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Fractional topological insulators of Cooper pairs induced by proximity effect

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Certain insulating materials with strong spin-orbit interaction can conduct currents along their edges or surfaces owing to the non-trivial topological properties of their electronic band-structure. This phenomenon is somewhat similar to the integer quantum Hall effect of electrons in strong magnetic fields. Topological insulators analogous to the fractional quantum Hall effect are also possible, but have not yet been observed in any material. Here we show that a quantum well made from a topological band insulator such as Bi_2Se_3 or Bi_2Te_3, placed in contact with a superconductor, can be used to realize a two-dimensional topological state with macroscopic many-body quantum entanglement whose excitations carry fractional amounts of electron's charge and spin. This fractional topological insulator is a "pseudogap" state of induced spinful p-wave Cooper pairs, a new strongly correlated quantum phase with possible applications to spintronic devices and quantum computing.

The recently discovered two-dimensional topological insulators (TI) with time-reversal (TR) symmetry[1–4] are band-insulators related to integer quantum Hall states in which electron spin plays the role of charge. They can be obtained in HgTe, Bi_2Te_3 and Bi_2Se_3 quantum wells owing to the strong spin-orbit coupling, and exhibit topologically protected gapless edge states despite the spin non-conservation[5]. The properties of quantum wells are linked to the topologically protected surface states of the extensively studied bulk materials[6–8].

Instabilities caused by interactions among electrons can establish unconventional quantum states in TIs, with broken symmetries[9,10] or topological order[11–14]. These envisioned forms of quantum matter could realize robust macroscopic entanglement between spatially separated electrons in the TI materials, which motivates both the fundamental research and the quest for applications in spintronics and quantum computing. Here we aim to realize a new class of strongly correlated TIs that exhibit phenomena reminiscent of the fractional quantum Hall effect (FQHE) in strong magnetic fields, but without its TR symmetry violation[15–22]. Such fractional TIs feature quasiparticles that carry fractional amounts of electron's charge and spin. Exotic states with non-Abelian statistics are also possible and promise the ability to perform quantum computation with a greater level of quantum control than in FQHE qubits, because both charge and spin can be manipulated and entangled.

One approach to obtaining fractional TIs, inspired by the FQHE, exploits Coulomb interactions among electrons in a partially populated band made narrow by the spin-orbit coupling[23–25]. It might be very difficult to find TI materials with sufficiently narrow bands and strong interactions, so the goal of this paper is to propose a different approach. Here we consider a heterostructure device in which a two-dimensional electron gas can be tuned near a quantum critical point (QCP). Every quantum critical system is sensitive to relevant perturbations that impose their energy scales on dynamics and define the phases that surround the critical point in the phase diagram. We will show that the spin-orbit coupling is characterized by a large "cyclotron" energy, and thus indeed represents a relevant perturbation that can dominate near the critical point and stabilize fractional topological states just like a strong magnetic field would. The proposed heterostructure is not only routinely achievable, but also provides the best platform to experimentally seek a variety of topologically non-trivial superconducting and insulating quantum states that have not been observed or hypothesized before, and whose existence is guaranteed by the fundamental principles discussed here.

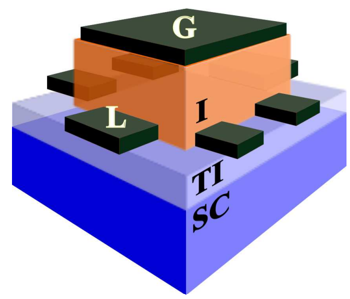

FIG. 1: The heterostructure device that can host fractional TR-invariant quantum states. A topological insulator (TI) quantum well is sandwiched between a conventional superconductor (SC) and a conventional insulator (I). The gate (G) voltage can be used to control the state of the TI, and the topological properties of the TI can be probed via a Hall-bar setup of leads (L).

