No-Hair Theorem for the Galileon

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A no-hair theorem for the galileon

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We consider a galileon field coupled to gravity. The standard no-hair theorems do not apply, because of the galileon's peculiar derivative interactions. We prove that, nonetheless, static spherically symmetric black holes cannot sustain non-trivial galileon profiles. Our theorem holds for trivial boundary conditions and for cosmological ones, and regardless of whether there are non-minimal couplings between the galileon and gravity of the covariant galileon type.

Black holes famously have no hair [1, 2]—except when they do [3]. No-hair theorems involve assumptions that can be violated. For instance, for scalar hair, Bekenstein's version [2] assumes that the action depends on the derivatives of the scalar (π) only through the combination $(\partial\pi)^2$. Several recent proposals for modifying gravity on long distance scales involve introducing scalar derivative interactions of the galileon type [4], which are not covered by existing no-hair theorems. The galileon can even violate the null energy condition, in a ghost-free manner [5]. In view of the observational and theoretical significance of the galileon—observational as an explanation for cosmic acceleration, theoretical as a generic ingredient of massive gravity [6, 7]—it is useful to investigate whether the no-hair theorem can be extended to the galileon. We will demonstrate that indeed black holes carry no galileon charge, at least for spherically symmetric ones. That this is true suggests an interesting experimental test of the galileon, namely that central massive black holes in galaxies are expected to be offset from the stars, which is presented in a separate paper [8]. A discussion of no-hair theorem for the galileon can also be found in an independent paper by Babichev and Zahariade [9]. See also [10] for related discussions.

Our proof is logically very simple, and uses little information about the theory. The main ingredients it relies on are the shift-symmetry of the galileon action, the symmetries of the solution, and the regularity of diff-invariant quantities at the horizon. It makes no use of Einstein's equations. We assume that we have a spherically symmetric, static black-hole, in the presence of a spherically symmetric, static scalar field $\pi(r)$. We find it convenient to choose the radial coordinate r such that $g_{tt} = -1/g_{rr}$, in which case the angular part of the metric has to be left generic:

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + \rho^2(r)d\Omega^2 \ . \tag{1}$$

Here f and ρ are generic functions. For the Schwarzschild solution one has $f = 1 - \frac{1}{r}$, $\rho = r$. For simplicity, we choose units in which the black-hole horizon sits at $r = 1$. We will not assume the Schwarzschild metric, but rather use the more general form eq. (1). Our proof can be schematically divided into four steps:

1. The galileon eom is a current conservation equation. The galileon equation of motion in the absence of sources can be written as the (covariant) conservation of a current:

$$\nabla_\mu J^\mu = 0 \ . \tag{2}$$

This follows directly from the shift invariance

$$\pi \to \pi + c \ , \tag{3}$$

which is exact for the galileon coupled to gravity, even in the presence of covariant galileon-type non-minimal couplings [11]. The Noether current associated with such a symmetry involves first and second (covariant) derivatives of π—as well as curvature tensors in the covariant galileon case. In flat space such a current takes the form [4, 12]

$$J^\mu = G^{\mu\nu}(\partial\partial\pi)\,\partial_\nu\pi \ , \tag{4}$$

where G is a tensor polynomial.

On the other hand, for generic spacetimes, galilean shifts of the form

$$\pi \to \pi + b_\mu x^\mu \tag{5}$$

are not a symmetry of the action, nor do they admit an obvious generalization [1]. As a consequence, their Noether currents [15] are not conserved on non-trivial gravitational backgrounds. Our proof does not use them, and can thus be applied to generic shift-invariant scalar theories.

2. J^r vanishes at the horizon. First, notice that in the coordinates that we are using, J^r is the only non-zero component of J^μ. This follows from the symmetries of the solution, for the metric as well as for the scalar. By

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[1] See [13, 14] for non-trivial generalizations to maximally symmetric spacetimes.

rotational invariance, there cannot be any angular components of J^μ. This is obvious, but here is a proof for the more fastidious among our readers. The current is a covariant quantity built out of the scalar and the metric, their derivatives, etc. The Killing vectors ξ^μ that generate rotations, define symmetries for the metric and for the scalar:

$$\mathcal{L}_\xi \, g_{\mu\nu} = 0 \; , \qquad \mathcal{L}_\xi \, \pi = 0 \; , \tag{6}$$

where $\mathcal{L}_\xi$ denotes the Lie-derivative along ξ. Any tensor constructed solely from these fields and their derivatives shares the same symmetries. In particular,

$$\mathcal{L}_\xi J^\mu = 0 \; . \tag{7}$$

However, we know that *any* regular two-vector field defined on a two-sphere must vanish at some point. Combining this with the spherical symmetry of J^μ, eq. (7), we see that the angular components of J^μ have to vanish everywhere.

