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Ontological models predictively inequivalent to quantum theory*

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Recently, it has been argued that no extension of quantum theory can have improved predictive power, under a strong assumption of free choice of the experimental settings, and validity of quantum mechanics. Here, under a different free choice assumption, we describe a model which violates this statement for almost all states of a bipartite two-level system, in a possibly experimentally testable way. From consistency with quantum mechanics and the non-signalling principle, we derive a bound on the local averages for the family of deterministic ontological theories our model belongs to.

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Introduction — In this work we address the following question: could a theory, which is conceived as a completion of quantum mechanics, be experimentally distinguishable from it? By *completion* we mean that the theory should be consistent with quantum mechanics, that is, it should fully reproduce all the quantum outcomes in a suitable regime, but it could provide a more refined description of the microscopic reality. By *experimentally distinguishable* we mean that there should not be physical principles making this deeper description fully unaccessible to any observer, irrespective of experimental complications. For instance, these are the features of classical mechanics when compared to statistical mechanics [20]. In practice, the completion involves the consideration of the so-called *ontic state* of the system, the (in principle) most accurate specification of its physical state. The request of the experimental testability of the completion does not require the precise knowledge of the ontic state, but only the accessibility of some information about it, which allows more accurate predictions than those implied by the mere knowledge of the quantum state vector.

Among others issues, the question we have raised at the beginning has been recently answered in the negative under a specific assumption of free measurement choice [2]. In this paper we exhibit a model which provides a positive answer, opening new perspectives on the debate of the completeness of quantum mechanics. For deterministic ontological models of this type describing a pair of two-level systems, we also derive constraints concerning how the local averages can differ from the quantum ones.

The family of completions of quantum mechanics has been usually denoted as *hidden variable models*, although more recently the term *ontological models* is preferred. In the past, the main motivation for their introduction was the attempt to provide a description of the micro-world as close as possible to that of the macro-world, interpreting all (or some of) the non-classical features of quantum mechanics (as probabilism and non-locality) as ignorance of the ontic state [1], supplying a richer information on the

state of the system than that given by the state vector of standard quantum theory. After the theorems of Bell [3] and Kochen-Specker [4], proving that non-locality and contextuality are unavoidable in these theories, other issues are investigated, noticeably the meaning of the quantum state, as describing an element of reality or rather a state of knowledge [5–7], the possibility to deviate from quantum mechanics [8–11], the role of the measurement independence assumption [12, 13], and the dimension of the ontic state space [14–16]. We describe in more detail the aspects which are relevant for this work.

In quantum mechanics, the state of a system is represented by a vector ψ in a suitable Hilbert space. Observable quantities are represented by Hermitian operators, whose spectra contain the only possible measurement outcomes. For general states, these outcomes are known only probabilistically. Completeness of the theory means that it is impossible to have a more detailed description of the state than that given by ψ , even in principle. In particular, only a probabilistic knowledge of measurement outcomes is possible. We consider a bipartite system, the separated subsystems being identified by A and B . The observables pertaining to these subsystems are denoted by $A(a)$, $B(b)$, and the corresponding Hermitian operators by $\hat{A}(a)$, $\hat{B}(b)$, where a and b are vectors which identify the specific observables taken into account.

In an ontological model of quantum mechanics there is a deeper specification of the state of the system, the ontic state λ , living in a space which, for what concerns us, is not relevant to identify precisely. For sake of simplicity we assume that λ are continuous variables, but our considerations apply to completely general cases. The ontic state represents the complete description of the state of the system [21], which, however, is not fully accessible. To each state ψ is associated a distribution $\rho_\psi(\lambda)$, with $\rho_\psi(\lambda) \geq 0$ and

$$\int \rho_\psi(\lambda) d\lambda = 1 \quad \text{for all } \psi. \quad (1)$$

These distributions might have overlapping supports for different ψ or not [1]. We limit our attention to deterministic ontological models, in which the measurement outcomes are fully specified by the ontic state (more generally, λ could determine only their probabilities). Contextuality implies that these outcomes can depend on the full context of measured observables; we will then denote them as $A_\psi(a, b, \dots, \lambda)$ and $B_\psi(a, b, \dots, \lambda)$, depending on the subsystem they refer to. Consistency of the ontological models with quantum mechanics means that all the quantum averages are reproduced, in particular