We engineer a QCP by placing a TI quantum well in contact with a conventional superconductor (SC) as shown in Fig.1. The SC's pairing glue induces a weak short-range attractive interaction between the TI's electrons, but the TI's two-dimensionality assures the formation of bound-state Cooper pairs for any interaction strength[27]. Electrons could then be pulled into the TI and immediately bound into pairs by applying a gate voltage, causing a bosonic mean-field quantum phase transition to a superconducting state in the TI[28–30]. The

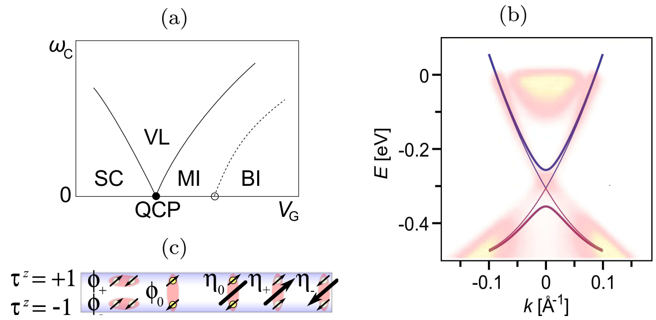

FIG. 2: (a) The qualitative zero-temperature phase diagram of attractively interacting electrons in a quantum well. Gate voltage V_{G} controls the electron gap in the quantum well and tunes the quantum critical point (QCP) between a superconductor (SC) and a bosonic Mott insulator (MI) of Cooper pairs. The lowest energy excitations in the MI are charge $2e$ bosons, but they disappear at the crossover (dashed line) to the band-insulator (BI) where only gapped fermionic quasiparticles with charge e exist. A spin-orbit coupling whose strength is measured by a "cyclotron" energy ω_Φ (defined in the text) introduces a vortex lattice in the superconducting state of spinful p-wave Cooper pairs. Quantum melting of such a vortex lattice gives rise to correlated "vortex liquid" (VL) states, which are the prime candidates for fractional TIs. (b) The energy spectrum $E(k)$ of the Hamiltonian (1) for $m \sim 2.5\cdot 10^{-31}$ kg, $v \sim 4\cdot 10^5$m/s and $\Delta \sim 100$ eV ($k = p/\hbar$). This two-orbital example approximates the ARPES spectrum from the Fig.3d of Ref.[26] when $\Delta = 0$. (c) The Cooper pairing channels in the TI include intra-orbital spin singlets ($\Phi_\pm$), inter-orbital spin singlets (Φ_0), and inter-orbital p-wave spin triplets (η_m, where $m \in \{0, \pm 1\}$ is the z-axis spin projection).

ensuing QCP could naively occur in any pairing channel, but the conventional proximity effect (order parameter leakage) washes out such a QCP of singlets. Only triplet Cooper pairs, made from two electrons in different TI's hybridized surface orbitals, are free to experience a true phase transition if they can be energetically favored. This is where the TI's Rashba spin-orbit coupling steps in. It gives the triplets a crucial boost, and then takes our system away from the QCP as in the Fig.2(a). We will argue that the triplet superconductor in the TI is a vortex lattice of spin-currents, whose quantum melting induced by a gate voltage likely yields a non-Abelian fractional TI.

This scenario can be derived from solid phenomenological arguments alone. We will use the symmetries of the minimal TI model to construct the effective action of the TI affected by the SC. We will then explain why the triplets form a vortex lattice, why such a vortex lattice is inevitably melted by applying a gate voltage, and why the resulting vortex liquid is a candidate for a fractional TI. At the end we discuss the properties of the possible fractional TIs in our system and limitations of our model.

The minimal model Hamiltonian of a non-interacting TI quantum well can be written as[31,32]:

$$H = \frac{(\mathbf{p} - \tau^z \boldsymbol{\mathcal{A}})^2}{2m} + \Delta\tau^x - \mu \quad , \quad \boldsymbol{\mathcal{A}} = -mv(\hat{\mathbf{z}}\times\mathbf{S}) \quad . \quad (1)$$

