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Controllability transition and nonlocality in network control

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A common goal in the control of a large network is to minimize the number of driver nodes or control inputs. Yet, the physical determination of control signals and the properties of the resulting control trajectories remain widely under-explored. Here we show that: (i) numerical control fails in practice even for linear systems if the controllability Gramian is *ill-conditioned*, which occurs frequently even when existing controllability criteria are satisfied unambiguously; (ii) the control trajectories are generally *nonlocal* in the phase space, and their lengths are strongly anti-correlated with the numerical success rate and number of control inputs; (iii) numerical success rate increases abruptly from zero to nearly one as the number of control inputs is increased, a transformation we term *numerical controllability transition*. This reveals a trade-off between nonlocality of the control trajectory in the phase space and nonlocality of the control inputs in the network itself. The failure of numerical control cannot be overcome by merely increasing numerical precision—successful control requires instead increasing the number of control inputs beyond the numerical controllability transition.

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A system is controllable if its state can be driven to different predefined states by a given set of input control signals, with controllability depending on both the number and the placement of the control inputs [1]. Control often relies on the promise that the direct manipulation of relatively few degrees of freedom can render the entire system controllable. This promise has special meaning in the study of complex networks, where the large total number of nodes contrasts with the limited number that can be directly controlled due to cost and physical constraints. The control of network systems is important in applications as diverse as the operation of infrastructure networks [2], coordination of moving sensors and robots [3], devise of therapeutic interventions [4], management of ecological networks [5], and control of cascading failures in general [6, 7], and it has received increased attention in the recent physics literature [7–12].

A number of significant recent studies have focused on networks with n -dimensional linear time-invariant dynamics [13–16],

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t), \quad (1)$$

with controllability usually determined by the Kalman’s controllability matrix $K = [B \ AB \ \dots \ A^{n-1}B]$ [17]. Given the matrices A and B , control inputs $u(t)$ exist for any initial state $x^{(0)}$ and target state $x^{(1)}$ if and only if K has full row rank. In particular, if all nodes in a network have intrinsic dynamics so that $A_{ii} \neq 0$ for all i , it follows that generically there exists $u(t)$ such that the entire network can in theory be controlled by a single control input [14, 18].

A fundamental question is, however, whether the control signals $u(t)$ can actually be constructed in practice. At first glance, this question may sound dull given that an explicit expression exists for $u(t)$ corresponding to the minimal-energy control trajectories in $t \in [t_0, t_1]$:

$$u(t) = B^T \Phi^T(t_0, t) W^{-1}(t_0, t_1) [\Phi(t_0, t_1) x^{(1)} - x^{(0)}], \quad (2)$$

where $W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, t) B B^T \Phi^T(t_0, t) dt$ is the controllability Gramian and $\Phi(t', t) = e^{(t'-t)A}$ [19]. Incidentally, matrix $W(t_0, t_1)$ being invertible [hence Eq. (2) being

well-defined] is equivalent to the commonly used Kalman’s rank condition mentioned above. Quite surprisingly, despite this explicit solution and formal equivalence, we show that the determination of $u(t)$ is fundamentally limited in networks with more than a handful of nodes. This calls for a careful re-interpretation of existing controllability criteria.

Specifically, in this Letter we show that control fails in practice if the controllability Gramian is ill-conditioned, which can occur even when the corresponding Kalman’s controllability matrix is well-conditioned. We also show that the control trajectory from an initial to a target state is generally non-local in the phase space and remains finite-size (in fact very long) even when the target state is brought arbitrarily close to the initial one. The length $\int_{t_0}^{t_1} \|\dot{x}(t)\| dt$ of such a trajectory generally increases with the condition number of the Gramian. Both the nonlocality of the control trajectory and control failure rate are reduced by increasing the number of control inputs. The latter manifests itself as a sharp transition as a function of the number of control inputs, below which numerical control always fails and above which it succeeds. Aside from its implications for control, the characterization of such a numerical controllability transition adds a new dimension to the research on structural [20], dynamical [21], and algorithmic complexity [22] transitions in networks, which has had broad impacts [23–25], with recent examples ranging from synchronization and percolation to epidemic processes [26–30].

There are known factors that can cause control to fail, including nonlinearity of the dynamics, parameter uncertainty, and stochasticity. Our results show that even in the most favorable case, in which the system is deterministic, autonomous, and linear, the disparity between theory and practice poses a fundamental limit on our ability to control large networks.

