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Jelena Klinovaja and Daniel Loss

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Fractional Fermions with Non-Abelian Statistics

Jelena Klinovaja and Daniel Loss

Department of Physics, University of Basel, Klingelbergstrasse 82, CH-4056 Basel, Switzerland

We introduce a novel class of low-dimensional topological tight-binding models that allow for bound states that are fractionally charged fermions and exhibit non-Abelian braiding statistics. The proposed model consists of a double (single) ladder of spinless (spinful) fermions in the presence of magnetic fields. We study the system analytically in the continuum limit as well as numerically in the tight-binding representation. We find a topological phase transition with a topological gap that closes and reopens as a function of system parameters and chemical potential. The topological phase is of the type BDI and carries two degenerate mid-gap bound states that are localized at opposite ends of the ladders. We show numerically that these bound states are robust against a wide class of perturbations.

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Introduction. Topological properties of condensed matter systems have attracted considerable attention in recent years. In particular, Majorana fermions [1], being their own antiparticles, are expected to occur in a number of systems, e.g. fractional quantum Hall systems [2, 3], topological insulators [4–6], optical lattices [7, 8], p -wave superconductors [9], nanowires with strong Rashba spin orbit interaction [10–15], and carbon-based systems [16–18]. Another class of topological systems is given by bound states of Jackiw-Rebbi type [19] containing fractional charge $e/2$ [20–27]. Such exotic quantum states are interesting in their own right, and due to their special robustness against many forms of perturbations they offer the possibility for applications in quantum computations, especially when they exhibit non-Abelian statistics such as Majorana fermions [28, 29].

In this Letter we introduce a surprisingly simple class of models supporting a topological phase with bound states that possess not only fractional charge but also exhibit non-Abelian statistics under braiding. These bound states behave in many ways similar to the well-studied Majorana fermions in superconducting-semiconducting nanowires [28], but in contrast to them they are complex fermions, and quite surprisingly, emerge in the absence of superconductivity and without BCS-like pairing.

The two non-interacting tight-binding models we propose consist of a double ladder containing spinless particles in a uniform magnetic field and a single ladder containing spinful particles in the presence of both uniform and spatially periodic magnetic fields. We find a topological phase transition in these systems when varying system parameters or the chemical potential, with a characteristic closing and re-opening of a topological gap. Inside the topological phase we find two degenerate bound states, one localized at the right and one at the left end of the system. These bound states are fractionally charged fermions and are shown to exhibit non-Abelian braiding statistics of the Ising anyon type. We study the systems analytically in a continuum approach, finding explicit solutions for the bound states, and confirm these findings

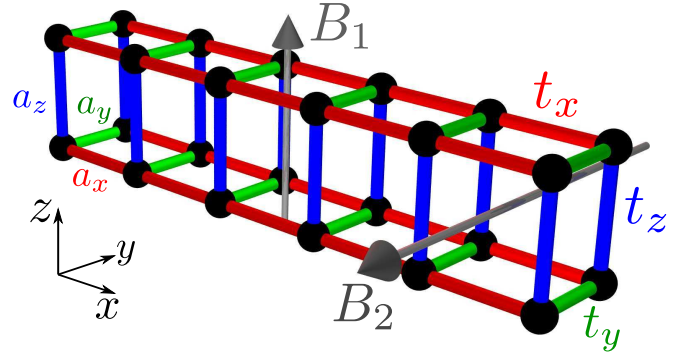


FIG. 1. Double-ladder tight-binding model, consisting of a lower ($\sigma = -1$) and an upper ($\sigma = 1$) ladder, each lying in the xy -plane and held at chemical potentials μ_σ . Here, t_x (red links) is the intra-chain, t_y (green links) the intra-ladder, and t_z (blue links) the inter-ladder hopping amplitude, and $a_{x,y,z}$ are the corresponding lattice constants. A uniform magnetic field $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$ is applied in the yz -plane. The associated magnetic flux results in phases ϕ_1 and ϕ_2 in the hopping amplitudes between different chains, see Eqs. (2, 3).

by independent numerics of the underlying tight-binding model. We further test the stability of these states numerically against a wide class of perturbations and show that the bound states are robust against most of them, except of local charge fluctuations, against which they are partly protected by charge neutrality.