It describes four electron states per momentum $\mathbf{p}$, labeled by the spin projection $S^z = \pm\frac{1}{2}$ (in the $\hbar = 1$ units), and the orbital index $\tau^z = \pm 1$ equivalent to the top/bottom surface of the quantum well. The vector spin operator is $\mathbf{S} = \frac{1}{2}\sigma^a \hat{\mathbf{r}}^a$, $a \in \{x, y, z\}$, and σ^a and τ^a are Pauli matrices that operate on the spin and orbital states respectively. The static Yang-Mills SU(2) gauge field $\boldsymbol{\mathcal{A}}$ embodies the Rashba spin-orbit coupling[33] $H_{\mathrm{so}} = v\,\hat{\mathbf{z}}(\mathbf{S}\times\mathbf{p})\tau^z$ and produces a massless Dirac spectrum if $\Delta = 0$. However, inter-surface tunneling $\Delta \neq 0$ opens a bandgap, assuming that the model applies only to momenta $p < \Lambda = \sqrt{(mv)^2 - (\Delta/v)^2}$. A natural cutoff Λ is provided by the lattice potential in materials. The mass m describes a small Dirac-cone curvature seen in ARPES measurements[26]. Fig.2(b) shows that (1) adequately approximates materials, with a relatively large fitted m. This model has the relativistic particle-hole symmetry when $\mu = 0$ and $m \to \infty$. Its many-body ground state is a band-insulator for $|\mu| < |\Delta|$, which is topological when Δ has a proper sign[2].

The spin-orbit SU(2) gauge field from (1) carries a nonzero "magnetic" Yang-Mills flux[34] ($\mu, \nu ... \in \{t, x, y\}$):

$$\Phi^\mu = \epsilon^{\mu\nu\lambda}(\partial_\nu \mathcal{A}_\lambda - i\tau^z \mathcal{A}_\nu \mathcal{A}_\lambda) = \frac{1}{2}(mv)^2 \delta_{\mu t}\,\tau^z\sigma^z \ . \quad (2)$$

Note that the SU(2) charge τ^z is required here by gauge invariance. Being a generalization of the U(1) magnetic flux responsible for the Hall effect, the SU(2) flux is the source of topological phenomena in TIs and sets their "cyclotron" energy scale $\omega_\Phi = mv^2$. Our construction of the effective action for interacting electrons will greatly benefit from exposing the SU(2) gauge symmetry of the idealized model (1). At the end, we will discuss the consequences of gauge symmetry violations in real materials.

The electron dynamics in the TI quantum well is altered by the proximity to the SC in the device from Fig.1. The SC is a fully gapped quantum liquid of Cooper pairs characterized by two energy scales, the pairing Δ_{p} and photon $\Delta_\gamma = \hbar c \lambda_{\mathrm{L}}^{-1}$ gaps, where c is the speed of light and λ_{L} is the London penetration depth. Fermionic quasiparticles have anomalously small or vanishing density of states below the pairing gap, which is $\Delta_{\mathrm{p}} = 2\hbar v_f/\pi\xi$ in conventional superconductors with Fermi velocity v_f and coherence length ξ. The smaller of the two gaps defines a cut-off energy for the low-energy dynamics in the TI that we shall discuss. The dynamics responsible for the triplet superconductor-insulator transition in the TI is indeed defined below this cut-off and hence can be captured by a two-dimensional effective theory whose degrees of freedom are decoupled from those of the SC. We will show that the resulting theory indeed features a triplet superconductor-insulator transition inside the TI across which $\Delta_{\mathrm{p}} \neq 0$.

Our effective TI model is given by the imaginary-time action $S = \int \mathrm{d}\widetilde{t}\,\mathrm{d}r^2\,\psi^\dagger(\partial_0 + H)\psi + S_{\mathrm{int}}$. Living near the conventional SC, all TI's electrons couple to its phonons and thus acquire BCS-like short-range attractive interactions among themselves, irrespective of their spin or or-