The vanishing of J^t is slightly trickier to ascertain. From invariance under time-translations, following the same logic as above we just get that J^t is constant in time, $\partial_t J^t = 0$. However, a non-zero J^t picks out a time direction—it can be future-directed or past-directed. There is nothing in the solution for the metric and for π picking a time-direction—they only depend on r. And J^t has to flip under time-reversal, because the galileon action does not contain an (odd number of) epsilon tensor(s): it only contains the scalar, the metric, and their derivatives, and it is invariant under time-reversal provided the scalar and the metric are even under it. We thus see that time-reversal invariance forces $J^t = 0$.

We are thus left with J^r only. It is immediate to see that this has to vanish *at the horizon*. As usual, the horizon $r = 1$ corresponds to a zero of $f(r)$. This is because, by definition, the horizon corresponds to a locus where the time-translational Killing vector,

$$\xi^\mu = (1, 0, 0, 0) \; , \tag{8}$$

becomes null: $g_{\mu\nu}\, \xi^\mu \xi^\nu = 0$. Then, assuming that the horizon be a regular locus—a locus where all scalar quantities, physical and geometrical ones, are regular—we see that for $J^\mu J_\mu = (J^r)^2/f$ to be regular there, J^r has to vanish.

3. J^r vanishes everywhere. We now use the current conservation equation (2) to bootstrap our way out of the near-horizon region, and show that, in fact, J^μ has to vanish everywhere. Covariant current conservation can always be rewritten as

$$\frac{1}{\sqrt{-g}} \partial_\mu \big(\sqrt{-g} J^\mu \big) = 0 \; . \tag{9}$$

In our case we have further simplifications. The only non-vanishing component of J^μ is J^r, and it depends on r only. Moreover, in our coordinates $\sqrt{-g}$ is simply $\rho^2(r) \sin^2\theta$. Thus we have $\rho^{-2}\partial_r(\rho^2 J^r) = 0$ which implies

$$\rho^2 J^r = \text{const} \; . \tag{10}$$

Notice that ρ^2 is expected to finite (neither infinite nor zero), even at the horizon, since it measures the area of constant-r spheres. We have shown previously that J^r vanishes at the horizon, and so the constant on the r.h.s. is in fact zero. We therefore arrive at the conclusion

$$J^r = 0 \quad \text{at all } r \; . \tag{11}$$

4. π vanishes everywhere. The final step in our proof involves integrating eq. (11), to find $\pi(r)$. Of course one possible solution is $\pi(r) = 0$. We want to prove that this is in fact the only possible solution. More precisely: it is the only solution that decays at infinity [2]. To see this, note that for a spherically symmetric, static configuration, the current takes the form

$$J^r = f \cdot \pi' \cdot F(\pi'; g, g', g'') \; , \tag{12}$$

where $f = g^{rr}$ as in eq. (1), $\pi' \equiv d\pi/dr$, and F is a polynomial of π', whose coefficients depend on the metric and its derivatives (to be justified below). The crucial property of F we will use is that it asymptotes to a *nonzero constant* (which does not depend on the metric) when π' goes to zero. This condition is obeyed by any non-degenerate galileon theory featuring a kinetic energy for π. The reason is simply that in the weak π limit, the action is well approximated by its quadratic terms and the shift-current reduces simply to $J^\mu \simeq \partial^\mu \pi$, up to an overall constant which defines π's normalization.

Now, by assumption π' vanishes at infinity. Let us imagine dialing the radius to progressively smaller values, starting from infinity. Imagine further that at some radius, π' starts deviating a little bit from zero. In that case, by continuity, F will still be different from zero. Since f does not vanish either (for $r > 1$), we therefore reach the conclusion $J^r \neq 0$ (eq. 12), contradicting eq. (11). The resolution is that π' in fact cannot deviate from zero, thus $\pi' = 0$ at all radii, from which we conclude $\pi = \text{const}$, or equivalently $\pi = 0$, completing our proof.

To round out our discussion, let us go back and justify the functional dependence of F. The expression for the current can be derived straightforwardly from the action via Noether's theorem. For instance, J^r has been computed explicitly for the galileon in flat space [4], where it takes the above form with $f = 1$ and

$$F = F(\pi'/r) \qquad \text{(flat space)}. \tag{13}$$

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[2] Alternatively, one can say that we are interested in a solution with vanishing first derivative at infinity, for $\pi =$ constant and $\pi = 0$ are equivalent solutions, related by the shift symmetry.

