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Recent advances in fields ranging from cosmology to computer science have hinted at a possible deep connection between intelligence and entropy maximization, but no formal physical relationship between them has yet been established. Here, we explicitly propose a first step toward such a relationship in the form of a causal generalization of entropic forces that we find can cause two defining behaviors of the human "cognitive niche" – tool use and social cooperation – to spontaneously emerge in simple physical systems. Our results suggest a potentially general thermodynamic model of adaptive behavior as a non-equilibrium process in open systems.

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Recent advances in fields ranging from cosmology to computer science have hinted at a possible deep connection between intelligence and entropy maximization. In cosmology, the causal entropic principle for anthropic selection has used the maximization of entropy production in causally connected space-time regions as a thermodynamic proxy for intelligent observer concentrations in the prediction of cosmological parameters [1]. In geoscience, entropy production maximization has been proposed as a unifying principle for non-equilibrium processes underlying planetary development and the emergence of life [2–4]. In computer science, maximum entropy methods have been used for inference in situations with dynamically revealed information [5], and strategy algorithms have even started to beat human opponents for the first time at historically challenging high look-ahead depth and branching factor games like Go by maximizing accessible future game states [6]. However, despite these insights, no formal physical relationship between intelligence and entropy maximization has yet been established. In this Letter, we explicitly propose a first step toward such a relationship in the form of a causal generalization of entropic forces that we show can spontaneously induce remarkably sophisticated behaviors associated with the human "cognitive niche," including tool use and social cooperation, in simple physical systems. Our results suggest a potentially general thermodynamic model of adaptive behavior as a non-equilibrium process in open systems.

A non-equilibrium physical system's bias towards maximum instantaneous entropy production [3] is reflected by its evolution toward higher-entropy macroscopic states [7], a process characterized by the formalism of entropic forces [8]. In the canonical ensemble, the entropic force $\mathbf{F}$ associated with a macrostate partition, $\{\mathbf{X}\}$, is given by

$$\mathbf{F}(\mathbf{X_0}) = T\,\nabla_{\mathbf{X}} S(\mathbf{X})\big|_{\mathbf{X_0}}, \tag{1}$$

where T is the reservoir temperature, $S(\mathbf{X})$ is the entropy associated with macrostate $\mathbf{X}$, and $\mathbf{X_0}$ is the present macrostate.

Inspired by recent developments [1–6] to naturally generalize such biases so that they uniformly maximize entropy production between the present and a future time horizon, rather than just greedily maximizing instantaneous entropy production, we can also contemplate generalized entropic forces over paths through configuration space rather than just over the configuration space itself. In particular, we can promote microstates from instantaneous configurations to fixed-duration paths through configuration space while still partitioning such microstates into macrostates according to the initial coordinate of each path. More formally, for any open thermodynamic system, we can treat the phase-space paths taken by the system $\mathbf{x}(t)$ over the time interval $0 \le t \le \tau$ as microstates and partition them into macrostates $\{\mathbf{X}\}$ according to the equivalence relation $\mathbf{x}(t) \sim \mathbf{x}'(t)$ iff $\mathbf{x}(0) = \mathbf{x}'(0)$, thereby identifying every macrostate $\mathbf{X}$ with a unique present system state $\mathbf{x}(0)$, as schematically illustrated in Fig. 1(a). We can then define the causal path entropy S_c of a macrostate $\mathbf{X}$ with associated present system state $\mathbf{x}(0)$ as the path integral

$$S_c(\mathbf{X},\tau) = -k_B \int_{\mathbf{x}(t)} \Pr(\mathbf{x}(t)|\mathbf{x}(0)) \ln \Pr(\mathbf{x}(t)|\mathbf{x}(0))\, \mathcal{D}\mathbf{x}(t), \tag{2}$$

where $\Pr(\mathbf{x}(t)|\mathbf{x}(0))$ denotes the conditional probability of the system evolving through the path $\mathbf{x}(t)$ assuming the initial system state $\mathbf{x}(0)$, integrating over all possible paths $\mathbf{x}^*(t)$ taken by the open system's environment during the same interval:

$$\Pr(\mathbf{x}(t)|\mathbf{x}(0)) = \int_{\mathbf{x}^*(t)} \Pr(\mathbf{x}(t), \mathbf{x}^*(t)|\mathbf{x}(0))\, \mathcal{D}\mathbf{x}^*(t). \tag{3}$$

