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Fractional Topological Insulators of Cooper Pairs Induced by the Proximity Effect

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Certain insulating materials with strong spin-orbit interaction can conduct currents along their edges or surfaces owing to the non-trivial topological properties of their electronic band-structure. This phenomenon is somewhat similar to the integer quantum Hall effect of electrons in strong magnetic fields. Topological insulators analogous to the fractional quantum Hall effect are also possible, but have not yet been observed in any material. Here we show that a quantum well made from a topological band insulator such as Bi$_2$Se$_3$ or Bi$_2$Te$_3$, placed in contact with a superconductor, can be used to realize a two-dimensional topological state with macroscopic many-body quantum entanglement whose excitations carry fractional amounts of electron’s charge and spin. This fractional topological insulator is a “pseudogap” state of induced spinful $p$-wave Cooper pairs, a new strongly correlated quantum phase with possible applications to spintronic devices and quantum computing.

The recently discovered two-dimensional topological insulators (TI) with time-reversal (TR) symmetry$^{1-4}$ are band-insulators related to integer quantum Hall states in which electron spin plays the role of charge. They can be obtained in HgTe, Bi$_2$Te$_3$ and Bi$_2$Se$_3$ quantum wells owing to the strong spin-orbit coupling, and exhibit topologically protected gapless edge states despite the spin non-conservation$^5$. The properties of quantum wells are linked to the topologically protected surface states of the extensively studied bulk materials$^{6-8}$.

Instabilities caused by interactions among electrons can establish unconventional quantum states in TIs, with broken symmetries$^{9,10}$ or topological order$^{11-14}$. These envisioned forms of quantum matter could realize robust macroscopic entanglement between spatially separated electrons in the TI materials, which motivates both the fundamental research and the quest for applications in spintronics and quantum computing. Here we aim to realize a new class of strongly correlated TIs that exhibit phenomena reminiscent of the fractional quantum Hall effect (FQHE) in strong magnetic fields, but without its TR symmetry violation$^{15-22}$. Such fractional TIs feature quasiparticles that carry fractional amounts of electron’s charge and spin. Exotic states with non-Abelian statistics are also possible and promise the ability to perform quantum computation with a greater level of quantum control than in FQHE qubits, because both charge and spin can be manipulated and entangled.

One approach to obtaining fractional TIs, inspired by the FQHE, exploits Coulomb interactions among electrons in a partially populated band made narrow by the spin-orbit coupling$^{23-25}$. It might be very difficult to find TI materials with sufficiently narrow bands and strong interactions, so the goal of this paper is to propose a different approach. Here we consider a heterostructure device in which a two-dimensional electron gas can be tuned near a quantum critical point (QCP). Every quantum critical system is sensitive to relevant perturbations that impose their energy scales on dynamics and define the phases that surround the critical point in the phase diagram. We will show that the spin-orbit coupling is characterized by a large “cyclotron” energy, and thus indeed represents a relevant perturbation that can dominate near the critical point and stabilize fractional topological states just like a strong magnetic field would. The proposed heterostructure is not only routinely achievable, but also provides the best platform to experimentally seek a variety of topologically non-trivial superconducting and insulating quantum states that have not been observed or hypothesized before, and whose existence is guaranteed by the fundamental principles discussed here.

We engineer a QCP by placing a TI quantum well in contact with a conventional superconductor (SC) as shown in Fig.1. The SC’s pairing glue induces a weak short-range attractive interaction between the TI’s electrons, but the TI’s two-dimensionality assures the formation of bound-state Cooper pairs for any interaction strength$^{27}$. Electrons could then be pulled into the TI and immediately bound into pairs by applying a gate voltage, causing a bosonic mean-field quantum phase transition to a superconducting state in the TI$^{28-30}$. The
It describes four electron states per momentum $\mathbf{p}$, labeled by the spin projection $S^z = \pm \frac{1}{2}$ (in the $\hbar = 1$ units), and the orbital index $\tau^z = \pm 1$ equivalent to the top/bottom surface of the quantum well. The vector spin operator is $\mathbf{S} = \frac{\sqrt{2}}{2} \sigma^a \mathbf{r}^a$, $a \in \{x, y, z\}$, and $\sigma^a$ are Pauli matrices that operate on the spin and orbital states respectively. The static Yang-Mills SU(2) gauge field $\mathbf{A}$ embodies the Rashba spin-orbit coupling $H_{\text{SO}} = \mathbf{v} \cdot (\mathbf{S} \times \mathbf{p}) \tau^z$ and produces a massless Dirac spectrum if $\Delta = 0$. However, inter-surface tunneling $\Delta \neq 0$ opens a bandgap, assuming that the model applies only to momenta $p < \Lambda = \sqrt{(mv)^2 - (\Delta / v)^2}$. A natural cutoff $\Lambda$ is provided by the lattice potential in materials. The mass $m$ describes a small Dirac-cone curvature seen in ARPES measurements. Fig.2(b) shows that (1) adequately approximates materials, with a relatively large fitted $m$. This model has the relativistic particle-hole symmetry when $\mu = 0$ and $m \to \infty$. Its many-body ground state is a band-insulator for $|\mu| < |\Delta|$, which is topological when $\Delta$ has a proper sign $^2$

