# Equivalent D=3 Supergravity Amplitudes from Double Copies of Three-Algebra and Two-Algebra Gauge Theories

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# Equivalent $D = 3$ Supergravity Amplitudes from Double Copies of Three-Algebra and Two-Algebra Gauge Theories

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We show that three-dimensional supergravity amplitudes can be obtained as double copies of either three-algebra super-Chern-Simons matter theory or that of two-algebra super-Yang-Mills theory, when either theory is organized to display the color-kinematics duality. We prove that only helicity-conserving four-dimensional gravity amplitudes have nonvanishing descendants when reduced to three dimensions; implying the vanishing of odd-multiplicity S-matrix elements, in agreement with Chern-Simons matter theory. We explicitly verify the double-copy correspondence at four and six points for $\mathcal{N} = 12, 10, 8$ supergravity theories and discuss its validity for all multiplicity.

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It has been a long held view that four-dimensional perturbative quantum gravity defined by the Einstein-Hilbert action, with its supersymmetric completion, is ill defined due to the expectation of a proliferation of new divergences at each order of perturbation theory. In recent years such view has been challenged due to explicit calculations in $D \geq 4$ dimensions [1–3] showing the absence of previously expected divergences in $\mathcal{N} = 8$ supergravity [4], as well as $\mathcal{N} = 4$ supergravity [5] where candidate counterterms that satisfy all known symmetries of the theory has been explicitly constructed [6].

The absence of divergences should imply the existence of either a hidden symmetry which the would-be-counterterm violates [7], or hidden structures of perturbative amplitudes that results in tamer ultraviolet (UV) behavior. A proposal for the latter was given by Bern, Carrasco and one of the current authors (BCJ) [8, 9], who conjectured that super-Yang-Mills (sYM) amplitudes can be reorganized such that the kinematic structure mirrors the Lie-algebra relations satisfied by the color group. Furthermore, it was proven [10] that once such a *color-kinematics-dual* representation is found, gravity amplitudes can be simply obtained by taking a *double copy* (squaring) of the duality-satisfying kinematic factors [8]. While this gauge-gravity relation is classically equivalent to the field-theory version of the Kawai-Levellen-Tye (KLT) open-closed string theory relations [11], its seamless extension to loop level is unrivaled. Renewed analysis has shown that the double-copy relationship between gauge and gravity theories is intimately tied to improved UV behavior of maximal [9, 12] and half-maximal supergravity theories [13].

The notion of a duality between color and kinematics is surprisingly universal – it might well be a fundamental principle of nature. Besides being present in a wide range of $D$-dimensional Yang-Mills theories [8], the same structure has been observed in string theory [14, 15], and in three-dimensional Chern-Simons matter (CSm) theory [16]. The latter case is the topic of this Letter.

In three dimensions, there are two important classes of gauge theories: (super-)YM and (superconformal) CSm theory. Unlike the case of sYM theory, where the color dependence is governed by structure constants of a Lie two-algebra, the color structure for CSm is governed by a Lie three-algebra [17–19]. While CSm theory can be equivalently formulated using ordinary Lie two-algebra, the properties relevant to this paper are better understood in the three-algebra formulation. Recently Bargheer, He and McLoughlin [16] showed that the amplitudes for the $\mathcal{N} = 8$ CSm, also known as Bagger-Lambert-Gustavsson (BLG) theory [17], can be rearranged such that the kinematic parts mirror the relations of the three-algebra color structure. They demonstrated, at four and six points, by squaring the kinematic factors one obtains the amplitudes of the $\mathcal{N} = 16$ supergravity theory constructed by Marcus and Schwarz [20].

In this Letter, we show that tree-level amplitudes of CSm and sYM theories, via respective three- and two-algebra double copy relations, gives identical supergravity tree amplitudes – clarifying the results of ref. [16]. This result is consistent with the statement [20] that the $\mathcal{N} = 16$ Marcus-Schwarz theory is equivalent to the dimensional reduction of four-dimensional $\mathcal{N} = 8$ supergravity, at least for the on-shell S-matrix. The equivalence of the two double copies is striking since, in contrast to CSm, odd-multiplicity S-matrix elements of sYM are nonvanishing. We show that R-symmetry constrains imply that the KLT relations give vanishing odd-multiplicity gravity amplitudes, and furthermore imply that only the helicity-conserving four-dimensional gravity amplitudes are nonvanishing upon dimensional reduction to $D = 3$. This holds independent of the amount of supersymmetry. It is easy to confirm that the first non-trivial case, the four-point amplitude, is identical for the two- and three-algebra double-copy constructions. Given this, on-shell recursion [21] can be used to show the equivalence of the two double- copy constructions [22]. We demonstrate that this result is valid for $\mathcal{N}=6,4,2$ CSm theories, leading to $\mathcal{N}=12,10,8$ supergravity theories.