	$\psi_{\tau\sigma}(\mathbf{k})$	$\phi_n(\mathbf{k})$	$\eta_m(\mathbf{k})$
$\mathcal{T}_\mathbf{r}$ translations	$\psi_{\tau\sigma}(\mathbf{k})$	$\phi_n(\mathbf{k})$	$\eta_m(\mathbf{k})$
$\mathcal{R}_\theta$ rotations	$\psi_{\tau\sigma}(\mathcal{R}_\theta\mathbf{k})$	$\phi_n(\mathcal{R}_\theta\mathbf{k})$	$\eta_m(\mathcal{R}_\theta\mathbf{k})$
$\mathcal{R}_i$ spatial reflect.	$\psi_{\tau\bar{\sigma}}(\mathcal{R}_i\mathbf{k})$	$-\phi_n(\mathcal{R}_i\mathbf{k})$	$\eta_{\bar{m}}(\mathcal{R}_i\mathbf{k})$
$\mathcal{I}_t$ time reversal	$\sigma\psi^\dagger_{\tau\bar{\sigma}}(-\mathbf{k})$	$-\phi^\dagger_n(-\mathbf{k})$	$(-1)^m\eta^\dagger_{\bar{m}}(-\mathbf{k})$
$\mathcal{C}$ charge U(1)	$e^{i\theta}\psi_{\tau\sigma}(\mathbf{k})$	$e^{i2\theta}\phi_n(\mathbf{k})$	$e^{i2\theta}\eta_m(\mathbf{k})$
$\mathcal{S}$ spin U(1)	$e^{i\sigma\theta}\psi_{\tau\sigma}(\mathbf{k})$	$\phi_n(\mathbf{k})$	$e^{i2m\theta}\eta_m(\mathbf{k})$
local spin SU(2)	$W_{\sigma\sigma'}\psi_{\tau\sigma'}(\mathbf{k})$	$\phi_n(\mathbf{k})$	$U_{mm'}\eta_{m'}(\mathbf{k})$

TABLE I: The symmetry transformations of electron $\psi_{\sigma\tau}$, singlet ϕ_n and triplet η_m fields in (4). W and U are SU(2) transformation matrices $\exp(i\gamma^a\theta^a)$ with SU(2) generators γ^a, $a \in \{x,y,z\}$ in the $S=\frac{1}{2}$ and $S=1$ representations respectively. (γ^a are related to the angular momentum matrices $S^a = \hbar\gamma^a$. Also, $\tau,\sigma=\pm1$; $n,m\in\{\pm1,0\}$; $\bar{l}\equiv -l$.)

bital state. This is generic, but overcoming the Coulomb repulsion in the TI requires a sufficiently strong pairing in the SC and a sufficiently thin quantum well. Without knowing the microscopic form and strength of these interactions, we must consider all channels:

$$S_{\text{int}} = \frac{1}{2}\int \mathrm{d}\tilde{t}\,\mathrm{d}^2r\,\Big(U_1\psi^\dagger_{\tau\sigma}\psi^\dagger_{\tau\sigma'}\psi_{\tau\sigma'}\psi_{\tau\sigma} \qquad (3)$$
$$+ U_2\psi^\dagger_{\tau\sigma}\psi^\dagger_{\bar{\tau}\sigma'}\psi_{\bar{\tau}\sigma'}\psi_{\tau\sigma} + U_3\psi^\dagger_{\tau\sigma}\psi^\dagger_{\bar{\tau}\sigma'}\psi_{\tau\sigma'}\psi_{\bar{\tau}\sigma}\Big) + \cdots$$

Here $\tau=\pm1$ and $\sigma=\pm1$ label the orbital τ^z and spin S^z states of the electron fields $\psi_{\tau\sigma}$ respectively ($\bar{\tau}=-\tau$), while the dots denote weak orbital-non-conserving forces. By applying the Hubbard-Stratonovich transformation on the path-integral, we can eliminate the interaction couplings (3) in favor of six Cooper pair fields displayed in Fig.2(c): two intra-orbital singlets $\phi_\pm$ (U_1), two inter-orbital $S^z=\pm1$ triplets $\eta_\pm$ ($U_{2/3}$ at $\sigma{=}\sigma'$), inter-orbital singlet ϕ_0 and $S^z=0$ triplet η_0 ($U_{2/3}$ at $\sigma{\neq}\sigma'$):