We first establish the nonlocality of control trajectories. We say that a state $x^{(0)}$ of system (1) is strictly locally controllable (SLC) if for any ball $B(x^{(0)}, \varepsilon)$ of radius $\varepsilon > 0$ centered at $x^{(0)}$ there is a radius $\delta > 0$ such that any target $x^{(1)}$ inside the concentric ball $B(x^{(0)}, \delta)$ can be reached from $x^{(0)}$ with a control trajectory entirely inside $B(x^{(0)}, \varepsilon)$ —see Fig. 1(a). Note that a state can be locally controllable, in the sense that

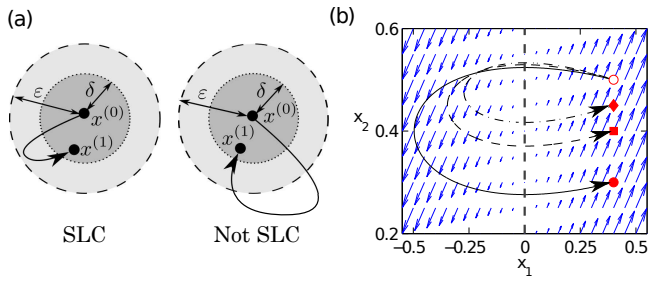


FIG. 1. (a) Illustration of a state that is SLC (left) and of a state that is not (right). (b) Example system $\dot{x}_1 = x_1 + u_1(t)$, $\dot{x}_2 = x_1$, where any state not on the line $x_1 = 0$ is not SLC; the curves indicate minimal-energy control trajectories for the given initial state (open symbol) and target states (solid symbols). The background arrows indicate the vector field in the absence of control. As this two-dimensional example suggests, almost all states of linear systems are not SLC whenever the number of control inputs is smaller than the number of dynamical variables.

control trajectories always exist for neighboring target states, and yet not be SLC. Figure 1(b) shows one such example in two dimensions: for a state in the $x_1 > 0$ half-plane, the control trajectories to any neighboring target state with a smaller x_2 -component necessarily cross into the $x_1 < 0$ half-plane (symmetric results apply to initial states in the other half-plane).

We show that this result is in fact general for any controllable system in which one or more components are not directly controlled. Indeed, if the k^{th} component is not directly controlled, the initial states in the half-space $(Ax)_k > 0$ can only be driven to neighboring target states with $x_k^{(1)} < x_k^{(0)}$ if the control trajectory crosses into the half-space $(Ax)_k < 0$, since otherwise $\dot{x}_k(t) = (Ax(t))_k$ would be nonnegative and $x_k(t)$ would never decrease (analogous argument applies to the other half-space). Therefore, all states outside the hyperplane $(Ax)_k = 0$ are not SLC, and hence almost all states are not SLC. The origin, on the other hand, is the only state that is SLC for any control matrix B when A is nonsingular. The control trajectories defined by Eqs. (1)-(2), which minimize the energy $\int_{t_0}^{t_1} \|u(t)\|^2 dt$, are given by

$$x(t) = \Phi(t, t_0)[x^{(0)} + M_{t_0, t_1, t}(\Phi(t_0, t_1)x^{(1)} - x^{(0)})], \quad (3)$$

where $M_{t_0, t_1, t} = W(t_0, t)W^{-1}(t_0, t_1)$ [19]. The SLC property of the origin then follows from taking the norm on both sides of Eq. (3) for $x^{(0)} = 0$ and upper-bounding the norm of the integral (exponential) terms by the integral (exponential) of the norm, which leads to $\|x(t)\| \leq C\|x^{(1)}\|$ for $C = (t_1 - t_0)\|BB^T\| \cdot \|W^{-1}(t_0, t_1)\|$. What sets the origin aside is that, for invertible A , it corresponds to the only fixed point of the system. Having established the nonlocality of control trajectories for typical states in general, we now study in detail the minimal-energy control trajectories.

We focus on undirected connected networks endowed with the dynamics in Eq. (1) for matrices A with nonzero diagonal elements. These networks generically satisfy the Kalman's controllability condition for a single control input. We consider Erdős-Rényi (ER) networks for a given number of nodes

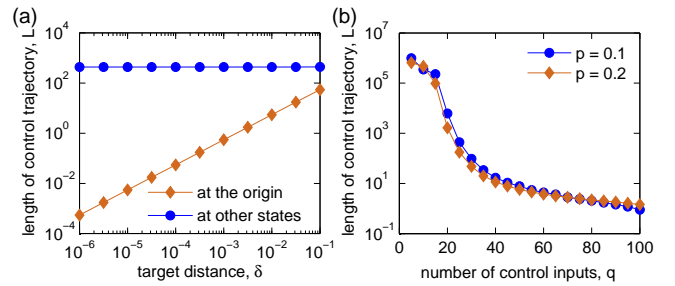


FIG. 2. (a) Average length of the control trajectories as function of the distance from the target when the initial states are away from the origin versus when they are at the origin. In the former case the initial states were chosen randomly on the unit sphere centered at the origin. In both cases the target states are randomly oriented δ apart and $q = 25$. (b) Average length of the control trajectories as function of the number of control inputs for initial states away from the origin and $\delta = 10^{-2}$. Each data point corresponds to 1,000 realizations, for ER networks with $n = 100$.