We give a brief discussion of three-algebra based [16] color-kinematics duality [8]. A three-algebra is constructed via a triple product, antisymmetric in the first two entries, and four-indexed structure constants [17],

$$[T^a, T^b; \bar{T}^{\bar{c}}] = f^{ab\bar{c}}{}_d T^d \,. \tag{1}$$

The structure constants satisfy the fundamental identity

$$f^{ab\bar{c}}{}_l f^{dl\bar{e}\bar{g}} + f^{ba\bar{e}}{}_l f^{dl\bar{c}\bar{g}} + f^{*\bar{c}\bar{e}b}{}_{\bar{l}} f^{da\bar{l}\bar{g}} + f^{*\bar{e}\bar{c}a}{}_{\bar{l}} f^{db\bar{l}\bar{g}} = 0 \,, \tag{2}$$

where indices are raised/lowered by the metric $h^{a\bar{b}} = \mathrm{Tr}(T^a \bar{T}^{\bar{b}})$. Imposing the structure constants to be real and totally antisymmetric leads to the $\mathcal{N}=8$ BLG theory. Relaxing the antisymmetry constraint leads to $\mathcal{N}=6,5,4$ theories [18, 19].

Tree amplitudes of superconformal CSm theories are naturally represented by quartic diagrams, at $m$ points

$$\mathcal{A}_m = i\Big(\frac{2\pi}{k}\Big)^{\frac{m-2}{2}} \sum_{i\in \text{quartic}} \frac{n_i c_i}{\prod_{\alpha_i} s_{\alpha_i}} \,, \tag{3}$$

where $i$ label the diagrams, $\alpha_i$ label propagators, and $k$ is the level. The color factors $c_i$ are constructed by dressing each quartic vertex by four-index structure constants. The "kinematic numerators", $n_i$, are (nonlocal) functions that encode the remaining state dependence.

Following the color-kinematics duality for sYM theory, one can impose a similar duality between the kinematic numerators and color factors of the quartic diagrams. One requires that the numerators satisfy the same symmetries and identities as the color factors, schematically

$$c_i \to -c_i \quad \Leftrightarrow \quad n_i \to -n_i \tag{4}$$
$$c_i + c_j + c_k + c_l = 0 \quad \Leftrightarrow \quad n_i + n_j + n_k + n_l = 0 \,.$$

The second line signifies the fundamental identity or generalized Jacobi identity. The double-copy principle state that once duality-satisfying numerators are found, the three-dimensional supergravity amplitude is given by

$$\mathcal{M}_m = i\Big(\frac{\kappa}{2}\Big)^{m-2} \sum_{i\in \text{quartic}} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}} \,, \tag{5}$$

where $\kappa$ is the gravity coupling. The $n_i, \tilde{n}_i$ may be identical or distinct CSm numerators depending on the theory under consideration. The formula is valid if at least one of the two sets of numerators satisfy the duality [9, 10].

This brings us to our main equation: stating the equivalence of the three-dimensional supergravity amplitudes obtained from either two-algebra or three-algebra constructions. Suppressing $i(\kappa/2)^{m-2}$ factors, we have

$$\mathcal{M}_m = \sum_{\substack{j\in\text{cubic}\\ N_j\in\text{2-algebra}}} \frac{N_j \tilde{N}_j}{\prod_{\beta_j} s_{\beta_j}} = \sum_{\substack{i\in\text{quartic}\\ n_i\in\text{3-algebra}}} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} s_{\alpha_i}} \,, \tag{6}$$

where $N_j, \tilde{N}_j$ are sYM numerators of cubic graphs and $n_i, \tilde{n}_i$ are CSm numerators of quartic graphs. $N_j, n_i$ satisfy the kinematic two- and three-algebra, respectively. The particular theories encoded by these numerators need to be properly identified, examples are given in table I. Obtaining the sYM numerators is explained in ref. [8], so it will not be discussed here. The first equality in eq. (6) has been proven [8, 10], and the second equality is the topic of the remainder of this paper.

