left(-\frac{\hbar^2}{2m} \sum_{i=1}^d \frac{d^2}{dy_i^2} + V(\vec{y}) \right) \text{Tr } \rho r^2 \right\} \quad (15)$$

$$= C[V(\vec{x})] \quad (16)$$

The first equality comes from applying the definition in Equation (9); the second comes from using the y -coordinates instead of x and rotational invariance of kinetic energy and r^2 ; finally, the third comes from applying the definition again. Thus, if we assume a minimizing

potential $V(\vec{x})$ exists, then $C[V(R\vec{x})]$ must also be minimal by rotational invariance, Equation (16). Moreover, if we assume the optimal potential to be also unique, then $V(R\vec{x}) = V(\vec{x})$. Hence, by assuming existence and uniqueness, we conclude the optimal potential must be a function of only the hyper-radius r . As we try to find the potential which minimizes $C[V(\vec{x})]$, we have in the problem only the parameters $\hbar$, m , and r to construct our optimal potential. Obviously, nature supplies us with more energy and length scales than what is considered here; for example, we could allow the interaction potential to include also electronic charges, spin, or speed of light, but such a formulation would take us beyond non-relativistic quantum theory. Thus, choosing to work with potentials that depend only on distances as indicated in Equation (1), we are led to conjecture the solution to be $V(r) = -W \frac{\hbar^2}{2mr^2}$ by dimensional analysis, where W is a dimensionless number possibly depending on S . It is also important to note that such is the only combination of $\hbar$, m , and r that yields an energy quantity. The larger W , that is the more attractive the potential, the smaller the orbitals and total energy, thus allowing us to save on the total cost without compromise. However, the attractive inverse square potential for $W > (1 - d/2)^2$ is unstable as it has negative infinity ground state energy, i.e. “fall to the center” [5–8]. Thus, the optimal potential occurs at the maximum allowed value of W before it collapses, namely

$$H_{\text{opt}} = -\frac{\hbar^2}{2m} \sum_{i=1}^d \frac{d^2}{dx_i^2} - \left(1 - \frac{d}{2}\right)^2 \frac{\hbar^2}{2mr^2} \quad (17)$$

Since we arrive at the optimal device by assuming existence and uniqueness, we do not claim to have proved it. Rather, it remains a conjecture for the rest of our paper.

IV. MINIMUM COST

Since H_{opt} minimizes $C[V(x)]$, by Equations (9) and (12), we have

$$P \geq \min_{\rho} \{ \text{Tr } \rho H_{\text{opt}} \text{Tr } \rho r^2 \}. \quad (18)$$

To evaluate the right hand side, let us introduce $\kappa > 0$ and define

$$\tilde{C} := \min_{\rho} \text{Tr } \rho \left(H_{\text{opt}} + \frac{\kappa}{2} r^2 \right) \quad (19)$$

with ρ subject to the entropy constraint Equation (10). The auxiliary variable κ allows us to convert the cost in space into an energy quantity so that in a single expression, $\tilde{C}$ is the joint costs of energy and space in the form of a sum. As to be shown in the subsection, we have

$$\tilde{C} \geq \hbar \sqrt{\frac{\kappa}{m}} d (\exp(S/d) - 1) \quad (20)$$

Hence, for the mixed state obtaining the minimum of the right hand side in Equation (19), we have

$$\text{Tr } \rho H_{\text{opt}} \geq \hbar \sqrt{\frac{\kappa}{m}} d (\exp(S/d) - 1) - \frac{\kappa}{2} \text{Tr } \rho r^2, \quad (21)$$

which yields

$$\begin{aligned} \text{Tr } \rho H_{\text{opt}} \text{Tr } \rho r^2 &\geq \\ \sqrt{\kappa} \left(\hbar \frac{d}{\sqrt{m}} (\exp(S/d) - 1) - \frac{\sqrt{\kappa}}{2} \text{Tr } \rho r^2 \right) \text{Tr } \rho r^2 \end{aligned}$$

for all non-negative values of κ . Since the right hand side is quadratic in $\sqrt{\kappa}$, at the critical κ the inequality yields

$$\text{Tr } \rho H_{\text{opt}} \text{Tr } \rho r^2 \geq \frac{\hbar^2}{2m} d^2 (\exp(S/d) - 1)^2, \quad (22)$$

which is now free of the introduced variable κ . Because of Inequality (18), the cost of storing information satisfies the bound

$$P \geq \frac{\hbar^2}{2m} d^2 (\exp(S/d) - 1)^2, \quad (23)$$

which is our main result.