Our proposed causal path entropy measure (2) can be seen as a special form of path information entropy – a more general measure that was originally proposed in the context of stationary states and the fluctuation theorem [9] – in which macrostate variables are restricted to the initial states of path microstates (hence, each path microstate can be seen as a "causal" consequence of the macrostate to which it belongs). In contrast, path information entropy by definition imposes no such causal macrostate restriction and allows temporally delocalized macrostate variables such as time-averaged quantities over paths [10]. By restricting the macrostate variables to the present ($t = 0$) state of the microscopic degrees of freedom, a causal entropy gradient force can always be defined that can be treated as a real microscopic force that acts directly on those degrees of freedom and that can prescribe well-defined flows in system state space. Again, in contrast, gradients of

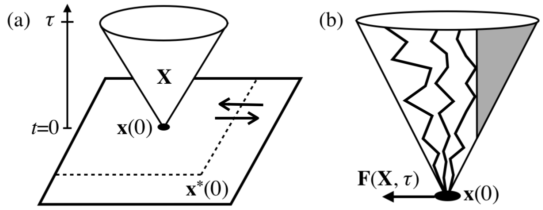

FIG. 1. Schematic depiction of a causal entropic force. (a) A causal macrostate $\mathbf{X}$ with horizon time τ, consisting of path microstates $\mathbf{x}(t)$ that share a common initial system state $\mathbf{x}(0)$, in an open thermodynamic system with initial environment state $\mathbf{x}^*(0)$. (b) Path microstates belonging to a causal macrostate $\mathbf{X}$, in which (for illustrative purposes) there is an environmentally imposed excluded path-space volume that breaks translational symmetry, resulting in a causal entropic force $\mathbf{F}$ directed away from the excluded volume.

path information entropy can only generically be interpreted as phenomenological forces that describe a system's macroscopic dynamics. The generic limitation of path information entropy to serving as a macroscopic description, rather than a microscopic physical prescription, for dynamics was noted recently [11].

A path-based causal entropic force $\mathbf{F}$ corresponding to (2), as schematically illustrated in Fig. 1(b), may then be expressed as

$$\mathbf{F}(\mathbf{X_0}, \tau) = T_{\mathrm{c}}\, \nabla_{\mathbf{X}} S_{\mathrm{c}}(\mathbf{X}, \tau)\Big|_{\mathbf{X_0}}, \tag{4}$$

where T_{c} is a causal path temperature that parameterizes the system's bias toward macrostates that maximize causal entropy. Alternatively, T_{c} can be interpreted as parameterizing the rate at which paths in a hypothetical dynamical ensemble of all possible fixed-duration paths transition into each other, in analogy to the transitions between configurational microstates of an ideal chain responsible for its entropic elasticity. Note that the force (4) is completely determined by only two free parameters, T_{c} and τ, and vanishes in three degenerate limits: (i) in systems with translationally symmetric dynamics; (ii) as $\tau \to 0$; and (iii) if the system and its environment are deterministic.

For concreteness, we will now explore the effect of selectively applying causal entropic forcing to the positions of one or more degrees of freedom of a classical mechanical system. Working with a bounded allowed region in unbounded classical phase space to break translational symmetry, we will denote system paths through position-momentum phase space as $\mathbf{x}(t) = (\mathbf{q}(t), \mathbf{p}(t))$ and denote the forced degrees of freedom by $\{j\}$, with effective masses m_j. As a simple means for introducing nondeterminism, let us choose the environment $\mathbf{x}^*(t)$ to be a heat reservoir at temperature T_{r} that is coupled only to the system's forced degrees of freedom, and that periodically with timescale ϵ rethermalizes those degrees of freedom via nonlinear Langevin dynamics with temporally discretized additive thermal noise and friction terms. Specifically, let us assume the overall energetic force on the forced degrees of freedom to be $g_j(\mathbf{x}(t), t) \equiv -p_j(\lfloor t/\epsilon \rfloor \epsilon)/\epsilon + f_j(\lfloor t/\epsilon \rfloor \epsilon) + h_j(\mathbf{x}(t))$, where $h_j(\mathbf{x}(t))$ collects any deterministic state-dependent internal system forces and $f_j(t)$ is a piecewise-constant random Gaussian force with mean $\langle f_j(t) \rangle = 0$ and correlation function $\langle f_j(t) f_{j'}(t') \rangle = m_j k_{\mathrm{B}} T_{\mathrm{r}} \delta_{j,j'} \delta_{\lfloor t/\epsilon \rfloor, \lfloor t'/\epsilon \rfloor}/\epsilon^2$.