Eq. 6 implies that $\mathcal{M}_4$ is a product of (color-stripped) CSm amplitudes: drooping all couplings, $\mathcal{M}_4 = A_4 \tilde{A}_4$. At six points, the appearance of propagators and the fundamental identity results in an intriguing interplay. Considering the $\mathcal{N} \le 6$ CSm theories one can form nine distinct color factors $c_i$ given that odd legs $1,3,5$ have bared color indices $\bar{a}$. These nine contributing diagrams can be identified from their three-particle channels

$$s_i \equiv (s_{123},\ s_{126},\ s_{134},\ s_{125},\ s_{146},\ s_{136},\ s_{145},\ s_{124},\ s_{156}) \,.$$

The duality implies that the number of independent color factors and independent numerator factors is one to one; $p=5$ at six points. Expressing the five color-stripped amplitudes in terms of the independent numerators gives

$$A_{(i)} = \sum_{j=1}^{p} \Theta_{ij} n_j \,. \tag{7}$$

This defines the $\Theta$ matrix, which is comprised of sums of products of propagators with $\pm 1$ coefficients. At four points this matrix is trivial, $\Theta = 1$, and at six points $\Theta$ is a five-by-five matrix, given by

$$\begin{pmatrix} \frac{1}{s_1} & \frac{1}{s_2}+\frac{1}{s_9} & \frac{1}{s_9} & -\frac{1}{s_9} & 0 \\ \frac{1}{s_8} & -\frac{1}{s_8} & \frac{1}{s_3} & \frac{1}{s_4}+\frac{1}{s_8} & 0 \\ \frac{1}{s_7} & -\frac{1}{s_7} & -\frac{1}{s_6}-\frac{1}{s_7} & \frac{1}{s_6}+\frac{1}{s_7} & \frac{1}{s_5}+\frac{1}{s_6}+\frac{1}{s_7} \\ 0 & -\frac{1}{s_9} & -\frac{1}{s_3}-\frac{1}{s_9} & \frac{1}{s_9} & -\frac{1}{s_5} \\ 0 & -\frac{1}{s_2} & \frac{1}{s_6} & -\frac{1}{s_4}-\frac{1}{s_6} & -\frac{1}{s_6} \end{pmatrix} , \tag{8}$$

where the five independent color-ordered amplitudes $A_{(i)}$ are chosen to be $A(\bar{1},2,\bar{3},4,\bar{5},6)$, $A(\bar{1},4,\bar{3},6,\bar{5},2)$, $A(\bar{1},6,\bar{3},2,\bar{5},4)$, $A(\bar{1},4,\bar{3},2,\bar{5},6)$ and $A(\bar{1},6,\bar{3},4,\bar{5},2)$. The five independent numerators are $n_1,n_2,n_3,n_4,n_5$.

Naively, inverting $\Theta$ would give duality-satisfying numerators. However, the matrix $\Theta$ is not invertible; at six points it has only rank 4. Although counterintuitive, this is a desirable property. It implies that at least one $n_i$ corresponds to "pure gauge" and can thus be set to any convenient value while still obtaining the correct amplitude. Using this property one can work out non-trivial relations between color-ordered amplitudes. As a side remark, we note that the matrix $\Theta$ has full rank if one employs either $D > 3$ or off-shell momenta. The former is in contrast to the sYM case, where the corresponding $\Theta$ cannot be inverted in any space-time dimension.

For the six-point case, one obtains a single amplitude relation via the kernel

$$\mathrm{Ker}(\Theta^T) \cdot A = \sum_{i=1}^{5} C_{ik} A_{(i)} = 0\,, \qquad (9)$$

where $C_{ik} = (-1)^{i+k} M_{ik}$ is the $(i,k)$ cofactor, and $M_{ik} = \mathrm{Det}(\Theta_{\hat{i}\hat{k}})$ is the $(i,k)$ minor of the matrix $\Theta$. Note that this formula is equivalent to replacing the $k$-th column of $\Theta$ by $A_{(i)}$ and then demanding that the resulting matrix has zero determinant. All choices for $k$ give the same relation. Up to an overall irrelevant factor, the coefficients $C_{ik}$ are degree-four polynomials in $s_i$.