$$\int A_\psi(a, b, \dots, \lambda) \rho_\psi(\lambda) d\lambda = \langle A(a) \rangle_\psi, \quad (2)$$

for the local averages, where $\langle A(a) \rangle_\psi = \langle \psi | \hat{A}(a) | \psi \rangle$ and similarly for B . But also quantum correlations should be obtained,

$$\int A_\psi(a, b, \dots, \lambda) B_\psi(a, b, \dots, \lambda) \rho_\psi(\lambda) d\lambda = \langle A(a) B(b) \rangle_\psi, \quad (3)$$

where $\langle A(a) B(b) \rangle_\psi = \langle \psi | \hat{A}(a) \otimes \hat{B}(b) | \psi \rangle$.

Notice that the ontic state λ is necessarily not fully accessible. In fact, in a theory like the one we are devising, it is just the ignorance of the precise value of λ which cancels non-locality in the averages (2), in accordance with the fact that quantum mechanics does not allow faster-than-light communication. Nonetheless, at least in principle, one can investigate whether it is possible to avoid superluminal communication while taking advantage of some information on λ . It is exactly this property which must characterize a completion of quantum theory which could be experimentally distinguishable from it, the crucial problem addressed in this paper.

With this in mind, suppose that λ is equivalently described by two variables (μ, τ) , where μ denotes the unaccessible part, i.e. the one which must be averaged over, and τ the accessible one. Knowledge of τ should not allow superluminal communication; therefore, by writing $\rho_\psi(\lambda) = \rho_{\psi, \tau}(\mu) \rho_\psi(\tau)$, we can compute

$$\begin{aligned} \int A_\psi(a, b, \dots, \lambda) \rho_{\psi, \tau}(\mu) d\mu &= f_\psi(a, \tau), \\ \int B_\psi(a, b, \dots, \lambda) \rho_{\psi, \tau}(\mu) d\mu &= g_\psi(b, \tau), \end{aligned} \quad (4)$$

which are the local averages of the theory at the intermediate level. They express the *non-signalling conditions* in our context. It remains true that the theory reproduces quantum mechanics when τ is averaged over,

$$\begin{aligned} \int f_\psi(a, \tau) \rho_\psi(\tau) d\tau &= \langle A(a) \rangle_\psi, \\ \int g_\psi(b, \tau) \rho_\psi(\tau) d\tau &= \langle B(b) \rangle_\psi, \end{aligned} \quad (5)$$

but, if the state vector ψ is supplemented with the information on τ , the theory could be experimentally distin-

guishable from quantum mechanics [22]. Theories having this structure have been initially introduced by A. Leggett [8], and then further analyzed in the case of maximally entangled states [9, 10]. Here we provide one model fitting this class, describing arbitrarily entangled states.

A model with non trivial local averages — The model is a generalization to arbitrary states of the famous Bell's model for the singlet state of a pair of two-level systems, and its details can be found elsewhere [18]. Here we review only the main results using a different notation, which is more convenient for the present purposes. An arbitrary state ψ is written as

$$|\psi\rangle = \sin \frac{\theta}{2} |00\rangle + \cos \frac{\theta}{2} |11\rangle, \quad (6)$$

with $\theta \in [0, \pi/2]$; if $\theta = 0$, $|\psi\rangle$ is a separable state; if $\theta = \pi/2$ it is a maximally entangled state. The hidden variable λ is a unit vector in the 3-dimensional real space and the pair $\{\psi, \lambda\}$ is identified with the ontic state. We consider local observables represented by the operators $\hat{A}(a) = \sigma \cdot a$ and $\hat{B}(b) = \sigma \cdot b$, where a and b are real, unit vectors and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. In particular, σ_z is defined so that $|0\rangle$ and $|1\rangle$ are its $+1$ and -1 eigenvectors respectively. For sake of simplicity we assume that a and b lie in the plane orthogonal to the direction of propagation of the entangled particles [23]. The possible outcomes are in the set $\{-1, 1\}$, and they are defined as

$$A_\psi(a, b, \lambda) = \begin{cases} +1, & \text{if } \hat{a} \cdot \lambda \geq \cos \xi, \\ -1, & \text{if } \hat{a} \cdot \lambda < \cos \xi, \end{cases} \quad (7)$$

and

$$B_\psi(b, \lambda) = \begin{cases} +1, & \text{if } b \cdot \lambda \geq \cos \chi, \\ -1, & \text{if } b \cdot \lambda < \cos \chi. \end{cases} \quad (8)$$

In the previous relations, $\hat{a} = \hat{a}(a, b)$ is in the plane of a and b , as detailed in [18]; moreover, $\cos \xi = -\langle A(a) \rangle_\psi$, and $\cos \chi = -\langle B(b) \rangle_\psi$.