However the generic schematic form (12) can be inferred immediately by generalizing to general metric an argument given in [4] for the flat-space case: For static, spherically symmetric configurations, the galileon equation of motion takes the form of the current conservation equation (9); Being a two-derivative eom, this cannot involve derivatives of π higher than π''; This then implies that J^r cannot involve derivatives of π higher than π'; Moreover, J^r cannot involve π directly without derivatives, because each π in the action is acted upon by at least one derivative. An analogous argument holds for the dependence of F on the metric, and guarantees that F depends at most on its first r-derivatives for the covariant galileon case [11], and at most on its second r-derivatives for the minimally coupled one [4]. We do not make use of this last fact—this is one of the reasons why the covariant galileon and the minimal galileon cases can be treated on equal footing in our proof.

Cosmological boundary conditions. For completeness, it is worth emphasizing that there can be in general non-linear solutions in which it is the F factor that vanishes exactly, but these solutions will not decay at infinity—as our proof shows. Indeed, consider for the moment the flat-space case, where F takes the form (13). In order to make π'/r constant and equal to a zero of F, these non-linear solutions have to behave as $\pi \sim r^2$ at large r, and correspond to non-trivial cosmological boundary conditions at infinity, rather than to scalar profiles "sourced" by the black-hole [4, 16]. Adding to this r^2 background field a "tail" that decays at infinity and that is somehow sourced by the black hole is not allowed, because that would violate the vanishing of the current. To see this, expand the full π field as

$$\pi(r) = \tfrac{1}{2} H_\pi^2 \, r^2 + \delta\pi(r) \; , \tag{14}$$

where H_π^2 is a zero of F— H_π of order of the present Hubble scale H_0, or smaller (due to the presence of matter for instance) [4]—and $\delta\pi(r)$ is the decaying perturbation sourced by the black hole. At large r, we can expand eq. (11) in powers of $\delta\pi(r)$. The leading contribution comes from the first non-vanishing derivative of F, and yields simply

$$\delta\pi'(r) = 0 \; , \tag{15}$$

at large but *finite* r, and therefore at any r.

This flat-space analysis is simplistic though, for it neglects two sources of background curvature: the presence of the black-hole, and the stress-energy tensor associated with π's non-trivial profile. In fact, even neglecting the latter, the former will certainly affect π's solution close to the black hole—the simple r^2 profile will be distorted, if for no other reason than that the r coordinate has no invariant meaning close to the black hole. Whether there is scalar hair now becomes a bit subtle. In certain cases [8, 17], we are just interested in whether the black hole experiences a π force, i.e. whether it falls in the presence of some long wavelength external π fluctuation. The question is thus whether far away from the black hole, there is a $1/r$ tail to the π profile on top of the r^2 cosmological background [3].

Consider for simplicity the limiting case of a black hole whose gravitational radius $r_S = 2GM$ is much much smaller than the curvature radius associated with the non-trivial (cosmological) boundary conditions—the Hubble radius H_0^{-1}. We focus on intermediate distances

$$r_S \ll r \ll H_0^{-1} \; , \tag{16}$$

where the metric can be well approximated by flat plus small corrections, the leading ones being a Newtonian $1/r$ tail and an FRW-like quadratic potential [4],

$$g_{\mu\nu}(r) = \eta_{\mu\nu} + h_{\mu\nu} \; , \qquad h_{\mu\nu} = \mathcal{O}\big(\tfrac{r_S}{r}, H_0^2 r^2\big) \; . \tag{17}$$

To focus on the fields generated by the black hole, we can go to distances such that the latter is negligible:

$$H_0^2 r^2 \ll \frac{r_S}{r} \; . \tag{18}$$

This is of course the relevant limit for observations on astrophysical black holes in our universe. We can now expand the equation $F = 0$ to first order in $\delta\pi$ and $h_{\mu\nu}$. We get, schematically,

$$\frac{\partial F}{\partial \pi'}\delta\pi' + \frac{\partial F}{\partial g}h + \frac{\partial F}{\partial g'}h' + \frac{\partial F}{\partial g''}h'' \simeq 0 \tag{19}$$

where the derivatives of F are evaluated at the flat-space solution $g_{\mu\nu} = \eta_{\mu\nu}$, $\pi_0(r) = \frac{1}{2}H_\pi^2 r^2$.