Under these assumptions, the components of the causal entropic force (4) at macrostate $\mathbf{X_0}$ with associated initial system state $(\mathbf{q}(0), \mathbf{p}(0))$ can be expressed as

$$F_j(\mathbf{X_0}, \tau) = T_{\mathrm{c}} \frac{\partial S_{\mathrm{c}}(\mathbf{X}, \tau)}{\partial q_j(0)}\bigg|_{\mathbf{X}=\mathbf{X_0}} \tag{5}$$

for the positions of the forced degrees of freedom. Since paths that run outside the allowed phase space region have probability zero, $\int_{-\infty}^{\infty} \partial \Pr(\mathbf{x}(t)|\mathbf{x}(0))/\partial q_j(0)\, \mathrm{d}q_j(0) = 0$, so bringing the partial derivative from (5) inside the path integral in (2) yields

$$F_j(\mathbf{X_0}, \tau) = -k_{\mathrm{B}} T_{\mathrm{c}} \int_{\mathbf{x}(t)} \frac{\partial \Pr(\mathbf{x}(t)|\mathbf{x}(0))}{\partial q_j(0)} \ln \Pr(\mathbf{x}(t)|\mathbf{x}(0))\, \mathcal{D}\mathbf{x}(t). \tag{6}$$

Since the system is deterministic within each period $[n\epsilon, (n+1)\epsilon)$, we can express any nonzero conditional path probability $\Pr(\mathbf{x}(t)|\mathbf{x}(0))$ as the first-order Markov chain

$$\Pr(\mathbf{x}(t)|\mathbf{x}(0)) = \left(\prod_{n=1}^{N-1} \Pr(\mathbf{x}(t_{n+1})|\mathbf{x}(t_n)) \right) \Pr(\mathbf{x}(\epsilon)|\mathbf{x}(0)), \tag{7}$$

where $t_n \equiv n\epsilon$ and $N \equiv \tau/\epsilon$, and therefore

$$\frac{\partial \Pr(\mathbf{x}(t)|\mathbf{x}(0))}{\partial q_j(0)} = \left(\prod_{n=1}^{N-1} \Pr(\mathbf{x}(t_{n+1})|\mathbf{x}(t_n)) \right) \frac{\partial \Pr(\mathbf{x}(\epsilon)|\mathbf{x}(0))}{\partial q_j(0)}. \tag{8}$$

Choosing ϵ to be much faster than local position-dependent $(\epsilon \ll (2m_j|\nabla_{\mathbf{q}(0)} h_j(\mathbf{x}(0))|^{-1})^{1/2})$ and momentum-dependent $(\epsilon \ll |\nabla_{\mathbf{p}(0)} h_j(\mathbf{x}(0))|^{-1})$ variation in internal system forces, the position $q_j(\epsilon)$ is given by

$$q_j(\epsilon) = q_j(0) + \frac{p_j(0)}{2m_j}\epsilon + \frac{f_j(0) + h_j(0)}{2m_j}\epsilon^2. \tag{9}$$