The spin-orbit SU(2) gauge field from (1) carries a non-zero “magnetic” Yang-Mills flux $^{34}$ $(\mu, \nu, \tau \in \{x, y, z\})$:

$$\Phi^\mu = \epsilon^{\mu\nu\tau} (\partial_\nu A_\tau - i \tau^z A_\nu A_\tau) = \frac{1}{2} (mv)^2 \delta_{\mu\nu} \tau^z \tau^\nu . \quad (2)$$

Note that the SU(2) charge $\tau^z$ is required here by gauge invariance. Being a generalization of the U(1) magnetic flux responsible for the Hall effect, the SU(2) flux is the source of topological phenomena in TIs and sets their “cyclotron” energy scale $\omega_b = mv^2$. Our construction of the effective action for interacting electrons will greatly benefit from exposing the SU(2) gauge symmetry of the idealized model (1). At the end, we will discuss the consequences of gauge symmetry violations in real materials.

The electron dynamics in the TI quantum well is altered by the proximity to the SC in the device from Fig.1. The SC is a fully gapped quantum liquid of Cooper pairs characterized by two energy scales, the pairing $\Delta_p$ and photon $\Delta_g = hc\lambda_L^{-1}$ gaps, where $c$ is the speed of light and $\lambda_L$ is the London penetration depth. Fermionic quasiparticles have anomalously small or vanishing density of states below the pairing gap, which is $\Delta_p = 2hef/\pi \xi$ in conventional superconductors with Fermi velocity $v_f$ and coherence length $\xi$. The smaller of the two gaps defines a cut-off energy for the low-energy dynamics in the TI that we shall discuss. The dynamics responsible for the triplet superconductor-insulator transition in the TI is indeed defined below this cut-off and hence can be captured by a two-dimensional effective theory whose degrees of freedom are decoupled from those of the SC. We will show that the resulting theory indeed features a triplet superconductor-insulator transition inside the TI across which $\Delta_p \neq 0$.

Our effective TI model is given by the imaginary-time action $S = \int d\tau d^2 \psi^\dagger (\partial_\tau + H) \psi + S_{\text{int}}$. Living near the conventional SC, all TI’s electrons couple to its phonons and thus acquire BCS-like short-range attractive interactions among themselves, irrespective of their spin or ort-
bital state. This is generic, but overcoming the Coulomb repulsion in the TI requires a sufficiently strong pairing in the SC and a sufficiently thin quantum well. Without knowing the microscopic form and strength of these interactions, we must consider all channels:

$$S_{\text{int}} = \frac{1}{2} \int d\mathbf{r} d\mathbf{r'} \left( U_1 \psi_{\tau\sigma}^\dagger \psi_{\tau\sigma}^\dagger \psi_{\tau\sigma} \psi_{\tau\sigma} + U_2 \psi_{\tau\sigma}^\dagger \psi_{\tau\sigma}^\dagger \psi_{\tau\sigma} \psi_{\tau\sigma} + U_3 \psi_{\tau\sigma}^\dagger \psi_{\tau\sigma}^\dagger \psi_{\tau\sigma} \psi_{\tau\sigma} + \cdots \right)$$

(3)

Here $\tau = \pm 1$ and $\sigma = \pm 1$ label the orbital $\tau^z$ and spin $S^z$ states of the electron fields $\psi_{\tau\sigma}$ respectively ($\tau^z = -\tau$), while the dots denote weak orbital-non-conserving forces.