In this expansion, however, we can treat H_0^2 and $H_\pi^2 \lesssim H_0^2$ as negligible in first approximation, because of (18). This is crucial, because of the way a curved metric and its derivatives enter F: they always accompany at least one π', either to contract or raise the associated indices, or via covariant derivatives of π. On the unperturbed solution, π' is proportional to H_π^2, i.e., suppressed in our limit. As a result we have

$$\frac{\partial F}{\partial g} \; , \; \frac{\partial F}{\partial g'} \; , \; \frac{\partial F}{\partial g''} \propto H_\pi^2 \frac{\partial F}{\partial \pi'} \tag{20}$$

where the proportionality factors involve powers of r, but no extra powers of H_π^2 or H_0^2. Plugging these estimates into our linearized equation above, and reinstating the appropriate powers of r via dimensional analysis, we get

$$\delta\pi(r) \sim \frac{r_S}{r} \cdot H_\pi^2 r^2 \; , \tag{21}$$

which is much smaller than the naively expected 'hair' ($\delta\pi \sim r_S/r$), does not scale as $1/r$, and is in fact much

[3] For the relation between the field sourced by an object, and the response of the object to an external field of the same kind, see e.g. ref. [17].

smaller than the fields associated with the cosmological background, $h \sim H_0^2 r^2$, whose 'tidal' effects on orbits about a black hole are utterly negligible (for r obeying eq. (18)). Note that there is in principle an ambiguity to the definition of a $1/r$ tail, due to coordinate freedom. Using the conventional definition of r such that the metric decays like $1/r$ away from the black hole, and goes like r^2 approaching the Hubble scale, we have shown that no such $1/r$ tail exists. Allowing for redefinitions of r that do not affect these asymptotic regimes in the metric cannot alter this conclusion.

To see all of the above happen in a concrete example, consider for simplicity the cubic galileon Lagrangian [4]

$$\mathcal{L} = c_2\big[-\tfrac{1}{2}(\partial\pi)^2 + \tfrac{1}{2H_\pi^2}(\partial\pi)^2\Box\pi\big] \,, \tag{22}$$

which in the absence of gravity admits the background solution $\pi_0(r) = \frac{1}{2}H_\pi^2 r^2$. The associated shift current is

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\pi)} - \nabla_\nu\frac{\partial\mathcal{L}}{\partial(\nabla_\nu\nabla_\mu\pi)} \tag{23}$$

$$= c_2\big[-\nabla^\mu\pi + \tfrac{1}{H_\pi^2}\big(\nabla^\mu\pi\Box\pi - \nabla_\nu\nabla^\mu\pi\nabla^\nu\pi\big)\big] \,. \tag{24}$$

Dropping the overall c_2 factor, and going to spherical coordinates with the metric (1), we get

$$J^r = f\cdot\pi'\cdot\big[-1 + \tfrac{1}{2H_\pi^2}\pi'\cdot\big(f' + 2f\rho'/\rho\big)\big] \,. \tag{25}$$

According to the decomposition (12), the terms inside the bracket make up our F. As predicted, the metric coefficients (f, ρ) and their derivatives enter only multiplying π'. For a nearly flat metric in spherical coordinates, we have $f = 1 + \mathcal{O}(h)$, $\rho = r\cdot(1+\mathcal{O}(h))$, so that

$$F = -1 + \tfrac{1}{H_\pi^2}\tfrac{\pi'}{r}\big(1+\mathcal{O}(h)\big) \,, \tag{26}$$

where we used that $h' \sim h/r$, which is appropriate for the distances we are interested in, for which $h \sim r_{\rm S}/r$. Setting $F = 0$ we finally get

$$\pi = \tfrac{1}{2}H_\pi^2 r^2 + \mathcal{O}(H_\pi^2 r^2\cdot h(r)) \,, \tag{27}$$

as predicted.

Concluding remarks. We have shown that static, spherically symmetric black-hole solutions for the gravity-galileon coupled system cannot sustain non-trivial galileon profiles. Our proof does not make use of Einstein's equations—it only uses the shift-symmetry of the galileon action, and the regularity of diff-invariant quantities at the horizon. In would be interesting to extend our analysis to stationary rotating black holes.

For vanishing boundary conditions for the galileon field at infinity, our theorem needs no qualifications. For non-trivial (cosmological) boundary conditions, the question of scalar hair is more subtle. We have shown that for black holes that are much smaller than the asymptotic curvature radius, the π profile contains no large-distance tail of order of the black-hole's gravitational field $h \sim r_{\rm S}/r$. The biggest the scalar hair can be is suppressed by an extra $(H_\pi r)^2$ factor, which makes it completely unobservable, being much smaller than the already unobservable fields associated with the cosmological background. It is worth noting that known black hole solutions in massive gravity are consistent with our results: they either have no galileon hair or suffer from singularities at the horizon (see [18] and references therein).

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