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Null Values and Quantum State Discrimination

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We present a measurement protocol for discriminating between two different quantum states of a qubit with high fidelity. The protocol, called *null value*, is comprised of a projective measurement performed on the system with small probability (a.k.a. partial-collapse), followed by a tuned postselection. We report on an optical experimental implementation of the scheme. We show that our protocol leads to an amplified signal-to-noise ratio (as compared with a straightforward strong measurement) when discerning between the two quantum states.

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The notion of "measurement" was part of the early framework of quantum mechanics. Since early developments, the discord between information acquisition on the system and the corresponding disturbance of the system's state became clear. The contest of obtaining information while keeping minimal disturbance is still an active and vibrant field of study that has branched off into many sub-topics. Of note and of great practical interest in quantum information processing is the study of quantum state discrimination [1–4]. The ability to optimally discriminate between non-orthogonal quantum states depends on the fidelity of the measurement apparatus and on the amount of prior knowledge one has on the states between which he wants to distinguish.

Here we introduce a novel procedure to enhance the discrimination-fidelity between two quantum states. Our procedure introduces the notion of quantum measurements with postselection in the field of quantum information processing. Our two-step measurement protocol is related, but differs from, the celebrated weak value (WV) measurement protocol, where postselected quantum measurements were first introduced [5]. We choose to demonstrate our new approach by focusing on a specific discrimination problem. In conjunction with our theoretical analysis, we report on experimental results involving classical light, which demonstrate the practicality of our measurement protocol, denoted "null value" (NV) measurement protocol.

In the original works on quantum state discrimination, the observer is handed a single copy of the state to be discriminated, which may be either one of the a-priori known pure states $|A\rangle$ and $|B\rangle$. Well adapted to this task is the approach known as *minimum error state discrimination* [1], for which it was shown that the minimum error is obtained by optimizing the axis of a standard two-outcome measurement. A second approach is the *unambiguous state discrimination* [6–8] where the measurement produces either an error-free or an inconclusive result, i.e., the measurement apparatus is oriented such that it has three-outcomes – the state is either A, B or unknown.

Developments on the original works led to many variants of state discrimination, such as discrimination between two a-priori unknown pure states [9, 10], as well as discrimination between mixed states [11, 12]. Further works (see e.g., Refs. [13, 14]) introduced also the notion of multi-copy state discrimination (employing a number of copies of the state to be discriminated). For such schemes, the notion of individual vs. collective measurements was introduced depending on whether the strategy consists of individual measurements each of which performed separately on a single copy, or a single measurement which is performed on all the copies as a whole. Notably, however, in most standard measurement procedures one performs individual measurements on N single copies. Thus, it is necessary to define statistical tests to quantify the fidelity of the discrimination [15].

In the present scheme, we study a specific variant of the quantum state discrimination problem: the observer prepares a device (a protocol) that should discriminate whether the provided state is equal to the known state $|\psi_0\rangle$ or not, i.e. is some other nearby state $|\psi_\delta\rangle$. Noting the context of earlier works on state discrimination, our variant applies to both single-copy and many-copy analyses [16]. In the former, due to the a-priori unknown orientation of $|\psi_\delta\rangle$, a minimum error state discrimination is under-constrained. Additionally, an unambiguous state discrimination is impossible as the unknown state would generate both erroneous and inconclusive results. Such a case, dubbed "intermediate discrimination scheme" has been treated for discrimination between two different states (see e.g., Refs. [17–19]).

We present our analysis henceforth for discrimination between two-level states (qubits). Assuming that the probability distribution of $|\psi_\delta\rangle$ is uniform on the Bloch sphere in some area around $|\psi_0\rangle$, we have analyzed the single-copy minimum error and the single-copy intermediate schemes vis-a-vis our discrimination problem [16]. For the former, we obtain that, regardless of the area of the distribution, minimum error is obtained for a standard measurement in the direction orthogonal to $|\psi_0\rangle$. For the latter, we recall that a three-outcome measurement on a qubit can be realized by two-consecutive measurements [3]. Their optimal orientations depend on the area of the distribution. For discrimination between nearby states ($|\psi_0\rangle$ or not $|\psi_0\rangle$), the optimal orientations of both measurements are nearly orthogonal to $|\psi_0\rangle$. By contrast, when the probability distribution of $|\psi_\delta\rangle$ covers the entire Bloch sphere the first of the two measurements is oriented in the direction of $|\psi_0\rangle$ itself.