By applying the Hubbard-Stratonovich transformation on the path-integral, we can eliminate the interaction couplings (3) in favor of six Cooper pair fields displayed in Fig.2(c): two intra-orbital singlets $\eta_{\pm}$ ($U_2/3$ at $\sigma = \sigma'$), inter-orbital singlet $\phi_0$ and $S_\parallel = 0$ triplet $\eta_{0}$ ($U_2/3$ at $\sigma \neq \sigma'$):

$$S_{\text{int}} = \int d\mathbf{r} d\mathbf{r'} \left\{ \sum_{\tau = \pm 1} \left( \frac{1}{2} u |\phi_0|^2 + \phi_0^\dagger \phi_{\sigma\tau}^\dagger \psi_{\tau\sigma} \psi_{\tau\sigma} + h.c. \right) \right. + u' |\phi_0|^2 + \phi_0^\dagger \phi_{\sigma\tau}^\dagger \psi_{\tau\sigma} \psi_{\tau\sigma} + h.c. \) + \sum_{\sigma = \pm 1} \left( \frac{1}{2} v |\eta_0|^2 + \eta_0^\dagger \eta_{\sigma\tau}^\dagger \psi_{\tau\sigma} \psi_{\tau\sigma} + h.c. \right) + \left( \frac{1}{2} v' |\eta_0|^2 + \eta_0^\dagger \eta_{\sigma\tau}^\dagger \psi_{\tau\sigma} \psi_{\tau\sigma} + h.c. \right) \}$$

(4)

The symmetry transformations of these fields are summarized in the Table I. In conjunction with (1), the SU(2) symmetry would imply $v = v'$. Fermionic excitations remain gapped across superconductor-insulator quantum phase transition in simple two-dimensional band-insulators with attractive interactions, and we will explain shortly why this also holds in TIs. Then, we may integrate-out the gapped fermion fields in the path-integral to obtain a purely bosonic effective action $S_{\text{eff}}$ that describes Cooper pair dynamics at energies below the pairing gap. We can avoid a complicated calculation by relying on symmetries to construct the Landau-Ginzburg form of $S_{\text{eff}}$. Since two electrons with the same spin from different orbitals have the same cyclotron chirality, the Cooper pairs $\eta_\pm$ with $S^z = \pm 1$ possess the SU(2) charge, unlike the singlet fields. Together with $\eta_0$ they form a triplet $\eta = (\eta_-, \eta_0, \eta_+)$ that minimally couples to the same SU(2) gauge field $\mathbf{A}$ as (1) but expressed in the $S = 1$ representation. This can be seen from the local SU(2) transformations in the Table I. Therefore,

$$S_{\text{eff}} = \int d\mathbf{r} d^2r \left\{ \phi^\dagger \partial_0 \phi + (\nabla \phi)^\dagger \tilde{K}_s (\nabla \phi) + \phi^\dagger \tilde{t}_s \phi \right. + \eta^\dagger \partial_0 \eta + K_1 |(\nabla - i\mathbf{A}) |\eta|^2 + \left( t_1 + \phi^\dagger \tilde{t}_s \phi \right) |\eta|^2 + U_1 |\eta|^4 + \left. U_{s,\sigma_1,\sigma_2,\sigma_3} \phi_{\sigma_1}^\dagger \phi_{\sigma_2}^\dagger \phi_{\sigma_3} + \Delta_1 \phi - \Delta_2 \phi^\dagger \right\}.$$  

(5)

Some Cooper pair modes may have energy in the two-electron continuum, and should be expelled from $S_{\text{eff}}$. We omitted Coulomb interactions, and used the most general non-relativistic dynamics. We organized the singlet fields into a vector $\phi = (\phi_-, \phi_0, \phi_+)$ and wrote their quadratic couplings in the matrix form. The vector $\Delta_\sigma$ depends on the SC’s order parameter and the SC-TI interface. The singlet matrices $K_s, \tilde{t}_s$ and tensor $U_s$ are TR-invariant, and realistic SU(2) symmetry violations can be captured by additional triplet couplings.

Inter-orbital triplets compete with singlets. One of the intra-orbital singlet channels has a stronger induced interaction than all inter-orbital channels for geometric reasons, which naively means that singlets should condense before triplets when electrons are drawn into the TI from the SC by the gate voltage. Here we neglect the intrinsic singlet condensation due to $\Delta_\sigma \neq 0$, made small by the TI’s bandgap. However, the Rashba spin-orbit coupling in $A_s$ mixes the triplets into two helical modes, analogous to the Dirac conduction and valence band eigenstates of (1). One helical mode has energy that decreases when its momentum grows (like the Dirac valence band), and thus “always” condenses at sufficiently large momenta according to (5). It has a natural advantage over singlets despite its origin in the weaker induced interaction.