In [18] it has been proved that this model is predictively equivalent to quantum mechanics when λ is uniformly distributed on the unit sphere, $\rho(\lambda) = 1/4\pi$. At this point, we express λ by using spherical coordinates (μ, τ) , whose pole is identified by the direction of the incoming particle; μ is the azimuthal angle and τ the polar angle. They represent the unaccessible and the accessible part of λ , respectively. In (4) we have $\rho_{\psi, \tau}(\mu) = 1/2\pi$, and, by construction, integration over μ cancels non-locality in local averages. We find that

$$f_\psi(a, \tau) = \frac{1}{\pi} \cos^{-1} \left(\frac{2\langle A(a) \rangle_\psi^2}{\sin^2 \tau} - 1 \right) - 1, \quad (9)$$

if $|\tau - \frac{\pi}{2}| \leq \xi$ and $f_\psi(a, \tau) = -1$ otherwise, and a similar relation (with ξ replaced by χ) for $g_\psi(b, \tau)$, which clearly shows that in general $f_\psi(a, \tau)$ and $g_\psi(b, \tau)$ differ

from the local quantum averages. While this toy-model is completely artificial, it provides evidence that models compatible with quantum mechanics, but in principle distinguishable from it, are indeed possible, with measurement settings a and b freely chosen.

On the local averages of any ontological theory — The requirements of equivalence with quantum mechanics and the non-signalling conditions put severe constraints on the probabilities at the intermediate level. In the following, we derive a bound on the local averages of any deterministic ontological model for quantum mechanics describing a pair of 2-level systems arbitrarily entangled, at the intermediate level. We adopt the aforementioned description of the ontic state in terms of the state vector, ψ , of an additional accessible part, τ , and a non-accessible one, μ , and assume that this splitting is independent of the local settings a and b .

While our derivation holds for both subsystems, for sake of simplicity we focus on the A part and its corresponding local average $f_\psi(a, \tau)$. As a measure of the distance between the local averages at the intermediate level of the ontological theory and quantum mechanics we introduce the quantity

$$\delta_\psi(a) = \int \left(f_\psi(a, \tau) - \langle A(a) \rangle_\psi \right)^2 \rho_\psi(\tau) d\tau, \quad (10)$$

which is the variance of the variable $f_\psi(a, \tau)$ over the distribution $\rho_\psi(\tau)$. We consider a generic observable $A(a)$, represented in quantum mechanics by $\hat{A}(a)$ with eigenvalues ± 1 . By elementary manipulations of (10), and considering that $f_\psi(a, \tau) \in [-1, 1]$, we can write

$$\delta_\psi(a) \leq \int |f_\psi(a, \tau)| \rho_\psi(\tau) d\tau - \langle A(a) \rangle_\psi^2. \quad (11)$$

We now focus on the integral in the r.h.s. of this inequality. Since $\hat{A}(-a) = -\hat{A}(a)$, it follows that $A_\psi(-a) = -A_\psi(a)$, and then $f_\psi(-a, \tau) = -f_\psi(a, \tau)$. Consider $2n + 1$ unit vectors γ_j , $j = 0, 1, \dots, 2n$ such that $\gamma_0 = a$ and $\gamma_{2n} = -a$. Further assume that $2n$ pairs of local measurements are performed according to the following scheme: (i) when j is even, the local observables are given by $A(\gamma_j)$ and $B(\gamma_{j+1})$; (ii) when j is odd, the local observables are given by $A(\gamma_{j+1})$ and $B(\gamma_j)$. The measurement outcomes for $A(\gamma_j)$ and $B(\gamma_j)$ are ± 1 . Therefore, by using that

$$|A_\psi(\gamma_j, \gamma_{j+1}, \lambda) - B_\psi(\gamma_j, \gamma_{j+1}, \lambda)| = 1 - A_\psi(\gamma_j, \gamma_{j+1}, \lambda) B_\psi(\gamma_j, \gamma_{j+1}, \lambda), \quad (12)$$

after multiplication of both sides by $\rho_{\psi, \tau}(\mu)$ and integration over μ , it follows that

$$|f_\psi(\gamma_j, \tau) - g_\psi(\gamma_{j+1}, \tau)| \leq 1 - E_{\psi, \tau}(\gamma_j, \gamma_{j+1}), \quad (13)$$

where we have defined the correlation at the intermediate level as

$$E_{\psi, \tau}(a, b) = \int A_\psi(a, b, \lambda) B_\psi(a, b, \lambda) \rho_{\psi, \tau}(\mu) d\mu \quad (14)$$