Tight-binding model. We consider a double-ladder system consisting of four coupled chains aligned along x -direction, see Fig. 1. Two upper (lower) chains form the upper (lower) ladder. Each chain is labeled by two indices τ and σ , where $\tau = \pm 1$ refers to the left/right chains, and $\sigma = \pm 1$ refers to the upper/lower ladders. The tight-binding Hamiltonian of a (τ, σ) -chain reads

$$H_{\tau,\sigma}^x = t_x \sum_n (c_{\tau,\sigma,n+1}^\dagger c_{\tau,\sigma,n} + h.c.) + \mu_\sigma \sum_n c_{\tau,\sigma,n}^\dagger c_{\tau,\sigma,n}, \quad (1)$$

where $c_{\tau,\sigma,n}$ ($c_{\tau,\sigma,n}^\dagger$) is the annihilation (creation) opera-

tor on site n of the (τ, σ) -chain. The sum runs over N sites composing the chain. Here, t_x is the hopping matrix element in x -direction, and μ_σ is the chemical potential of the (τ, σ) -chain.

The intra-ladder coupling is given by

$$H_\sigma^y = t_y \sum_n (e^{in\phi_1} c_{1,\sigma,n}^\dagger c_{\bar{1},\sigma,n} + h.c.), \quad (2)$$

where the phase ϕ_1 accompanies the hopping matrix element t_y in y -direction. This phase arises from the magnetic flux through the unit cell produced by a magnetic field \mathbf{B}_1 in z -direction. The inter-ladder coupling between two left or two right chains is given by

$$H_\tau^z = t_z \sum_n (e^{in\phi_2} c_{\tau,1,n}^\dagger c_{\tau,\bar{1},n} + h.c.). \quad (3)$$

Similarly, the phase ϕ_2 arises from the flux through the unit cell produced by a magnetic field \mathbf{B}_2 in y -direction. We note that in practice only one total field, $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$, needs to be applied in the yz -plane, see Fig. 1. The total tight-binding Hamiltonian for the double-ladder model is given by

$$H = \sum_{\tau,\sigma} [H_{\tau,\sigma}^x + (H_\sigma^y + H_\tau^z)/2]. \quad (4)$$

Now we focus on a particular case of the above model. First, we fix the chemical potentials on the upper and lower ladders to be of opposite signs, $\mu_1 = -\mu_{\bar{1}}$. Second, we assume the upper ladder to be at quarter-filling, i.e. $\mu_1 = \sqrt{2}t_x$. The magnetic fields are chosen such that $\phi_1 = \pi/2$ and $\phi_2 = \pi$, or in terms of field strengths, $B_1 = \Phi_0/4a_xa_y$ and $B_2 = \Phi_0/2a_xa_z$, where Φ_0 is the flux quantum, and $a_{x,y,z}$ are the corresponding lattice constants. Assuming that $t_x \gg t_y, t_z$, we treat from now on inter-chain hoppings as small perturbations.

Continuum model. The most convenient way to analyze the tight-binding Hamiltonian in Eq. (4) is to go to the continuum limit. For this we first derive the spectrum via Fourier transformation along the x -axis for each (τ, σ) -chain, $c_{\tau,\sigma,n} = \frac{1}{\sqrt{N}} \sum_k e^{ikna_x} c_{\tau,\sigma,k}$, where $c_{\tau,\sigma,k}$ is the annihilation operator of the electron with momentum k . The Hamiltonian $H_{\tau,\sigma}^x$ has the well-known spectrum, $\epsilon_{k,\sigma} = \mu_\sigma + 2t_x \cos(ka_x)$. At quarter-filling ($\mu_{\pm 1} = \pm\sqrt{2}t_x$) the Fermi momenta are given by $k_{F,1} = \pi/4a_x$ and $k_{F,\bar{1}} = 3\pi/4a_x$, see Fig. 2. We emphasize here that the system is charge neutral, and the shifted chemical potentials just redistribute electrons between chains.

Next, we linearize the spectrum around the Fermi momenta by expressing the annihilation operators $\Psi(x)$ that act on the states close to the Fermi level in terms of slowly varying right ($R_{\tau,\sigma}$) and left ($L_{\tau,\sigma}$) moving fields as

$$\Psi(x) = \sum_{\tau,\sigma} [e^{ik_{F,\sigma}x} R_{\tau,\sigma}(x) + e^{-ik_{F,\sigma}x} L_{\tau,\sigma}(x)]. \quad (5)$$

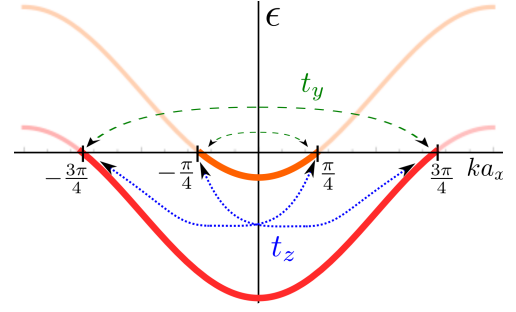


FIG. 2. The spectrum of the upper (orange) and lower (red) chains in the first Brillouin zone. The chemical potentials are chosen such that the system is at the charge-neutrality point. The filled (empty) states are indicated by dark-colored (light-colored) lines. The two Fermi wave vectors are given by $k_{F,1} = \pi/4a_x$ and $k_{F,\bar{1}} = 3\pi/4a_x$. The intra-ladder hopping t_y (green dashed line) and the inter-ladder hopping t_z (blue dotted lines) lead to opening of gaps at the Fermi level ($\epsilon = 0$).