$$S'_{\text{int}} = \int \mathrm{d}\tilde{t}\,d^2r\,\Bigg\{\sum_{\tau=\pm1}\Big(u|\phi_\tau|^2 + \phi_\tau\epsilon_{\sigma\sigma'}\psi^\dagger_{\tau\sigma}\psi^\dagger_{\tau\sigma'} + h.c.\Big)$$
$$+ u'|\phi_0|^2 + \phi_0\frac{1}{\sqrt{2}}\Big(\psi^\dagger_{+\uparrow}\psi^\dagger_{-\downarrow} - \psi^\dagger_{+\downarrow}\psi^\dagger_{-\uparrow}\Big) + h.c.$$
$$+ \sum_{\sigma=\pm1}\Big(v|\eta_\sigma|^2 + \eta_\sigma\psi^\dagger_{+\sigma}\psi^\dagger_{-\sigma} + h.c.\Big) \qquad (4)$$
$$+ v'|\eta_0|^2 + \eta_0\frac{1}{\sqrt{2}}\Big(\psi^\dagger_{+\uparrow}\psi^\dagger_{-\downarrow} + \psi^\dagger_{+\downarrow}\psi^\dagger_{-\uparrow}\Big) + h.c.\Bigg\}$$

The symmetry transformations of these fields are summarized in the Table I. In conjunction with (1), the SU(2) symmetry would imply $v=v'$.

Fermionic excitations remain gapped across superconductor-insulator quantum phase transition in simple two-dimensional band-insulators with attractive interactions[29], and we will explain shortly why this also holds in TIs. Then, we may integrate-out the gapped fermion fields in the path-integral to obtain a purely bosonic effective action S_{eff} that describes Cooper pair dynamics at energies below the pairing gap. We can avoid a complicated calculation by relying on symmetries to construct the Landau-Ginzburg form of S_{eff}. Since two electrons with the same spin from different orbitals have the same cyclotron chirality, the Cooper pairs $\eta_\pm$ with $S^z=\pm1$ possess the SU(2) charge, unlike the singlet fields. Together with η_0 they form a triplet $\eta=(\eta_-,\eta_0,\eta_+)$ that minimally couples to the same SU(2) gauge field $\boldsymbol{\mathcal{A}}$ as (1) but expressed in the $S=1$ representation. This can be seen from the local SU(2) transformations in the Table I. Therefore,

$$S_{\text{eff}} = \int \mathrm{d}\tilde{t}\,\mathrm{d}^2r\,\Big\{\phi^\dagger\partial_0\phi + (\boldsymbol{\nabla}\phi)^\dagger\hat{K}_{\mathrm{s}}(\boldsymbol{\nabla}\phi) + \phi^\dagger\hat{t}_{\mathrm{s}}\phi$$
$$+\eta^\dagger\partial_0\eta + K_{\mathrm{t}}\Big|(\boldsymbol{\nabla}-i\boldsymbol{\mathcal{A}})\eta\Big|^2 + \Big(t_{\mathrm{t}} + \phi^\dagger\hat{t}'_{\mathrm{s}}\phi\Big)|\eta|^2 + U_t|\eta|^4$$
$$+U_{\mathrm{s},\sigma_1\sigma_2\sigma_3\sigma_4}\phi^\dagger_{\sigma_1}\phi^\dagger_{\sigma_2}\phi_{\sigma_3}\phi_{\sigma_4} - \Delta^\dagger_{\mathrm{s}}\phi - \Delta_{\mathrm{s}}\phi^\dagger\Big\}\,. \qquad (5)$$

Some Cooper pair modes may have energy in the two-electron continuum, and should be expelled from S_{eff}. We omitted Coulomb interactions, and used the most general non-relativistic dynamics. We organized the singlet fields into a vector $\phi=(\phi_-,\phi_0,\phi_+)$ and wrote their quadratic couplings in the matrix form. The vector Δ_{s} depends on the SC's order parameter and the SC-TI interface. The singlet matrices $\hat{K}_{\mathrm{s}},\hat{t}_{\mathrm{s}},\hat{t}'_{\mathrm{s}}$ and tensor $\hat{U}_{\mathrm{s}}$ are TR-invariant, and realistic SU(2) symmetry violations can be captured by additional triplet couplings.