When j is odd, a similar argument leads to a relation analogous to (13), with γ_j and γ_{j+1} exchanged. By summing all these relations, and considering that $f_\psi(-a, \tau) = -f_\psi(a, \tau)$, we find that

$$|f_\psi(a, \tau)| \leq n - \frac{1}{2} \sum_{k=0}^{n-1} \left(E_{\psi, \tau}(\gamma_{2k}, \gamma_{2k+1}) + E_{\psi, \tau}(\gamma_{2k+2}, \gamma_{2k+1}) \right). \quad (15)$$

By further multiplying by $\rho_\psi(\tau)$ and integrating over τ ,

$$\int |f_\psi(a, \tau)| \rho_\psi(\tau) d\tau \leq \min_{\gamma_1, \dots, \gamma_n; n \in \mathbb{N}} \Omega_\psi(a, n), \quad (16)$$

where we have defined

$$\Omega_\psi(a, n) = n - \frac{1}{2} \sum_{k=0}^{n-1} \left(\langle A(\gamma_{2k}) B(\gamma_{2k+1}) \rangle_\psi + \langle A(\gamma_{2k+2}) B(\gamma_{2k+1}) \rangle_\psi \right), \quad (17)$$

and the minimum is taken by arbitrarily varying the vectors $\gamma_1, \dots, \gamma_{2n-1}$, for any n . Therefore, by taking into account the joint correlations arising in an arbitrary number of measurements, we can write

$$\delta_\psi(a) \leq \min_{\gamma_1, \dots, \gamma_n; n \in \mathbb{N}} \Omega_\psi(a, n) - \langle A(a) \rangle_\psi^2. \quad (18)$$

Notice that, when ψ is a maximally entangled state, both terms in the r.h.s. of (18) vanish, and then $f_\psi(a, \tau) = \langle A(a) \rangle_\psi$ for all a (a rigorous proof, using analogous arguments, can be found in [11]). For arbitrary states, it is in general difficult to find the minimum in (18), therefore we have resorted to a numerical analysis, providing strong evidence that this minimum is given by $\cos \theta$. Therefore, we conjecture that (18) can be simply written as

$$\delta_\psi(a) \leq \cos \theta - \langle A(a) \rangle_\psi^2. \quad (19)$$

In Fig. 1 we show how this constraint for $\delta_\psi(a)$ varies with entanglement, and plot the corresponding curve for the aforementioned model.

Optimality of the bound — Our generalization of the Bell's model does not saturate the bound on the local average at the intermediate level and this rises the question whether this constraint is optimal or not. In general it is not, because in (11) we have used the fact that $f_\psi(a, \tau)^2 \leq |f_\psi(a, \tau)|$, which is a strict inequality unless $f_\psi(a, \tau) \in \{-1, 0, 1\}$. However, we can provide a model which saturates the bound when the local observable is σ_z . This is done by modifying our model, such that the supports on the unit sphere, where the two outcomes A_ψ and B_ψ take the values $+1$ and -1 , remain consistent with (2), (3) and (4). We define $A_\psi(a, b, \lambda) = -\text{sign}\langle A(a) \rangle_\psi$ if $|\mu - \mu_a| \leq \pi$ and $|\tau - \frac{\pi}{2}| \leq \eta$, and $A_\psi(a, b, \lambda) = \text{sign}\langle A(a) \rangle_\psi$ otherwise, where μ_a