A. Central Lemma

We now proceed to prove Lemma (20). Simplifying the dimensionful quantities in $\tilde{C}$ allows us to rewrite Equation (19) as

$$\tilde{C} = \hbar \sqrt{\kappa/m} \min_{\rho} \text{Tr } \rho H, \quad (24)$$

where

$$H := \frac{1}{2} \left(\sum_{i=1}^d -\frac{d^2}{dq_i^2} - \frac{W}{q^2} + q^2 \right) \quad (25)$$

is dimensionless as is $\vec{q} := \vec{x}(m\kappa)^{1/4}/\sqrt{\hbar}$. Let

$$\rho = \sum_i p_i |\Psi_i\rangle \langle \Psi_i| \quad (26)$$

with eigenvalues p_i and Ψ_i orthogonal and not necessarily eigenstates of H . Thus, we would like to minimize the expression

$$\sum_i p_i \langle \Psi_i | H | \Psi_i \rangle \quad (27)$$

by varying $|\Psi_i\rangle$ and p_i subject to the entropy condition (10). Without loss of generality, let us assume p_i is non-increasingly ordered,

$$p_1 \geq p_2 \geq p_3 \geq \dots \quad (28)$$

Obviously, we would have the smallest sum possible if $\langle \Psi_i | H | \Psi_i \rangle$ is in the opposite ordering,

$$\langle \Psi_1 | H | \Psi_1 \rangle \leq \langle \Psi_2 | H | \Psi_2 \rangle \leq \langle \Psi_3 | H | \Psi_3 \rangle \leq \dots \quad (29)$$

By the mini-max principle, Ψ_i are eigenstates of H with increasing eigen-energies [9]. Optimization over p_i with the constraints of entropy and normalization can be treated with Lagrange multipliers in a manner identical to that typically found in textbooks, thus resulting in the Boltzmann distribution. In other words, the minimizing density matrix of Equation (24) is diagonal in the eigenstates of H with $p_i \propto \exp(-\beta E_i)$ with E_i being eigen-energies of H and β a constant depending on S . The spectrum of H can be computed by casting the Laplacian into hyper-spherical coordinates and solving the ensuing hypergeometric differential equation as in [5]. Eigenenergies are determined by two quantum numbers $n, l \in \mathbb{N}$

$$E(n, l) = 2n + \sqrt{l(l+d-2)}, \quad (30)$$

with degeneracy

$$g(l) = \frac{(d+2l-2)(d+l-3)!}{l!(d-2)!} \quad (31)$$

Thus, Equation (24) becomes

$$\tilde{C} = \frac{\hbar}{Z} \sqrt{\frac{\kappa}{m}} \sum_{n \in \mathbb{N}} \sum_{l \in \mathbb{N}} \exp(-\beta E(n, l)) g(l) E(n, l) \quad (32)$$

where

$$Z := \sum_{n \in \mathbb{N}} \sum_{l \in \mathbb{N}} g(l) \exp(-\beta E(n, l)). \quad (33)$$

Although in our context Equation (24) $\tilde{C}$ means the cost of energy and space, it is also equivalent to the internal energy of a thermodynamic system at a temperature proportional to $1/\beta$. Moreover, the separation of n and l in the eigen-energy equation allows the partition function to be factorized

$$Z = \underbrace{\sum_{n \in \mathbb{N}} \exp(-2\beta n)}_{Z_n} \underbrace{\sum_{l \in \mathbb{N}} \exp(-\beta \sqrt{l(l+d-2)}) g(l)}_{Z_l} \quad (34)$$

into two uncoupled sub-systems n and l , which, as we would see if we followed the derivation of Equation (30), are simply the radial and angular parts of the Hamiltonian. Thus, the total internal energy and entropy are sums of those in the subsystems; that is,

$$\tilde{C} = \hbar \sqrt{\frac{\kappa}{m}} (U_n + U_l) \quad \text{and} \quad (35)$$

$$S = S_n + S_l, \quad (36)$$

where U_n , U_l , S_n , and S_l are internal energies and entropies of the n and l subsystems. As is well known in

thermodynamics, the internal energy and entropy can be calculated via

$$\tilde{C} = -\frac{\partial \log Z}{\partial \beta} \quad (37)$$

$$S = \log Z + \beta \tilde{C}, \quad (38)$$

which would also allow us to calculate U_n , S_n , and U_l , S_l via their respective partition functions Z_n and Z_l . Because Z_n , being a geometric sum, can be computed in closed form, we get the following exact results for the thermodynamic quantities

$$\beta = \frac{1}{2} \log \left(1 + \frac{2}{U_n} \right) \quad (39)$$

$$S_n = \log \left(1 + \frac{U_n}{2} \right) + \frac{U_n}{2} \log \left(1 + \frac{2}{U_n} \right). \quad (40)$$