It follows from (9) that $q_j(\epsilon) - \langle q_j(\epsilon) \rangle = f_j(0)\epsilon^2/(2m_j)$ and $\langle q_j^2(\epsilon) \rangle - \langle q_j(\epsilon) \rangle^2 = k_{\mathrm{B}} T_{\mathrm{r}} \epsilon^2/(4m_j)$, and since the distribution $\Pr(\mathbf{x}(\epsilon)|\mathbf{x}(0))$ is Gaussian in $q_j(\epsilon)$, we can therefore write

$$\frac{\partial \Pr(\mathbf{x}(\epsilon)|\mathbf{x}(0))}{\partial q_j(0)} = -\frac{\partial \Pr(\mathbf{x}(\epsilon)|\mathbf{x}(0))}{\partial q_j(\epsilon)} = \frac{2 f_j(0)}{k_{\mathrm{B}} T_{\mathrm{r}}} \Pr(\mathbf{x}(\epsilon)|\mathbf{x}(0)). \tag{10}$$

Finally, substituting (10) into (8), and (8) into (6), we find that

$$F_j(\mathbf{X_0}, \tau) = -\frac{2T_{\mathrm{c}}}{T_{\mathrm{r}}} \int_{\mathbf{x}(t)} f_j(0) \Pr(\mathbf{x}(t)|\mathbf{x}(0)) \ln \Pr(\mathbf{x}(t)|\mathbf{x}(0))\, \mathcal{D}\mathbf{x}(t). \tag{11}$$

Qualitatively, the effect of (11) can be seen as driving the forced degrees of freedom (j) with a temperature-dependent strength $(T_{\mathrm{c}}/T_{\mathrm{r}})$ in an average of short-term directions $(f_j(0))$ weighted by the diversity of long-term paths $(-\Pr(\mathbf{x}(t)|\mathbf{x}(0)) \ln \Pr(\mathbf{x}(t)|\mathbf{x}(0)))$ that they make reachable, where path diversity is measured over all degrees of freedom of the system, and not just the forced ones.

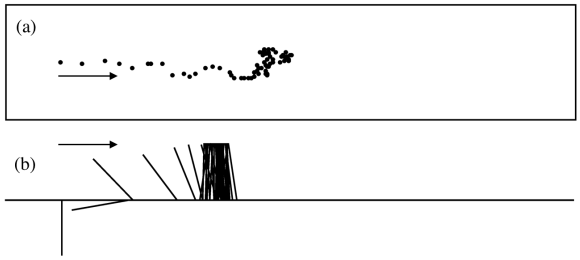

FIG. 2. Illustrative examples of causal entropic forcing. (a) A particle in a box is forced toward the center of its box. (b) A cart and pole system in which only the cart is forced successfully swings up and stabilizes an initially downward-hanging pole. (See also Supplemental Movies 1 and 2 [13], respectively.)

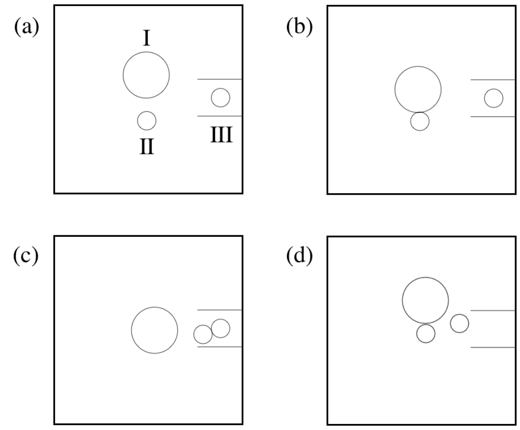

FIG. 3. Effect of causal entropic forcing on a tool use puzzle. (a) Disc I is too large to fit into a tube containing Disc III, whereas Disc II is small enough. (b) Disc I collides with Disc II. (c) Disc II enters the tube and collides with Disc III. (d) Disc III has been successfully released from its initially fixed position in the tube, and is now free for direct manipulation by Disc I. (See also Supplemental Movie 3 [13].)

To better understand this classical-thermal form (11) of causal entropic forcing, we simulated its effect [12, 13] on the evolution of the causal macrostates of a variety of simple mechanical systems: (i) a particle in a box, (ii) a cart and pole system, (iii) a tool use puzzle, and (iv) a social cooperation puzzle. The first two systems were selected for illustrative purposes. The latter two systems were selected because they isolate major behavioral capabilities associated with the human "cognitive niche" [14]. The widely-cited cognitive niche theory proposes that the unique combination of "cognitive" adaptive behavioral traits that are hyperdeveloped in humans with respect to other animals have evolved in order to facilitate competitive adaptation on faster time scales than natural evolution [15].