Setting, say, $n_5$ to be zero, and omitting, say, $A_{(5)}$, we can now invert the reduced matrix $\Theta_{ij}$ to express $n_1, n_2, n_3, n_4$ in terms of color ordered $\mathcal{N} = 6, 4, 2$ CSm amplitudes. Inserting the resulting $n_i$ into eq. (6), with appropriate pairing, we have explicitly checked that one obtains correct $\mathcal{N} = 12, 10, 8$ supergravity amplitudes.

## Explicit amplitudes

In order to clarify the bookkeeping details for the various supergravities we now list the four-point double-copy amplitudes. The maximal $\mathcal{N} = 16$ supergravity case is a square of the $f^{abcd}$-stripped BLG $\mathcal{N} = 8$ amplitude [16]. Suppressing couplings and $i$'s henceforth, we have

$$\begin{aligned}\mathcal{M}_4^{\mathcal{N}=16} &= \frac{\delta^{(16)}(\sum_i \lambda^\alpha \eta_i^I)}{s_{12}s_{23}s_{31}} \\ &= \left(\frac{\delta^{(8)}(\sum_i \lambda^\alpha \eta_i^I)}{\langle 1\,2\rangle \langle 2\,3\rangle \langle 3\,1\rangle}\right)^2 = (A_4^{\mathcal{N}=8})^2\,, \end{aligned} \qquad (10)$$

where the square of the fermionic delta function is understood as a tensor product for the Grassmann-valued $\eta_i^I$, with R-charge indices $I \in (1, \ldots, \mathcal{N}/2)$ for each theory. We use standard spinor products $\langle i\, j\rangle = \lambda^\alpha \lambda^\beta \epsilon_{\alpha\beta} = \sqrt{s_{ij}}$. With maximal supersymmetry the on-shell states form a single multiplet, but for fewer supersymmetries the states split into two multiplets $\Phi^{\mathcal{N}}$ and $\overline{\Phi}^{\mathcal{N}}$ similar to that defined in refs. [23, 24]. These are chiral and antichiral multiplets of SU$(\mathcal{N}/2) \subset$ SO$(\mathcal{N})$. Here we take $\mathcal{N}$ to be even; theories with odd $\mathcal{N}$ we leave for future work. Taking legs 1 and 3 to be antichiral multiplets, the $\mathcal{N} = 12$ amplitude is

$$\mathcal{M}_4^{\mathcal{N}=12}(\bar{1}, 2, \bar{3}, 4) = (A_4^{\mathcal{N}=6})^2 = \left(\frac{\delta^{(6)}(\sum_i \lambda^\alpha \eta_i^I)}{\langle 1\,2\rangle \langle 2\,3\rangle}\right)^2, \qquad (11)$$

where $A_4^{\mathcal{N}=6} = A_4^{\mathcal{N}=6}(\bar{1}, 2, \bar{3}, 4)$ is the color-stripped four-point amplitude [23] of the $\mathcal{N} = 6$ theory constructed by Aharony, Bergman, Jafferis and Maldacena (ABJM) [25]. One may also construct the $\mathcal{N} = 12$ amplitude as a heterotic double copy $A_4^{\mathcal{N}=8} \times A_4^{\mathcal{N}=4}$. This gives the correct result, as explicitly verified up to six points, even though the structure constants and hence the numerators of the two theories obey different symmetries. For $\mathcal{N} = 10$ supergravity the four-point amplitude is given by

$$\mathcal{M}_4^{\mathcal{N}=10}(\bar{1}, 2, \bar{3}, 4) = \frac{\delta^{(10)}(\sum_i \lambda^\alpha \eta_i^I) \langle 1\,3\rangle}{\langle 1\,2\rangle^2 \langle 2\,3\rangle^2}\,. \qquad (12)$$

It can be constructed as heterotic double copies $A_4^{\mathcal{N}=8} \times A_4^{\mathcal{N}=2}(\bar{1}, 2, \bar{3}, 4)$ or $A_4^{\mathcal{N}=6}(\bar{1}, 2, \bar{3}, 4) \times A_4^{\mathcal{N}=4}(\bar{1}, 2, \bar{3}, 4)$.