Alas, a single-copy approach is unfit for most experimental situations due to measurement device imperfections and noise. One then resorts to a multi-copy approach. Here one considers a statistical test (signal-to-noise ratio, SNR) that, given N replicas of the state, would result in a discrimination outcome ($|\psi_0\rangle$ or not $|\psi_0\rangle$) with some given fidelity. Below we define such SNRs and employ them to compare a multi-copy version of both the minimum error scheme and the intermediate scheme, focusing on a discrimination between nearby states. For the intermediate scheme we consider a correlated signal, dubbed *null value signal*, for a reason that will be made clear below. The SNR obtained by a NV signal proves to be higher than that obtained by single von Neumann measurements. We further show that the analysis in terms of optimal SNR fully agrees with a minimization of error probability in the single-copy cases [16].

Let us begin with analyzing the SNR of the discrimination, achieved through individual standard strong measurement, M_s, on N copies of a qubit. In this benchmark case, the occupation of the state $|M\rangle$, defined by polar angle θ_M, is measured [20]. The probabilities to detect the qubit states $|\psi_0\rangle$ and $|\psi_\delta\rangle$ with polar angles 0 and δ in $|M\rangle$ in any single attempt are $P(M_{s,0}) = |\langle M|\,\psi_0\rangle|^2$ and $P(M_{s,\delta}) = |\langle M|\,\psi_\delta\rangle|^2$, respectively. We define a statistical measure to be the difference between the number of positive detections

$$S_{\rm std} = N\,|P(M_{s,\delta}) - P(M_{s,0})| \cong |N_{s,\delta} - N_{s,0}|\ , \qquad (1)$$

where the right hand side is the measured estimator. The signal is a function of two variables $S_{\rm std}(N_{s,\delta}, N_{s,0})$. The uncertainty in the signal is then given by

$$\Delta S_{\rm std} = \sqrt{\left(\frac{\partial S_{\rm std}}{\partial N_{s,\delta}}\right)^2 \Delta N_{s,\delta}^2 + \left(\frac{\partial S_{\rm std}}{\partial N_{s,0}}\right)^2 \Delta N_{s,0}^2}\,. \qquad (2)$$

We assume Poissonian noise (which is dominant for coherent light experiments discussed below), i.e., $\Delta N_{s,\delta}^2 = N_{s,\delta}$ and $\Delta N_{s,0}^2 = N_{s,0}$. Thus, $\Delta S_{\rm std} = \sqrt{N_{s,\delta} + N_{s,0}}$, and the obtained SNR is

$$\mathrm{SNR}_{\rm std} = \frac{S_{\rm std}}{\Delta S_{\rm std}} \approx \sqrt{2}|\sin[\theta_M]|\delta\sqrt{N}\,, \qquad (3)$$

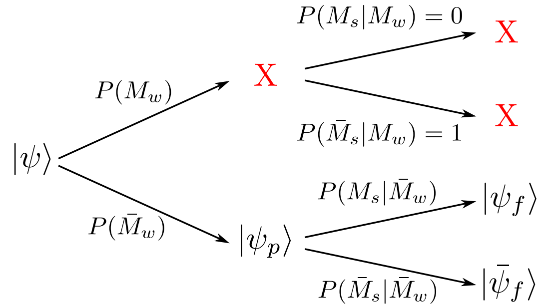

FIG. 1. A tree diagram of the qubit state evolution under subsequent partial-collapse measurements; the respective probabilities are indicated: $P(M_w)$ [$P(\bar{M}_w)$] is the probability that the detector "clicks" [no "click"] upon the first measurement. If it does "click", the system is destroyed, hence there are no clicks upon further measurements [this is marked by a (red) X]. Note that following $P(\bar{M}_w)$ (null detection of the qubit), the back action rotates $|\psi\rangle$ into $|\psi_p\rangle$.

where the approximation is for $\delta \ll 1$. Indeed, in this approach the maximal SNR is obtained when the measurement orientation, $|M\rangle$, is orthogonal to $|\psi_0\rangle$. This corresponds to the optimal measurement orientation obtained by the single-copy analysis.