Inter-orbital triplets compete with singlets. One of the intra-orbital singlet channels has a stronger induced interaction than all inter-orbital channels for geometric reasons, which naively means that singlets should condense before triplets when electrons are drawn into the TI from the SC by the gate voltage. Here we neglect the intrinsic singlet condensation due to $\Delta_{\mathrm{s}}\neq0$, made small by the TI's bandgap. However, the Rashba spin-orbit coupling in $\mathcal{A}_\mu$ mixes the triplets into two helical modes, analogous to the Dirac conduction and valence band eigenstates of (1). One helical mode has energy that decreases when its momentum grows (like the Dirac valence band), and thus "always" condenses at sufficiently large momenta according to (5). It has a natural advantage over singlets despite its origin in the weaker induced interaction.

The helical condensate locally gains Rashba energy $\hat{\mathbf{z}}(\mathbf{S}\times\mathbf{p})<0$ in (5) by coupling to $\mathcal{A}_\mu$ the TR-invariant currents of properly oriented spin (perpendicular to the current flow). Such a state is globally in equilibrium only if the currents flow in loops. The optimal configuration is always a vortex lattice[35,36], illustrated in Fig.3, and its existence also gives birth to fractional TIs. Imagine tuning the gate voltage to reduce the superconducting stiffness ρ_s toward zero. The vortex kinetic energy due to zero-point quantum motion can be estimated from the Heisenberg uncertainty as $E_{\text{kin}}\sim l_\Phi^{-2}m_{\mathrm{v}}$, where l_Φ is the SU(2) "magnetic length", and m_{v} is the effective vortex mass. In (charged) superconductors, m_{v} is roughly constant as $\rho_{\mathrm{s}}\to0$, but turns into $m_{\mathrm{v}}\sim|\log(\rho_{\mathrm{s}})|$ when

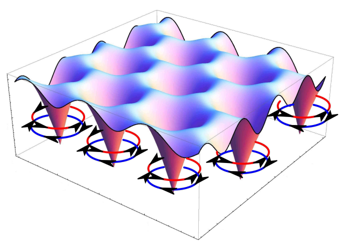

FIG. 3: A TR-invariant Abrikosov lattice in the helical triplet condensate, $\eta_+ = \eta_-^*$, $\eta_0 = -\eta_0^*$. Coinciding vortices in η_+ (red circles) and antivortices in η_- (blue circles), comprise an equilibrium state without charge currents and spin texture, which gains energy by its Rashba-coupled spin current loops.

the screening length $\lambda_{\mathrm{L}} \sim \rho_{\mathrm{s}}^{-1/2}$ diverges[37]. The potential energy due to the vortex lattice stiffness scales as $E_{\mathrm{pot}} \sim \rho_{\mathrm{s}}$ per vortex (ρ_s^2 if the spectrum has Landau levels), which easily follows from the free energy expansion[38] in powers of ρ_s. There is a critical *finite* ρ_{s} at which the vortex lattice melts in a first order transition because $E_{\mathrm{kin}} \geq E_{\mathrm{pot}}$. This happens at the solid line that separates the SC and VL regions in Fig.2(a). Since ρ_s also measures the quasiparticle pairing gap, its finite value implies that the transition is shaped by the Cooper pair dynamics below the fermion excitation gap. The resulting insulator is a quantum vortex liquid of uncondensed Cooper pairs, whose qualitative properties are captured by the purely bosonic theory (5).

Quantum liquids of SU(2) vortices are the prime candidates for fractional TIs when their density is comparable or larger than the density of Cooper pairs (otherwise, Mott or density-wave insulators are stable). This expectation is based on the transitions from vortex lattice condensates to fractional quantum Hall states in the analogous system of bosons in (effective) magnetic fields[39–43]. The mass m in (1) can be estimated from the curvatures of the Dirac cones in ARPES experiments[26], and it is larger than the "spin-orbit" mass $m_{\mathrm{so}} = \Delta/v^2$ by a factor of $\lambda = \frac{m}{m_{\mathrm{so}}} \approx 5 - 10$ ($v \approx 5 \cdot 10^5$ m/s). The cyclotron energy $\omega_\Phi = \Phi/m = \lambda\Delta$ is not small in quantum wells with bandgaps $\Delta = 10 - 100$ meV that can be engineered with a few quintuple layers[4]. The density of "magnetic" SU(2) flux quanta is $n_\Phi = \Phi/h^2 = \lambda^2\Delta^2/(vh)^2 \approx \lambda^2 \times 2 \cdot 10^{15}\ \mathrm{m}^{-2}$. These estimates look promising if we compare them with typical flux-quantum densities $n_\phi = B(hc/e)^{-1} \approx 2.5 \cdot 10^{15}\ \mathrm{m}^{-2}$ (in $B = 10$ T) and cyclotron scales $\omega_\phi = \hbar eB/mc \approx 1$ meV of electrons in fractional quantum Hall states. The TI's Cooper pair density is controlled by the gate voltage, and can be brought near and below n_Φ to stabilize a *fractional incompressible quantum liquid* in a finite parameter range of size ω_Φ surrounding the QCP in Fig.2(a). Detecting fractional charge and statistics in the absence of magnetic fields, by shot-noise or quantum interferometry methods from FQHE experiments[44,45], would provide clear evidence of an established fractional TI in the quantum well.