As discussed in ref. [26] all supergravity theories with $\mathcal{N} > 8$ supersymmetry are believed to be unique, while beginning with $\mathcal{N} = 8$ one can have $n$ matter multiplets, corresponding to $16n$ states. The dimensional reduction of pure half-maximal $D = 4$ supergravity corresponds to $\mathcal{N} = 8$ with $n = 2$; the four-point amplitude is

$$\mathcal{M}_{4,n=2}^{\mathcal{N}=8}(\bar{1}, 2, \bar{3}, 4) = (A_4^{\mathcal{N}=4})^2 = \left(\frac{\delta^{(4)}(\sum_i \lambda^\alpha \eta_i^I) \langle 1\,3\rangle}{\langle 1\,2\rangle \langle 2\,3\rangle}\right)^2, \qquad (13)$$

with $A_4^{\mathcal{N}=4} = A_4^{\mathcal{N}=4}(\bar{1}, 2, \bar{3}, 4)$. One can also write this as $A_4^{\mathcal{N}=6} \times A_4^{\mathcal{N}=2}$. For $n = 1$, the four-point amplitude is

$$\mathcal{M}_{4,n=1}^{\mathcal{N}=8} = \frac{1}{2} \frac{\delta^{(8)}(\sum_i \lambda^\alpha \eta_i^I)(s_{12}^2 + s_{23}^2 + s_{13}^2)}{\langle 1\,2\rangle^2 \langle 2\,3\rangle^2 \langle 1\,3\rangle^2}\,. \qquad (14)$$

This is given by a direct product of $A_4^{\mathcal{N}=8} \times A_4^{\mathcal{N}=0}$, where $\mathcal{N} = 0$ denotes CS theory plus a single minimally-coupled scalar. We summarize these results in table I.

## Vanishing of odd-multiplicity amplitudes

The validity of double-copy formula (6) for all multiplicities can be proven [22] utilizing on-shell recursion formulas for the amplitudes. The proof is similar to that of the two-algebra double-copy relations given in ref. [10]. It relies on having a well-behaved three-dimensional on-shell recursion formula. One such was given for the case of $\mathcal{N} = 6$ CSm theory in ref. [21]. This recursion extends straightforwardly for theories with only even-multiplicity S-matrix elements, and for amplitudes that vanish at large complex deformation. The latter property can be shown to be inherited from four-dimensional supergravity amplitudes; here, we show the former property for supergravity using R-symmetry arguments.

First we consider the constraint from R-symmetry in four dimensions (see [27] for similar discussion). Through the KLT relations, that is, the two-algebra double-copy formula, maximally supersymmetric $\mathcal{N} = 8$ gravity inherits an enhanced SU(8) R-symmetry. This includes

TABLE I: Examples of explicitly-confirmed double-copy constructions of supergravity theories with half-maximal or more supersymmetry. $\mathcal{N}=8$ CSm theory has 16 states while $\mathcal{N}=6,4,2$ CSm have $(8,\bar{8})$, $(4,\bar{4})$, $(2,\bar{2})$ states in the (chiral, antichiral) multiplet, respectively. For $\mathcal{N}=8,4,2,0$ sYM we use the (four-dimensional) state counts $16,8,4,2$. The single state "1" denotes pure $D=3$ YM, or single-scalar CSm theory, in the respective columns. Here $n$ counts matter multiplets.

| SG theory | $\text{CSm}_\text{L}\times\text{CSm}_\text{R}$ = supergravity | $\text{sYM}_\text{L}\times\text{sYM}_\text{R}$ = supergravity | coset |
|---|---|---|---|
| $\mathcal{N}=16$ | $16^2=256$ | $16^2=256$ | $\text{E}_{8(8)}/\text{SO}(16)$ |
| $\mathcal{N}=12$ | $8^2+\bar{8}^2 = 16\times(4+\bar{4}) = 128$ | $16\times 8=128$ | $\text{E}_{7(-5)}/\text{SO}(12)\otimes\text{SO}(3)$ |
| $\mathcal{N}=10$ | $8\times 4+\bar{8}\times\bar{4} = 16\times(2+\bar{2}) = 64$ | $16\times 4=64$ | $\text{E}_{6(-14)}/\text{SO}(10)\otimes\text{SO}(2)$ |
| $\mathcal{N}=8,\, n=2$ | $4^2+\bar{4}^2=8\times 2+\bar{8}\times\bar{2}=32$ | $16\times 2=32$ | SO(8,2)/SO(8)⊗SO(2) |
| $\mathcal{N}=8,\, n=1$ | $16\times 1=16$ | $16\times 1=16$ | SO(8,1)/SO(8) |

the following U(1) generator

$$R=\sum_{i=1}^{m}\eta_i^{I_\text{L}}\frac{\partial}{\partial\eta_i^{I_\text{L}}}-\tilde{\eta}_i^{I_\text{R}}\frac{\partial}{\partial\tilde{\eta}_i^{I_\text{R}}}\,, \tag{15}$$