As a first example of causal entropic forcing applied to a mechanical system, we considered a particle in a two-dimensional box [13]. We found that applying forcing to both of the particle's momentum degrees of freedom had the effect of pushing the particle toward the center of the box, as shown in Fig. 2(a) and Supplemental Movie 1 [13]. Heuristically, as the overall farthest position from boundaries, the central position maximized the diversity of causal paths accessible by Brownian motion within the box. In contrast, if the particle had merely diffused from its given initial state, while its expectation position would relax towards the center of the box, its maximum likelihood position would remain stationary.

As a second example, we considered a cart and pole (or inverted pendulum) system [13]. The upright stabilization of a pole by a mobile cart serves as a standard model for bipedal locomotion [16, 17], an important feature of the evolutionary divergence of hominids from apes [15] that may have been responsible for freeing up prehensile hands for tool use [14]. We found that causal entropic forcing of the cart resulted in the successful swing-up and upright stabilization of an initially downward-hanging pole, as shown in Fig. 2(b) and Supplemental Movie 2 [13]. During upright stabilization, the angular variation of the pole was reminiscent of the correlated random walks observed in human postural sway [18]. Heuristically, again, swinging up and stabilizing the pole made it more energetically favorable for the cart to subsequently swing the pole to any other angle, hence maximizing the diversity of causally accessible paths. Similarly, while allowing the pole to circle around erratically instead of remaining upright might make diverse pole angles causally accessible, it would limit the diversity of causally accessible pole angular momenta due to its directional bias. This result advances beyond previous evidence that instantaneous noise can help to stabilize an inverted pendulum [19, 20] by effectively showing how potential future noise can not only stabilize an inverted pendulum but can also swing it up from a downward hanging position.

As a third example, we considered a tool use puzzle [13] based on previous experiments designed to isolate tool use abilities in non-human animals such as chimpanzees [21] and crows [22], in which inanimate objects are used as tools to manipulate other objects in confined spaces that are not directly accessible. We modeled an animal as a disc (Disc I) undergoing causal entropic forcing, a potential tool as a smaller disc (Disc II) outside of a tube too narrow for Disc I to enter, and an object of interest as a second smaller disc (Disc III) resting inside the tube, as shown in Fig. 3(a) and Supplemental Movie 3 [13]. We found that Disc I spontaneously collided with Disc II (Fig. 3(b)), so as to cause Disc II to then collide with Disc III inside the tube (Fig. 3(c)), successfully releasing Disc III from its initially fixed position and making its degrees of freedom accessible for direct manipulation and even a sort of "play" by Disc I (Fig. 3(d)).

As a fourth example, we considered a social cooperation puzzle [13] based on previous experiments designed to isolate social cooperation abilities in non-human animals such as chimpanzees [23], rooks [24], and elephants [25], in which a pair of animals cooperate to perform synchronized pulling on both ends of a rope or string in order to retrieve an object from an inaccessible region. We modeled a pair of animals as

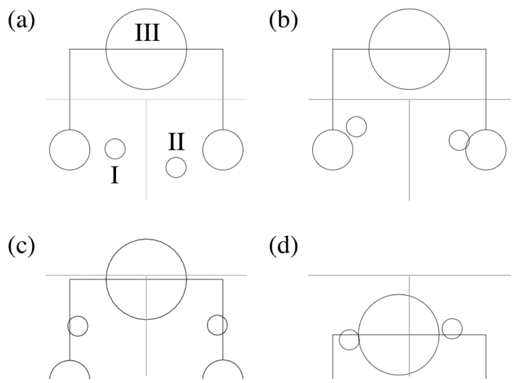

FIG. 4. Effect of causal entropic forcing on a social cooperation puzzle. (a) Initial configuration. (b) Discs I and II spontaneously synchronize their positions and push down in tandem on both string handles. (c) Disc III has been successfully pulled to an accessible position for direct manipulation. (d) Discs I and II directly manipulate Disc III. (See also Supplemental Movie 4 [13].)