n and connection probability p . We address the impact of network structure by also considering networks generated using the configuration model [24] for scale-free (SF) degree distributions $P(k) \sim k^{-\beta}$ for $k \geq k_{\min}$, where in our simulations the minimum degree k_{\min} is chosen to keep the average degree fixed as β is varied. The edges and the diagonal elements A_{ii} are assigned weights drawn from a uniform distribution in $[-1, 1]$. Without loss of generality, we assume that the nodes of the networks are one-dimensional dynamical systems [i.e., $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$, where $x_i(t)$ is a scalar variable representing the state of node i] and that the control matrix B is diagonal upon row permutation, so that there is a one-to-one relationship between control inputs and driver nodes. For each network, number q of driver nodes, and given initial and target states, we calculate numerically the minimal-energy control trajectories given by Eq. (3). Unless noted otherwise, we choose the driver nodes randomly at each independent realization and consider the control time window $0 \leq t \leq 1$. Numerical control is declared successful if the numerically calculated state $x(t_1)$ is within a given distance $\eta \ll \|x^{(1)} - x^{(0)}\|$ of the target state $x^{(1)}$, where in our simulations we use $\eta = 10^{-6}$.

Figure 2 shows the average length L of the control trajectories. For typical initial states, L does not approach zero (and in fact remains essentially constant) as the distance to the targets is reduced, in stark contrast with the case in which the initial state is at the origin [Fig. 2(a)]. This behavior becomes more pronounced when the number of driver nodes is small, reaching $L = 10^6$ for $q/n = 0.05$. The average length decreases as q is increased [Fig. 2(b)]. The latter is expected since (following our analytical argument above) the smaller $n - q$ the fewer hyperplanes $(Ax)_k = 0$ the control trajectory has to cross in order to acquire the right sign of $\dot{x}_k(t)$ for every k not directly controlled.

Given the mathematical equivalence of the Kalman's rank condition and the invertibility of the controllability Gramian, one might be tempted to assume that an ill-conditioned controllability Gramian $W(t_0, t_1)$ is a consequence of an ill-conditioned Kalman's controllability matrix K . Here we

disprove this conjecture by showing that the controllability Gramian can be nearly singular even when the corresponding controllability matrix is well-conditioned. This is best characterized by the reciprocal condition number γ . As a mathematically treatable example, we consider a directed linear chain $C(n)$ containing n nodes and no self-loops: $1 \rightarrow 2 \rightarrow 3 \dots \rightarrow n$. We examine the control of this network through the direct control of the root node, i.e., node 1. The control matrix is such that $B_i = \delta_{1,i}$ and, assuming for simplicity that the network is unweighted, matrix A is given by $A_{ij} = \delta_{i,j+1}$ for $i, j = 1 \dots n$. It follows from Eq. (1) that K is the $n \times n$ identity matrix. Therefore, K has full rank, has reciprocal condition number $\gamma = 1$, and is in fact the best conditioned of all matrices. Now, consider $W(t_0, t_1)$. Taking for convenience $t_0 = 0$, we can show that $W(0, t_1) = \int_0^{t_1} g(\tau)g^T(\tau)d\tau$, where $g(\tau) = \left[1, -\tau, \frac{(-\tau)^2}{2!}, \dots, \frac{(-\tau)^{i-1}}{(i-1)!}, \dots, \frac{(-\tau)^{n-1}}{(n-1)!}\right]^T$. Thus, we can calculate the analytic expression for each entry of the Gramian: $W(0, t_1)_{ij} = -\frac{(-t_1)^{i+j-1}}{(i+j-1)(i-1)(j-1)!}$. For fixed control time t_1 , the reciprocal condition number γ of $W(0, t_1)$ decreases exponentially as a function of the number of nodes in the linear chain [Fig. 3(b), inset]. Therefore, as the size of the linear chain increases, the controllability Gramian quickly becomes nearly singular, making the numerical control of system (1) nearly impossible, even though the Kalman's controllability matrix remains well-conditioned.