where $I_\text{L},I_\text{R}\in 1,\cdots,4$. Applied to the amplitude, the generator $R$ counts the $\eta$ degree minus the $\tilde{\eta}$ degree, or, as the $\eta$'s are charged under helicity, the difference of helicity weight between left and right amplitudes. Denoting the KLT map as $\mathcal{M}=K[\mathcal{A}_\text{L},\mathcal{A}_\text{R}]$, R-symmetry invariance thus requires that the two $\mathcal{N}=4$ sYM amplitudes must have the same helicity weight:

$$K[\mathcal{A}_\text{L}^{\text{N}^k\text{MHV}},\mathcal{A}_\text{R}^{\text{N}^{k'}\text{MHV}}]\quad\begin{cases}=0 & \text{for } k\neq k'\\ \neq 0 & \text{for } k=k'\end{cases}\,, \tag{16}$$

where $\text{N}^k$MHV stands for (next-to-)$^k$maximally-helicity-violating amplitude. Note that since one can consistently truncate supersymmetry on both sides of the KLT formula to obtain reduced supersymmetric theories, the above condition is valid for all tree-level pure (super) gravity amplitudes.

Reducing four-dimensional $\mathcal{N}=q$ supergravity to three dimensions, one obtains an enhanced SO($2q$) symmetry. The SO($2q$) generators are built out of quadratic forms $\sim\eta^2$, $\eta\partial_\eta$ and $(\partial_\eta)^2$, among these one can identify the U(1) generator $Y=Y_\text{L}+Y_\text{R}$, where

$$Y_\text{L}=\frac{1}{2}\left(\sum_{i=1}^{m}\eta_i^{I_\text{L}}\frac{\partial}{\partial\eta_i^{I_\text{L}}}\right)-m\,, \tag{17}$$

and $Y_\text{R}$ is similarly defined in terms of the $\tilde{\eta}$ variables. As $R=2(Y_\text{L}-Y_\text{R})$ it follows that $Y_\text{L}$ and $Y_\text{R}$ must vanish individually. This freezes the number of $\eta$'s, or $\tilde{\eta}$'s, to be $2m$, corresponding to helicity weight $m$. Additionally, any $D=4$ sYM amplitude carries overall helicity weight $-m$ not accounted for by the $\eta$'s (cf. Park-Taylor denominator). Thus, in total, only helicity-conserving Yang-Mills amplitudes – present exclusively at even multiplicity – can give nonvanishing gravity amplitudes in the KLT or double copy formula. Equivalently, four-dimensional gravity amplitudes have nonvanishing three-dimensional descendant only for helicity-conserving configurations. We have checked this explicitly for all $\text{N}^k$MHV sectors of graviton tree amplitudes up to 10 points. Unitarity of the S-matrix suggests that the vanishing of odd-multiplicity and helicity non-conserving amplitudes continues at loop level, however, the need for regularization of potential UV and IR divergences may complicate the details.

In conclusion, we have shown that the three-algebra based double-copy formula relates a large class of CSm amplitudes to $D=3$ supergravity amplitudes. Remarkably, the same gravity amplitudes can be obtained from two-algebra based double-copy of sYM amplitudes, as been previously shown [8, 10]. This is striking as CSm and sYM amplitudes have conspicuously distinct properties, such as (non)vanishing odd-multiplicity S-matrix elements. We have also clarified that amplitude relations arise from the fact that the matrix $\Theta_{ij}$ is of lower rank, which is only true for $D=3$. It would be interesting if the resulting amplitude relations have a string-theory explanation, as was the case for sYM [14]. Finally, we note that the existence of a three-algebra double-copy formula may have intriguing consequences for the UV behavior of three-dimensional supergravity, which, just as in four dimensions, is nonrenormalizable by naive power counting. Loop-level numerators that satisfy three-algebra color-kinematics must necessarily be nonlocal, due to the existence of soft poles in the four-point amplitudes (10)-(14). Such a nonlocal behavior, at each four-point vertex, has the potential to improve the naive UV power counting of supergravity. Viewing three-dimensional supergravity as a decoupling limit of string theory [28], also suggests a better UV behavior. Together, these clues suggest that a construction of explicit duality-satisfying loop-level numerators could advance our understanding of the detailed UV structure of gravity theories.

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