two discs (Discs I and II) undergoing independent causal entropic forcing and residing in separate compartments, which they initially shared with a pair of lightweight "handle" discs that were connected by a string, as shown in Fig. 4(a) and Supplemental Movie 4 [13]. The string was wound around a horizontal bar free to move vertically and supporting a target object (Disc III), which could move horizontally on it and which was initially inaccessible to Discs I and II. Disc masses and drag forces were chosen such that synchronized pulling on both ends of the string resulted in a much larger downward force on Disc III than asynchronous or single-sided pulling. Moreover, the initial positions of Discs I and II were set asymmetrically such that coordinated timing of the onset of pushing down on handle discs was required in order for the string not to be pulled away from either Disc I or II. We found that independent causal entropic forcing of Discs I and II caused them to first spontaneously align their positions and push down in tandem on both handles of the string, as shown in Fig. 4(b). As a result, Disc III was successfully pulled toward the compartments containing Discs I and II, as shown in Fig. 4(c), allowing Discs I and II to directly manipulate Disc III, as shown in Fig. 4(d). Heuristically, by cooperating to pull Disc III into their compartments and thereby making it accessible for direct manipulation through collisions, Discs I and II maximized the diversity of accessible causal paths.

To the best of our knowledge, these tool use puzzle and social cooperation puzzle results represent the first successful completion of such standard animal cognition tests using only a simple physical process. The remarkable spontaneous emergence of these sophisticated behaviors from such a simple physical process suggests that causal entropic forces might be used as the basis for a general – and potentially universal – thermodynamic model for adaptive behavior. Namely, adaptive behavior might emerge more generally in open thermodynamic systems as a result of physical agents acting with some or all of the systems' degrees of freedom so as to maximize the overall diversity of accessible future paths of their worlds (causal entropic forcing). In particular, physical agents driven by causal entropic forces might be viewed from a Darwinian perspective as competing to consume future histories, just as biological replicators compete to consume instantaneous material resources [26]. In practice, such agents might estimate causal entropic forces through internal Monte Carlo sampling of future histories [13] generated from learned models of their world. Such behavior would then ensure their uniform aptitude for adaptiveness to future change due to interactions with the environment, conferring a potential survival advantage, to the extent permitted by their strength (parameterized by a causal path temperature, T_c) and their ability to anticipate the future (parameterized by a causal time horizon, τ). Consistent with this model, non-trivial behaviors were found to arise in all four example systems when: (i) the characteristic energy of the forcing ($k_B T_c$) was larger than the characteristic energy of the system's internal dynamics (e.g., for the cart and pole example, the energy required to lift a downward-hanging pole); and (ii) the causal horizon τ was longer than the characteristic time scale of the system's internal dynamics (e.g., for the cart and pole example, the time the pole would need to swing through a semicircle due to gravity).

These results have broad physical relevance. In condensed matter physics, our results suggest a novel means for driving physical systems toward self-organized criticality [27]. In particle theory, they suggest a natural generalization of entropic gravity [8]. In econophysics, they suggest a novel physical definition for wealth based on causal entropy [28, 29]. In cosmology, they suggest a path entropy-based refinement to current horizon entropy-based anthropic selection principles that might better cope with black hole horizons [1]. Finally, in biophysics, they suggest new physical measures for the behavioral adaptiveness and sophistication of systems ranging from biomolecular configurations to planetary ecosystems [2, 3].

In conclusion, we have explicitly proposed a novel physical connection between adaptive behavior and entropy maximization, based on a causal generalization of entropic forces. We have examined in detail the effect of such causal entropic forces for the general case of a classical mechanical system partially connected to a heat reservoir, and for the specific cases of a variety of simple example systems. We found that some of these systems exhibited sophisticated spontaneous behaviors associated with the human "cognitive niche," including tool use and social cooperation, suggesting a potentially general thermodynamic model of adaptive behavior as a non-equilibrium process in open systems.

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