The helical condensate locally gains Rashba energy $\mathbf{z} \times \mathbf{p} < 0$ in (5) by coupling to $A_s$ the TR-invariant currents of properly oriented spin (perpendicular to the current flow). Such a state is globally in equilibrium only if the currents flow in loops. The optimal configuration is always a vortex lattice, illustrated in Fig.3, and its existence also gives birth to fractional TIs. Imagine tuning the gate voltage to reduce the superconducting stiffness $\rho_s$ toward zero. The vortex kinetic energy due to zero-point quantum motion can be estimated from the Heisenberg uncertainty as $E_{\text{kin}} \sim \hbar^2/2m_v$, where $\hbar$ is the SU(2) “magnetic length”, and $m_v$ is the effective vortex mass. In (charged) superconductors, $m_v$ is roughly constant as $\rho_s \rightarrow 0$, but turns into $m_v \sim \log(\rho_s)$ when

<table>
<thead>
<tr>
<th>$T_\tau$ translations</th>
<th>$\psi_{\tau\sigma}(\mathbf{k})$</th>
<th>$\phi_\sigma(\mathbf{k})$</th>
<th>$\eta_\sigma(\mathbf{k})$</th>
</tr>
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<tbody>
<tr>
<td>$R_\theta$ rotations</td>
<td>$\psi_{\tau\sigma}(R_\theta \mathbf{k})$</td>
<td>$\phi_\sigma(R_\theta \mathbf{k})$</td>
<td>$\eta_\sigma(R_\theta \mathbf{k})$</td>
</tr>
<tr>
<td>$R_\pi$, spatial reflect.</td>
<td>$\psi_{\tau\sigma}(R_\pi \mathbf{k})$</td>
<td>$-\phi_\sigma(R_\pi \mathbf{k})$</td>
<td>$\eta_\sigma(R_\pi \mathbf{k})$</td>
</tr>
<tr>
<td>$T_\tau$ time reversal</td>
<td>$\psi_{\tau\sigma}(\mathbf{k})$</td>
<td>$\phi_\sigma(\mathbf{k})$</td>
<td>$\eta_\sigma(\mathbf{k})$</td>
</tr>
<tr>
<td>$C$ charge U(1)</td>
<td>$e^{i\varphi} \psi_{\tau\sigma}(\mathbf{k})$</td>
<td>$e^{i\varphi} \phi_\sigma(\mathbf{k})$</td>
<td>$e^{i\varphi} \eta_\sigma(\mathbf{k})$</td>
</tr>
<tr>
<td>$S$ spin U(1)</td>
<td>$e^{i\varphi} \psi_{\tau\sigma}(\mathbf{k})$</td>
<td>$e^{i\varphi} \phi_\sigma(\mathbf{k})$</td>
<td>$e^{i\varphi} \eta_\sigma(\mathbf{k})$</td>
</tr>
<tr>
<td>local spin SU(2)</td>
<td>$W_{\tau\sigma} \psi_{\tau\sigma}(\mathbf{k})$</td>
<td>$\phi_\sigma(\mathbf{k})$</td>
<td>$U_{\alpha\mu\tau\sigma}(\mathbf{k})$</td>
</tr>
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the screening length $\lambda_L \sim \rho_x^{-1/2}$ diverges\textsuperscript{37}. The potential energy due to the vortex lattice stiffness scales as $E_{\text{pot}} \sim \rho_x$ per vortex ($\rho_x^2$ if the spectrum has Landau levels), which easily follows from the free energy expansion\textsuperscript{38} in powers of $\rho_x$. There is a critical finite $\rho_x$ at which the vortex lattice melts in a first order transition because $E_{\text{kin}} \geq E_{\text{pot}}$. This happens at the solid line that separates the SC and VL regions in Fig.2(a). Since $\rho_x$ also measures the quasiparticle pairing gap, its finite value implies that the transition is shaped by the Cooper pair dynamics below the fermion excitation gap. The resulting insulator is a quantum vortex liquid of uncondensed Cooper pairs, whose qualitative properties are captured by the purely bosonic theory (5).