Turning to the multi-copy intermediate discrimination, we define a SNR by constructing a correlated outcome out of the three-outcome measurement. Recall that such a measurement is implemented by measuring the qubit state twice (cf. Fig. 1) [16]. The first measurement M_w is a strong (projective) measurement which is performed on the system with small probability. Here the basis states $\{|\bar{M}\rangle, |M\rangle\}$ are measured with probabilities $\{p_0, p_1\}$, respectively. For simplicity, hereafter, we assume that only the state $|M\rangle$ is measured with probability $p_1 = p$ and $p_0 = 0$. If the detector "clicks" (the measurement outcome is positive), the qubit state is destroyed. Very importantly, having a "null outcome" (no click) still results in a back action on the system. We refer to this stage of the measurement process as "partial-collapse" [21]. Subsequently the qubit state is (strongly) measured a second time (postselected), M_s, to be in the state $|\psi_f\rangle$ (click) or $|\bar{\psi}_f\rangle$ (no click), where $|\psi_f\rangle$ has a polar angle of θ_f. We propose to discriminate between the two possible initial qubit states by individual application of this measurement protocol on N copies of $|\psi_0\rangle$ and $|\psi_\delta\rangle$. Motivated by WVs, the compared observables are the counter-causal conditional outcome of [having a click the first time conditional to *not* having a click the second time], denoted by $P(M_{w,0}|\bar{M}_{s,0})$ and $P(M_{w,\delta}|\bar{M}_{s,\delta})$, respectively. Events in which the qubit is measured strongly (in the second measurement), M_s, are discarded. In other words, we define our signal to be

$$S_{\rm NV} \equiv N\left|P(M_{w,\delta}|\bar{M}_{s,\delta}) - P(M_{w,0}|\bar{M}_{s,0})\right|\,. \qquad (4)$$

Note that this procedure can also be written as a statistical correlation between outcomes of a positive-operator valued measure (POVM) [16].

Our protocol takes advantage of the statistical correlations between the partial-collapse and strong measurements. To shed some light on its outcome we calculate explicitly the conditional probabilities following the measurement procedure sketched in Fig. 1. For example, if the first measurement results in a "click" the system's state is destroyed and any subsequent measurement on the system results in a null-result. This represents a classical correlation between the two measurements. By contrast, $P(\bar{M}_s|\bar{M}_w)$ embeds non-trivial quantum correlations [22]. Using Bayes theorem, we can write $P(M_{w,\delta}|\bar{M}_{s,\delta}) \cong N_{w,\delta}/(N_{w,\delta}+N_{p,\delta})$, where we used the measured estimator for the conditional probability, namely, we denoted $N_{w,\delta} \cong NP(M_{w,\delta})$ as the number of clicks in the (first) partial-collapse measurement and $N_{p,\delta} \cong NP(\bar{M}_{w,\delta})P(\bar{M}_{s,\delta}|\bar{M}_{w,\delta})$ as the number of no-clicks in the (second) postselection [16]. This finally leads to the measured signal

$$S_{\mathrm{NV}} \cong N\left|\frac{N_{w,\delta}}{N_{w,\delta}+N_{p,\delta}}-\frac{N_{w,0}}{N_{w,0}+N_{p,0}}\right|. \qquad (5)$$

In complete analogy with the case of a single strong measurement, the signal is now a function of four variables $S_{\mathrm{NV}}(N_{w,\delta}, N_{p,\delta}, N_{w,0}, N_{p,0})$, and we can define the uncertainty, ΔS_{NV}, in the statistical test [cf. Eq. (2)] [16].

We focus on obtaining a large $\mathrm{SNR}_{\mathrm{NV}} = S_{\mathrm{NV}}/\Delta S_{\mathrm{NV}}$ for discriminating between the two states. It depends on the choice of the measurement orientations, $|M\rangle$ and $|\psi_f\rangle$. We propose to perform a first measurement that will have a back action on both states $|\psi_0\rangle$ and $|\psi_\delta\rangle$ but is nearby the optimal orientation of the single measurement case, i.e. taking $\theta_M = \pi/2 + \Delta_M$. We propose two possible measurement schemes for obtaining a large $\mathrm{SNR}_{\mathrm{NV}}$. In the first scheme we choose the postselection such that the reference state satisfies $|\langle\bar{\psi}_f|\psi_0\rangle|^2 = 0$. This means that the reference state $|\psi_0\rangle$ would have always clicked in the second measurement had it not been first measured by the partial-collapse. We call this *scheme* $\boldsymbol{A}$. Alternatively, in *scheme* $\boldsymbol{B}$, we choose the postselection such that $|\langle\bar{\psi}_f|\psi_{0,p}\rangle|^2 = 0$: the null outcome rotates the reference state and it always clicks in the second measurement. For both schemes, we obtain

$$\mathrm{SNR}_{\mathrm{NV}}(N) \sim \frac{\sin^2[\delta]}{\sin[\Delta_M+\delta]\sqrt{p}}\sqrt{N}, \qquad (6)$$