While the n sum is trivial, we can make progress with the l -sum only when making approximations in certain regimes. In the following we will work in the limits $d \gg 1$ and $\beta \ll 1$, which corresponds to high entropy and many degrees of freedom. Using Stirling's formula and the method of steepest descent to compute the integral version of the discrete sum, we get

$$Z_l \approx 2\beta^{-d+1} \quad (41)$$

with the error governed by

$$|\log Z_l - \log 2\beta^{-d+1}| = \mathcal{O} \left(\frac{\beta}{d} \right) \quad (\text{See Figure 1}) \quad (42)$$

By the canonical equations yielding internal energy and entropy from the partition function, we then obtain

$$U_l = \frac{d-1}{\beta} + \mathcal{O} \left(\frac{1}{d} \right) \quad (43)$$

$$S_l = (d-1) \left(\log \frac{U_l}{d-1} + 1 \right) + \mathcal{O} \left(\frac{\beta}{d} \right) \quad (44)$$

Since we work in the regime $\beta \ll 1$, Equation (39) implies $U_n \gg 1$. Hence, keeping only the dominant terms in the n -subsystem, we obtain

$$U_n = \frac{1}{\beta} - 1 + \mathcal{O}(\beta) \quad (45)$$

$$S_n = \log \left(1 + \frac{U_n}{2} \right) + \mathcal{O}(1) \quad (46)$$

When we combine the above results for the separate sub-systems and keep only the dominant terms, we get

$$\tilde{C} = \hbar \sqrt{\frac{\kappa}{m}} (dU_n - (d-1) + \mathcal{O}(1/d)) \quad \text{and} \quad (47)$$

$$S = d \log U_n + \mathcal{O}(1), \quad (48)$$

We now obtain the equation relating the amount of encoded information to the associated cost,

$$\frac{\tilde{C}}{d} = \hbar \sqrt{\frac{\kappa}{m}} \left(\exp(S/d) - \frac{d-1}{d} + \mathcal{O}(1/d^2) \right), \quad (49)$$

which yields Lemma (20) in the limit of large d .

FIG. 1: We computed $\log Z$ in two different ways for $d = 10$. The $\log Z$ -plot is the result of carrying out the sum in Equation (34) to 1600 terms at which it seems stable. The approximation-plot is Equation (41) resulting from the method of steepest descent. The error plot shows the difference $\log Z - \log 2\beta^{-d+1}$, which is of the order β/d .

V. CONCLUSION.

Using non-relativistic quantum theory, we find the minimal physical resources needed to store information based on a number of plausible hypotheses. Defining the cost in the form of a product of energy and space, we proceed to find the interaction potential that minimizes the cost when storing a given amount of information. From Equation (16) we notice that the cost functional attains the same value for a potential and its rotated cousin. Hence, assuming the optimizing potential $V(\vec{x})$ exists, we conclude its rotated cousin $V(R\vec{x})$ must also be optimal. By further assuming uniqueness, we can conclude $V(\vec{x}) = V(R\vec{x})$, thus implying $V(\vec{x})$ actually depends only on the hyper-radius. We could also by fiat make the weaker assumption that the optimal potential is rotationally symmetric, but it is worth noting that rotational symmetry follows existence and uniqueness. Since we work with only Hamiltonians of the type in Equation (1), we have a limited set of parameters to construct the optimal potential, and dimensional analysis leaves but one

form of the optimal device, which is given in Equation (17). As our main result we arrive at Equation (23), which provides a lower bound to any device's energy and surface area when it encodes S amount of information in d degrees of freedom in the limits $d, S \gg 1$. According to our intuition we expect information to require no energy and space, and this only appears so because the factor $\hbar/m$ is extremely small in the classical regime. For comparison with some existing rough estimates in the literature, let us consider a spherical memory device made up of ordinary matter of total mass 1 kilogram and volume 1 liter. Since the energy in our result is not the rest-mass energy but the excitation energies relative to the ground, let us take the energy to be the ionization energy of typical atoms, which is 10 eV. Thus, our memory looks like a soup of many electrons at a macroscopic distance from a positively charged center. Our inequality yields $S/d \approx 20$, about 20 bits per atom, which agrees with Bekenstein's estimate for electronic levels but falls far short compared to 10^6 when energy is taken to be the rest-mass energy [10]. Hence, the whole device can hold about 10^{31} bits, which is the same as Lloyd's ultimate laptop, [11, 12]. Moreover, the cost is proportional to the square of the number of degrees of freedom but exponential in the information density, S/d , thus showing analog storage to be much more expensive than digital. Given a device storing one bit of information, analog storage would keep the same device and use its remaining internal states to store additional bits, and this is shown to be exponentially more expensive than bringing in more copies of the same device, which is digital storage. Our current state-of-the-art technology stores gigabytes of information in a macroscopic device consisting of an Avogadro number of entities; that is, we are operating at the low density regime, $S/d \approx 10^{-14}$. As we try to create smaller devices with greater information capacity, we are pushing S/d to greater values. However, our result implies this endeavour will become infeasible due to the exponential scaling.

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