Quantum liquids of SU(2) vortices are the prime candidates for fractional TIs when their density is comparable or larger than the density of Cooper pairs (otherwise, Mott or density-wave insulators are stable). This expectation is based on the transitions from vortex lattice condensates to fractional quantum Hall states in the analogous system of bosons in (effective) magnetic fields\textsuperscript{39–43}. The mass $m$ in (1) can be estimated from the curvatures of the Dirac cones in ARPES experiments\textsuperscript{26}, and it is larger than the “spin-orbit” mass $m_{so}$ of Dirac quanta $n_\phi = \Phi/h^2 = \lambda^2 \Delta^2/(\omega h)^2 \sim \lambda^2 \times 2 \times 10^{15} \text{ m}^{-2}$. These estimates look promising if we compare them with typical flux-quantum densities $n_\phi = B(\hbar c/e)^{-1} \approx 2.5 \times 10^{15} \text{ m}^{-2}$ (in $B = 10$ T) and cyclotron scales $\omega_\phi = eB/mc \approx 1 \text{ meV}$ of electrons in fractional quantum Hall states. The TI’s Cooper pair density is controlled by the gate voltage, and can be brought near and below $n_\phi$ to stabilize a fractional incompressible quantum liquid in a finite parameter range of size $\omega_\phi$ surrounding the QCP in Fig.2(a). Detecting fractional charge and statistics in the absence of magnetic fields, by shot-noise or quantum interferometry methods from FQHE experiments\textsuperscript{44,45}, would provide clear evidence of an established fractional TI in the quantum well.

Without microscopic modeling and experimental data we cannot rule out a possibility that singlets would condense before triplets in a particular device. But even then, a further raise of the gate voltage would eventually condense triplets. Singlets cannot completely screen out the gate from triplets because they repel each other stronger than they repel the triplets, by the Pauli exclusion principle. Future experimental probes of topological spin dynamics may be able to reveal fractional $\eta$ vortex liquids even if they coexisted with a singlet superconducting state of the $\phi$ fields (which cannot screen spin).

Finding the precise nature of the fractional TIs goes beyond the scope of this paper as it requires the exact diagonalization of a microscopic model. Instead, we can illustrate their bosonic character by a simple example, such as the bosonic Laughlin wavefunction\textsuperscript{23} of 2$N$ triplet Cooper pairs $\eta_{\pm}$ whose coordinates are $z_{i\pm}$:

$$\Psi = \prod_{i<j} \left( z_{i+} - z_{j+} \right)^n \left( z_{i-} - z_{j-} \right)^n \prod_{i} e^{-\frac{|z_{i+}|^2 + |z_{i-}|^2}{\Delta^2}}.$$  

The integer $n$ is even, and this Abelian TR-invariant state has excitations with fractional charge $2e/n$, spin $h/n$ and spin-Hall conductivity $\sigma^x_{\phi} = 4e^2/h(n^2)$. Since $\langle |\eta_{\pm}|^2 \rangle = \Phi/(2\pi n)$ and $\Phi$ can be calculated from (5) and (2), one can find $n$ in any ground state and identify Laughlin states by integer-valued $n$. The wavefunctions of hierarchical quantum spin-Hall states can also be constructed\textsuperscript{21,22}. They all describe TR-invariant vortex liquids of spinful bosons (with vortex density $l^{-2}$), and thus are not far from being good candidates for the fractional TIs in our system. However, they are not adequate either because the $S^z$ spin component is not conserved. It is presently unknown how to write a proper wavefunction for a fractional TI shaped by the Rashba spin-orbit coupling, but an effective field-theory description is available and points to the naturally non-Abelian character of the ensuing incompressible quantum liquids.\textsuperscript{36,46,47}

Instead of $S^z$, the spin quantum number in an ideal Rashba-based TI is the eigenvalue of $\pm \hat{z} (S^z \times \mathbf{p})$ as evident from (1). If it were conserved, measuring its average on the fractional TI’s quasiparticles in the momentum $\mathbf{p}$ eigenstate would yield a fraction of $\pm \hbar$. However, the realistic complete spin non-conservation, manifested as a gauge symmetry violation in (5), spoils the measurements of fractional spin. At least there is no obstacle to observing the conserved fractional charge, so the fractional TIs can exist. The fractional spin is a degree of freedom rather than a quantum number of quasiparticles (which has a mixed spin and orbital character). Combining an integer number of fractional quasiparticles must reconstitute a triplet Cooper pair, so the quasiparticles must inherit from it a degree of freedom that transforms like spin under time-reversal and spans multiple basis states. Its fractional quantization is guaranteed by the fundamental properties of vortex dynamics in incompressible quantum liquids, and its spin-orbit coupling may yield new topological orders not found in FQHE systems.

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4. Y. Zhang et al., Nature Physics 6, 584 (2010).
36. See Supplementary Material for a detailed calculation of the vortex lattice character in the helical condensate, and a field-theoretical perspective on the fractional TIs obtained upon this vortex lattice melting.
43. N. R. Cooper, Advances in Physics 57, 539 (2008).