The kinetic part of the Hamiltonian corresponding to $\sum_{\tau,\sigma} H_{\tau,\sigma}^x$ [see Eq. (1)] is rewritten as

$$H^x = -i\hbar v_F \sum_{\tau,\sigma} \int dx (R_{\tau,\sigma}^\dagger \partial_x R_{\tau,\sigma} - L_{\tau,\sigma}^\dagger \partial_x L_{\tau,\sigma}), \quad (6)$$

where we dropped the fast oscillatory terms, and where $v_F = \sqrt{2}t_x a_x / \hbar$ is the Fermi velocity. The intra-chain couplings, given by H_τ^z and H_σ^y [see Eqs. (2) and (3)], lead to mixing between $R_{\tau,\sigma}$ and $L_{\tau,\sigma}$ belonging to different chains. The intra-ladder hopping yields

$$H^y = \int dx t_y (R_{1,1}^\dagger L_{\bar{1},1} + R_{\bar{1},\bar{1}}^\dagger L_{1,\bar{1}} + h.c.), \quad (7)$$

while the inter-ladder hopping yields

$$H^z = \int dx t_z (R_{1,1}^\dagger L_{1,\bar{1}} + R_{\bar{1},\bar{1}}^\dagger L_{\bar{1},1} + R_{\bar{1},\bar{1}}^\dagger L_{\bar{1},1} + R_{1,1}^\dagger L_{1,1} + h.c.). \quad (8)$$

Next, we introduce a new basis $\phi = (R_{1,1}, L_{1,1}, R_{1,\bar{1}}, L_{1,\bar{1}}, R_{\bar{1},1}, L_{\bar{1},1}, R_{\bar{1},\bar{1}}, L_{\bar{1},\bar{1}})$ to rewrite the total Hamiltonian $H = H^x + H^y + H^z$ as $H = \int dx \phi^\dagger(x) \mathcal{H} \phi(x)$ in terms of the Hamiltonian density \mathcal{H} ,

$$\mathcal{H} = \hbar v_F \hat{k} \eta_3 + t_z \sigma_1 \eta_1 + \frac{t_y}{2} (\tau_1 \eta_1 - \tau_2 \sigma_3 \eta_2) + \delta \mu \sigma_3, \quad (9)$$

where the Pauli matrix η_i acts on the right- and left-mover subspace, and the Pauli matrices σ_i and τ_i act on the chain subspaces. The momentum operator is defined as $\hbar \hat{k} = -i\hbar \partial_x$, with eigenvalue k counted henceforth from the corresponding Fermi points. Here, we assume again that the chemical potentials of the upper and lower chains are opposite in sign, however, small deviations from quarter-filling, $\delta \mu = \delta \mu_1 = -\delta \mu_{\bar{1}}$, are taken into

account. In addition, we neglect any constant shifts of the spectrum.

The Hamiltonian density \mathcal{H} allows us to determine the topological class of the system [30]. The system is invariant under the time-reversal operation \mathcal{T} , defined by $U_T^\dagger \mathcal{H}^*(-k) U_T = \mathcal{H}(k)$. Indeed, $U_T = \tau_1 \eta_1$ satisfies this relation. Similarly, the charge-conjugation symmetry operation \mathcal{C} , defined by $U_C^\dagger \mathcal{H}^*(-k) U_C = -\mathcal{H}(k)$, can be satisfied by $U_C = \tau_1 \sigma_1 \eta_3$. Thus, the system belongs to the topological class BDI [30]. The one-dimensional systems of this class are allowed to have an arbitrary number of bound states inside the energy bulk gap [30]. To determine if bound states are present in the system for some given set of parameters, we follow the method developed in Refs. 27 and 31.