which becomes large for $p \to 0$ (weak partial-collapse) [16]. This is because the condition $N_{w,0} \gg N_{p,0}$ is satisfied vis-a-vis the NV signal of the reference state. Varying δ such that $|\langle\psi_\delta|\psi_0\rangle|^2$ decreases corresponds to a decrease of $N_{w,\delta}$ and an increase of $N_{p,\delta}$. A large $\mathrm{SNR}_{\mathrm{NV}}$ is obtained when $P(M_{w,\delta}|\bar{M}_{s,\delta})$ crosses to a regime where $N_{w,\delta} \leq N_{p,\delta}$. This happens first with scheme $\boldsymbol{A}$. Hence, scheme $\boldsymbol{A}$ produces a larger $\mathrm{SNR}_{\mathrm{NV}}$ for smaller δ; scheme $\boldsymbol{B}$ leads to far larger $\mathrm{SNR}_{\mathrm{NV}}$ for larger δ. Note, also, that taking the partial-collapse measurement to be more orthogonal to $|\psi_0\rangle$ (Δ_M) increases the SNR for $\delta \ll 1$.

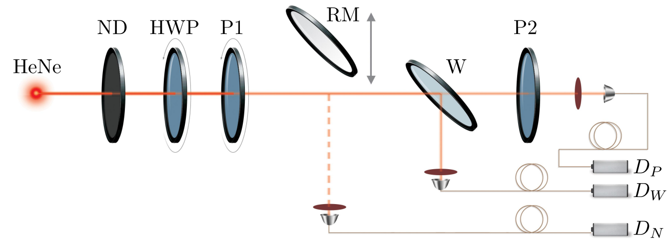

FIG. 2. A sketch of the experimental apparatus. Single spatial mode light from a helium-neon laser (HeNe) passes through a neutral density filter (ND) followed by a half-wave plate (HWP) and polarizer (P1) to prepare the initial state. During data acquisition, the HWP is used to maintain a constant photon flux which is measured using a removable mirror (RM). A glass window (W) weakly reflects vertically polarized light. Photons that pass through the window are then projected onto a linear polarization state with a second polarizer (P2). The photons in each spatial mode are passed through colored glass filters to block background, collected via multimode fiber and sent to single photon counting modules (D_N, D_W and D_P).

The postselection measurement orientations, which produce the high SNR, coincide with those obtained in the single-copy analysis, i.e. $|\langle M|\psi_0\rangle| \sim 0$, $p \ll 1$, for $\delta \ll 1$ [16]. This suggests that though the spirit of the present multi-copy analysis is quite different from the single-copy analysis, both analyses give similar guidance for optimally discriminating between non-orthogonal states. We reiterate, however, that (as compared with the single-copy approach) the statistical SNR approach (based on NV) is better suited to most experimental settings in which noise and experimental imperfections are present.

We measure the NV signal and its amplified SNR using an optical technique sketched in Fig. 2. Here, the qubits are replaced by photons from a dramatically attenuated coherent beam, and the measurement device consists of polarization optics and single-photon detectors. We encode the states in the polarization degree of freedom by passing the beam through a polarizer (P1), giving $|\psi_\delta\rangle = \cos[\delta-\Delta_M]|0\rangle + \sin[\delta-\Delta_M]|1\rangle$, where $\{|0\rangle, |1\rangle\}$ correspond to the horizontal and vertical polarization states, respectively. We perform a (weak) partial-collapse measurement by sending the photons through a glass window (W) set at Brewster angle. The window therefore weakly reflects vertically polarized light, with probability $p = 0.15$, and passes horizontal light with near unit probability. We set the second polarizer (P2) in the transmitted arm to strongly project the photon into the state $|\bar{\psi}_f\rangle$ which is represented by scheme $\boldsymbol{A}$ or $\boldsymbol{B}$, as desired [23]. From the resulting photon detections we obtain the values of $N_{w,\delta}$, $N_{p,\delta}$, $N_{w,0}$ and $N_{p,0}$ and