Figure 3 shows that this behavior is indeed general for the ER networks we consider. The reciprocal condition number of W decreases exponentially as the number of control inputs is reduced [Fig. 3(a)] or the size of the network is increased [Fig. 3(b)], while the reciprocal condition number of K (not shown) is generally larger than $\sqrt{\gamma(W)}$ [31]. Moreover, it follows that the average length of the control trajectories is strongly correlated with the condition number of the controllability Gramian [Fig. 3(a), inset]. This can be rationalized by noting that both L and $\gamma(W)$ are measures of how difficult it is to actually control the system in practice. Therefore, although the Kalman's controllability matrix K has attractive analytical properties, its use in practice requires caution. In particular, since the minimum-energy control involves the inversion of the controllability Gramian W (rather than of the Kalman's controllability matrix), we claim that it is the numerical rank of W that is often most relevant.

A full rank matrix is *numerically* rank deficient if one or more of its singular values fall below the predefined numerical threshold. In a matrix with a bounded largest singular value, a numerical rank smaller than the actual rank is necessarily related to a small reciprocal condition number γ , since γ is the ratio of the smallest to the largest singular values. While leading to a deficient numerical rank for matrix W when q/n is small, this is not a factor for matrix K because this matrix is reasonably well-conditioned in the networks we consider, indicating that the numerical calculation of its rank is reliable. Indeed, for all networks considered in Fig. 3 we verified that the numerical rank of K is n for any numerical threshold smaller than 10^{-3} .

Figure 4(a) shows that the numerical success rate increases

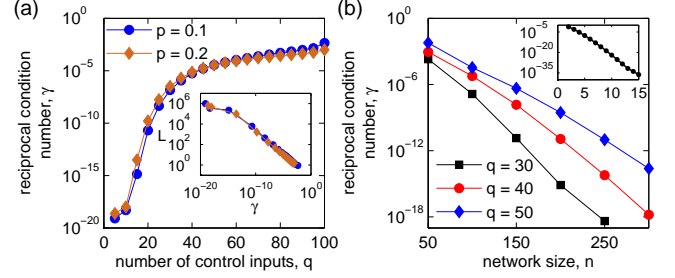


FIG. 3. Reciprocal condition number of the controllability Gramian W as function of (a) the number of control inputs for $n = 100$ and (b) the network size for ER networks with $np = 10$. Insets: relation between the average length of the control trajectories and reciprocal condition number of W [panel (a)]; reciprocal condition number of W of the (analytically solvable) chain networks $C(n)$ for $q = 1$ [panel (b)]. The statistics and parameters not shown are the same used in Fig. 2(b).

sharply from zero to one as the number of driver nodes is increased. According to the Kalman's rank condition, all networks we simulate are controllable in theory for q as small as 1. The transition in Fig. 4(a) is a direct consequence of the decrease in the condition number of the controllability Gramian and the limitation it imposes on numerical calculations. The numerical error in computing Eq. (3) is dominated by the round-off errors in the calculation of W^{-1} . Taking $t = t_1$ and using tilde to denote numerically computed values, we obtain

$$\|\tilde{x}^{(1)} - x^{(1)}\| \lesssim D \|W(\tilde{W}^{-1} - W^{-1})\|, \quad (4)$$

where $z = \Phi(t_0, t_1)x^{(1)} - x^{(0)}$ and $D = \|\Phi(t_1, t_0)\| \cdot \|z\|$. If $W + \Delta W$ is the exact inverse of \tilde{W}^{-1} , then the right side of Eq. (4) is bounded from above by $D \|\Delta W\| \cdot \|\tilde{W}^{-1}\| \approx D \|W\| \cdot \|W^{-1}\| \cdot \frac{\|\Delta W\|}{\|W\|}$, where $\|W\| \cdot \|W^{-1}\| = 1/\gamma(W)$ and $\frac{\|\Delta W\|}{\|W\|}$ is of the order of the numerical precision ϵ [32]. Thus, we predict that the transition to successful numerical control will occur in general as $\gamma(W)$ decreases past

$$\gamma_c \approx D' \frac{\epsilon}{\eta}, \quad (5)$$

where D' is a constant determined by t_1 , $x^{(0)}$, $x^{(1)}$, and A , and η is the radius of convergence. For double precision and $\eta = 10^{-6}$, as considered in our simulations, the transition is predicted to occur around $\gamma = 10^{-10}$ for D' of order unity, which agrees with our numerics.