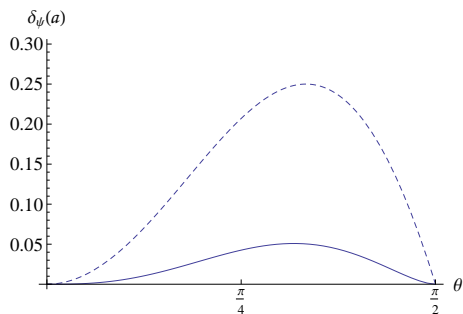


FIG. 1: Constraint for $\delta_\psi(a)$ as a function of entanglement, parameterized by θ (dashed line), when $\hat{A}(a) = \sigma_z$. The solid line represents $\delta_\psi(a)$ for the generalization of the Bell's model. The bound can be saturated in the special case of the local observable here considered, by redefining the outcomes supports.

is the azimuthal angle of a , $\sin \eta = 1 - |\langle A(a) \rangle_\psi|$, and $\tilde{a} = \tilde{a}(a, b)$ plays the same role of \hat{a} previously defined. Its explicit expression is not relevant here. Similarly, $B_\psi(b, \lambda) = -\text{sign}\langle B(b) \rangle_\psi$ if $|\mu - \mu_b| \leq \pi$ and $|\tau - \frac{\pi}{2}| \leq \zeta$, and $B_\psi(b, \lambda) = \text{sign}\langle B(b) \rangle_\psi$ otherwise, where μ_b is the azimuthal angle of b and $\sin \zeta = 1 - |\langle B(b) \rangle_\psi|$. We find that $f_\psi(a, \tau) = 0$ for $|\tau - \frac{\pi}{2}| \leq \eta$, and $f_\psi(a, \tau) = \text{sign}\langle A(a) \rangle_\psi$ otherwise, leading to

$$\delta_\psi(a) = |\langle A(a) \rangle_\psi| - \langle A(a) \rangle_\psi^2, \quad (20)$$

which reproduces (19) when σ_z is measured.

It turns out that the bound (19) is optimal for an arbitrary local measurement when the state ψ is factorized, that is, $\theta = 0$. Also in this case, this can be proven by redistributing the outcomes domains on the unit sphere, in particular by defining $A_\psi(a, \lambda) = -\text{sign}\langle A(a) \rangle_\psi$ if $|\tau - \frac{\pi}{2}| \leq \eta'$, and $A_\psi(a, \lambda) = \text{sign}\langle A(a) \rangle_\psi$ otherwise, where $\sin \eta = 2 \sin \eta'$, and $B_\psi(b, \lambda)$ is defined in order to reproduce the quantum averages. The model is local, and the non-signalling conditions are automatically satisfied. Therefore, (19) expresses the optimal bound for $\delta_\psi(a)$ in the extremal cases of maximal or null entanglement, for generic local measurements.

Conclusions — In this work we have proven that ontological models of quantum theory which are compatible with it, but possibly experimentally distinguishable from it, are possible. In our model, the standard free will assumption, which involves the measurements of A and B , is satisfied, and superluminal communication is impossible. Our result seems to contradict a recent result implying that quantum mechanics is maximally informative [2], but this is not the case, since in our model we do not rely on the same mathematical expression of the free will assumption. Moreover, our model provides the first example of a crypto-nonlocal theory in which the local averages differ from the quantum mechanical ones for arbitrary non-maximally entangled states of a pair of qubits, and it is consistent with former results on maxi-

mally entangled states [9–11, 19]. It proves that the local part of the hidden variables can be nontrivial, and suggests that possible deviations from quantum mechanics on the local averages could be observed also in the case of non-maximally entangled states.

Finally, we have derived an explicit upper bound on the local averages of any deterministic ontological theory for quantum mechanics, when the system is given by a pair of qubits. This constraint is determined by the requirements that: (i) the theory respects the non-signalling condition, when one takes into account the accessible part of λ , and (ii) it is compatible with quantum mechanics, that is, its predictions are the standard ones when the full average over λ is performed. In the case of general local measurements, our bound is optimal only for factorized or maximally entangled states. For arbitrary entanglement, it is optimal when $\hat{A}(a) = \sigma_z$.

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- [20] Obviously, in this case the deeper description is, in principle, completely accessible.
- [21] In this work, it will be apparent whether λ or rather $\{\psi, \lambda\}$ is considered as the ontic state of the system.
- [22] We assume that τ could be prepared at the source, and then possibly communicated to A and B . But different

scenarios are possible, for instance, the accessible and unaccessible parts of the ontic state could be contextual, that is, they could depend on a and b .

[23] Nonetheless, the model can be built in the general case of

arbitrary directions a and b , and non necessarily traceless local operators.