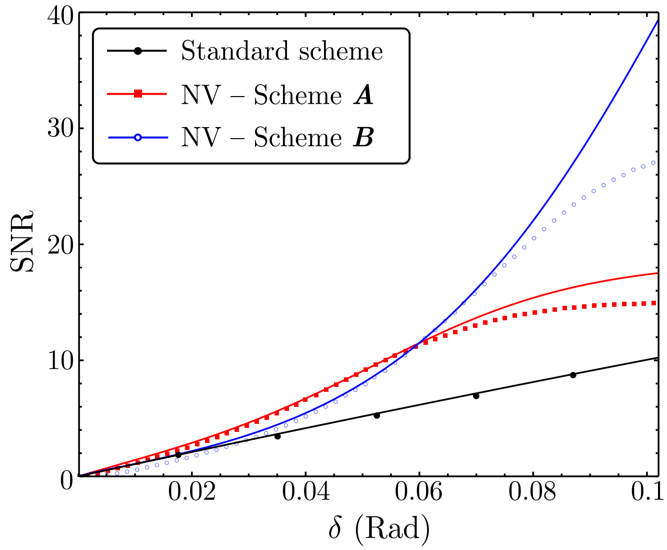

FIG. 3. A graph of the theoretical and experimental SNR obtained for different measurement schemes. Scheme $\boldsymbol{A}$ (red squares) and $\boldsymbol{B}$ (blue hollow circles) correspond to the null value technique ($\mathrm{SNR}_{\mathrm{NV}}$). The parameter δ denotes the distance between the measured and the reference state; it is varied by changing the angles for the input polarizer P1. For a given P2 and W (cf. Fig. 2) the reference state is determined by finding P1 for which $|\langle \bar{\psi}_f | \psi_0 \rangle|^2$, $|\langle \bar{\psi}_f | \psi_p \rangle|^2$ is minimal for schemes $\boldsymbol{A}$, $\boldsymbol{B}$, respectively. The standard scheme (black circles) is that defined by Eq. (3), and is represented by a single polarizer with no weak measurement. Dots correspond to calculations from data and lines correspond to the theoretical predictions. Each scheme used approximately the same number of photons, with $N \approx 11250$ per measurement.

their variances [16].

We consider schemes $\boldsymbol{A}$ and $\boldsymbol{B}$ for $\Delta_M = 0.1$ rad and plot the results in Fig. 3. We find that, for scheme $\boldsymbol{A}$, we can discriminate between the two states with a higher SNR than the standard scheme nearly over the whole range of angles considered. Similarly, while the SNR of the standard technique almost coincides with that of scheme $\boldsymbol{B}$ for small angles, we see that the sensitivity of the two schemes diverges quickly for larger angles; in this regime ($\delta \approx \Delta_M$), the NV scheme $\boldsymbol{B}$ is significantly better. The discrepancy between theory and experiment is due to a small amount of ellipticity incurred from the glass window not included in the theory plot [16].

The described NV procedure leading to large SNR is based on the conditional outcome of a quantum measurement. As such, it resembles the well established protocol of WV measurement [5]. The WV protocol consists of weakly measuring an operator $\hat{A}$ of a *system* prepared in an initial state $|i\rangle$ by weakly coupling it to a *detector*. The detector output is kept only if the system is eventually measured to be in a chosen final state, $|\psi_f\rangle$—*postselection*. The obtained conditional average of $\hat{A}$, $\langle f | \hat{A} | i \rangle / \langle f | i \rangle$, is named *weak value*, and can be anomalously large [5]. This property has been exploited for amplifying small signals both in quantum optics [24–28] and in solid state physics [29]. It is important to stress that the NV protocol is different from the WV protocol. The former makes use of a partial-collapse measurement of the operator $\hat{A}$, in which the system experiences back action only for a subset of all possible measurement outcomes, while a strong projection takes place for the remaining outcomes. This is not a weak measurement, which is used by the WV protocol. The obtained conditional average of $\hat{A}$ is now the NV, $(1/p)P(M_w|\bar{M}_s) = \langle i | \hat{A} | i \rangle / P(\bar{M}_s)$. It is quantitatively different from the WV even when p is explicitly "weak" [30]. Moreover, while a large WV leads to an amplification of the SNR for systems where the noise is dominated by an external (technical) component [26, 29], the method presented here leads to high fidelity discrimination between quantum states on the background of quantum fluctuations.

In conclusion we have presented here a new protocol based on a partial-collapse measurement followed by a tuned postselection. Our protocol enables one to discern between quantum states with better accuracy than a standard measurement would allow. By contrast to earlier protocols [3] tuned to discriminate between two prescribed states, the present one facilitates the study of an amplified SNR for a wide range of possible polarizations of one of the states, which is not a-priori known. We have demonstrated the feasibility and effectiveness of our protocol by employing an optical setup for discriminating between different polarization states of light. Notably, our present approach is based on a statistical analysis, which makes it particularly suitable for experiments.

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