To further characterize this transition, we analyze its width within the network ensemble, defined as $\Delta q \equiv (q_{c'} - q_c)/n$, where q_c and $q_{c'}$ mark the integer number of control inputs right below 5% and right above 95% success rate, respectively. As shown in Fig. 4(b), the transition becomes increasingly sharp as the size of the network increases. The transition point, which we take as being q_c/n for the purpose of this discussion, is around 0.20 and increases slowly as n increases.

To address the impact of degree heterogeneity, we have also analyzed the controllability transition in SF networks. As shown in Fig. 4(c,d), the transition becomes wider as the variance of the degree distribution increases (i.e., β decreases),

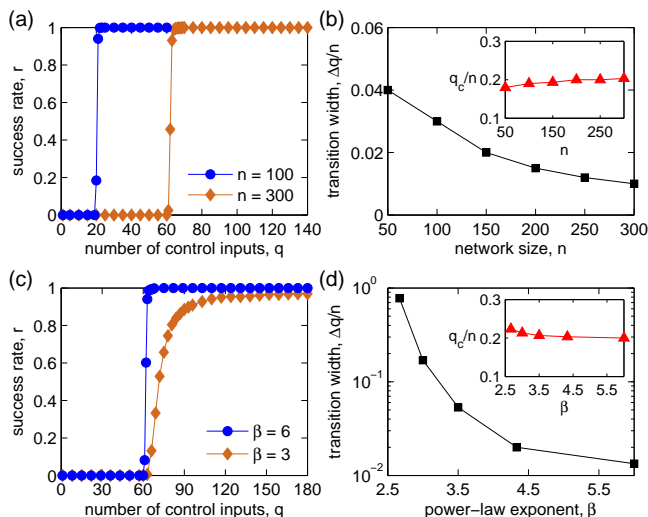


FIG. 4. Numerical controllability transitions for (a,b) ER networks of different sizes and (c,d) SF networks of different scaling exponents. (a,c) Success rate as function of the number of control inputs. (b,d) Transition width (main panel) and transition point (inset). The average degree was set to 10 and, for the SF networks, $n = 300$. The statistics and parameters not shown are the same used in Fig. 2(b).

indicating that control fails more often the more heterogeneous the degree distribution [13], but the starting point of the transition is very insensitive to the degree distribution and is essentially the same for ER and SF networks with the same network size and average degree [e.g., Fig. 4(c) with 4(a) for $n = 300$]. Moreover, these conclusions do not depend sensitively on how the driver nodes are selected: we have verified that the curves in Fig. 4(a,c) shift horizontally by less than the size of the symbols when the driver nodes are selected not randomly but instead as the lowest- or the highest-degree nodes in the network.

A few observations are in order. First, one may ask whether the impossibility of controlling the system with a reduced number of driver nodes could be avoided by increasing the precision of the numerical calculations. Because the reciprocal condition number $\gamma(W)$ decrease exponentially as q is reduced while γ_c decreases only linearly as ϵ is reduced, any realistic increase in precision will only have a small effect in

reducing the critical fraction of driver nodes q_c/n at which the numerical controllability transition takes place. Thus, tantamount to the impossibility of long-term predictability in chaotic dynamics, this fundamental limitation cannot be overcome by any realistic increase of precision, and becomes even more pronounced for larger networks.

Second, subtle differences in the formulation of the dynamics in Eq. (1) have led to very different conclusions about the minimal number of driver nodes according to the Kalman's rank condition, being generically one if the diagonal elements of A are nonzero [14, 18] and generally much larger than one if they are not [13, 33]. It is thus satisfying to find that in practice the results are far more robust against small changes in the model parameters.

Third, the nonlocality of linear control trajectories identified here reveals an intriguing mechanism of failure in applying control results from linearized dynamics to their nonlinear counterparts [34]: such an approach fails because the control trajectories are required to go outside the neighborhood in which the linearization is valid. Moreover, there are known examples of nonlinear systems that are globally controllable while their linearization are not [35], and it is straightforward to identify network systems too which have this property [36]. It is thus natural to speculate that nonlinear control approaches (see, e.g., [37]), although generally more involved and system-specific, can help lead to a framework better suited to address such systems.

Our demonstration that either the control trajectory is nonlocal in the phase space or the control inputs are nonlocal in the network has several implications. In practice, the former leads to failure of the numerical control and the latter points to an overhead that has to be accounted for in optimizing the number of driver nodes. For the random networks considered here, this gives rise to a sharp transition as a function of the number of driver nodes, below which numerical control always fails and above which it succeeds. These findings suggest the need of a controllability criterion that accounts not only for the existence but also for the actual computability of the control interventions. We suggest that by applying such criterion future research may reveal a rich set of relations between controllability and network structure even when no dependence is predicted by the Kalman's rank condition.

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