Without microscopic modeling and experimental data we cannot rule out a possibility that singlets would condense before triplets in a particular device. But even then, a further raise of the gate voltage would eventually condense triplets. Singlets cannot completely screen out the gate from triplets because they repel each other stronger than they repel the triplets, by the Pauli exclusion principle. Future experimental probes of topological spin dynamics may be able to reveal fractional η vortex liquids even if they coexisted with a singlet superconducting state of the ϕ fields (which cannot screen spin).

Finding the precise nature of the fractional TIs goes beyond the scope of this paper as it requires the exact diagonalization of a microscopic model. Instead, we can illustrate their bosonic character by a simple example, such as the bosonic Laughlin wavefunction[21] of $2N$ triplet Cooper pairs $\eta_\pm$ whose coordinates are $z_{i\pm}$:

$$\Psi = \prod_{i<j}^{1\dots N} \left(z_{i+} - z_{j+}\right)^n \left(z_{i-}^* - z_{j-}^*\right)^n \prod_{i}^{1\dots N} e^{-\frac{|z_{i+}|^2 + |z_{i-}|^2}{4l^2}} \ .$$

The integer n is even, and this Abelian TR-invariant state has excitations with fractional charge $2e/n$, spin $\hbar/n$ and spin-Hall conductivity $\sigma_{xy}^{\mathrm{s}} = 4e\hbar/(nh)$. Since $\langle|\eta_\pm|^2\rangle \equiv \Phi/(2\pi n)$ and Φ can be calculated from (5) and (2), one can find n in any ground state and identify Laughlin states by integer-valued n. The wavefunctions of hierarchical quantum spin-Hall states can also be constructed[21,22]. They all describe TR-invariant vortex liquids of spinful bosons (with vortex density l^{-2}), and thus are not far from being good candidates for the fractional TIs in our system. However, they are not adequate either because the S^z spin component is not conserved. It is presently unknown how to write a proper wavefunction for a fractional TI shaped by the Rashba spin-orbit coupling, but an effective field-theory description is available and points to the naturally non-Abelian character of the ensuing incompressible quantum liquids.[36,46,47]

Instead of S^z, the spin quantum number in an ideal Rashba-based TI is the eigenvalue of $\hat{\mathbf{z}}(\mathbf{S} \times \hat{\mathbf{p}})$ as evident from (1). If it were conserved, measuring its average on the fractional TI's quasiparticles in the momentum $\mathbf{p}$ eigenstate would yield a fraction of $\pm\hbar$. However, the realistic complete spin non-conservation, manifested as a gauge symmetry violation in (5), spoils the measurements of fractional spin. At least there is no obstacle to observing the conserved fractional charge, so the fractional TIs can exist. The fractional spin is a degree of freedom rather than a quantum number of quasiparticles (which has a mixed spin and orbital character). Combining an integer number of fractional quasiparticles must reconstitute a triplet Cooper pair, so the quasiparticles must inherit from it a degree of freedom that transforms like spin under time-reversal and spans multiple basis states. Its fractional quantization is guaranteed by the fundamental properties of vortex dynamics in incompressible quantum liquids, and its spin-orbit coupling may yield new topological orders not found in FQHE systems.

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