The eight spectrum branches of \mathcal{H} are given by

$$\begin{aligned} (\epsilon_{1,2}^\pm)^2 &= (\hbar v_F k \pm \delta\mu)^2 + t_z^2, \\ (\epsilon_{3,4}^\pm)^2 &= (\hbar v_F k)^2 + t_z^2 + t_y^2 + \delta\mu^2 \\ &\quad \pm 2\sqrt{t_z^2 t_y^2 + \delta\mu^2 [(\hbar v_F k)^2 + t_y^2]}. \end{aligned} \quad (10)$$

The system is gapless only for one particular set of parameters, $t_y^2 = t_z^2 + \delta\mu^2$. Otherwise, the spectrum is gapped, and there is a possibility for the existence of bound states inside this gap. Further, we are interested in states exactly in the middle of the gap, *i.e.* at zero energy. In addition, we focus on semi-infinite chains, so the boundary conditions are imposed only on the left (right) end. This implies that the chain length L is much larger than the localization length ξ of the bound states we find.

In order to address the existence of bound states, we first find four fundamental decaying solutions of the system of linear differential equations, following from the Schrodinger equation associated with \mathcal{H} [see Eq. (9)]. Second, the dimension of the null space of the corresponding Wronskian leads us to a topological criterion [27, 31] that separates a topological phase (with bound states) from a trivial phase (without bound states). We find that bound states exist provided the following *topological criterion* is satisfied,

$$t_y^2 > t_z^2 + \delta\mu^2. \quad (12)$$

Working in the operator basis $\Psi(x) = (\Psi_{1,1}, \Psi_{\bar{1},1}, \Psi_{\bar{1},\bar{1}}, \Psi_{1,\bar{1}})$, where $\Psi_{\tau,\sigma}(x)$ is the annihilation operator on the (τ, σ) -chain, we find the wavefunction of the state localized at the left end of the double-ladder system explicitly,

$$\psi_L(x) = \begin{pmatrix} if(x) \\ f^*(x) \\ -i(-1)^n f^*(x) \\ -(-1)^n f(x) \end{pmatrix}, \quad (13)$$

and the wavefunction of the state localized at the right end, $\psi_R(x) = \psi_L^*(L-x)$, where n labels the site, $x = na_x$

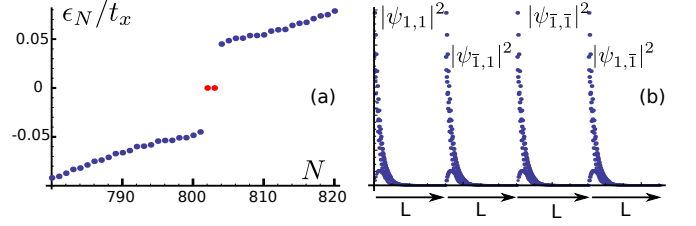


FIG. 3. (a) The relevant part of the spectrum for a double-ladder system of length $L/a_x = 401$ found by numerical diagonalization of the tight-binding Hamiltonian H [see Eq. (4)]. The energy ϵ_N corresponds to the N th energy level. The parameters $t_z/t_x = 0.05$, $t_y/t_x = 0.1$, and $\delta\mu = 0$ are chosen to satisfy the topological criterion Eq. (12). In the middle of the gap, $\epsilon = 0$, there are two degenerate bound states (red dots). (b) The probability density $|\psi_{\tau,\sigma}|^2$ of the left state in the (τ, σ) -chain. The density patterns are the same for all chains, in full agreement with the continuum solution Eq. (13).

[32]. Here, we have introduced the notations,

$$\begin{aligned} f(x) &= e^{i\theta/2} (e^{ik_F x} e^{-x/\xi_1 + i\delta\mu x/\hbar v_F} - e^{-ik_F x} e^{-x/\xi_2}), \\ e^{i\theta} &= \left(\sqrt{t_y^2 - \delta\mu^2} + i\delta\mu \right) / t_y. \end{aligned} \quad (14)$$

The localization length of the bound states is given by $\xi = \max\{\xi_1, \xi_2\}$, where $\hbar v_F \xi_1 = t_z$, and $\hbar v_F \xi_2 = \sqrt{t_y^2 - \delta\mu^2} - t_z$. Close to the phase transition point the localization length is determined by ξ_2 , whereas deep inside the topological phase by ξ_1 . If ξ is comparable with L , the two bound states localized at opposite ends overlap. As a result, the energy levels are split from zero energy, and the corresponding wavefunctions are given by the symmetric and antisymmetric combinations,

$$\psi_{s/a}(x) = \psi_L(x_L) \pm i\psi_R(x_R). \quad (15)$$

We note that these right or left localized bound states are fractionally charged fermions of charge $e/2$ [20–22].

The results obtained above in the continuum model are in good agreement with the numerical results obtained by direct diagonalization of the tight-binding Hamiltonian H given in Eq. (4), see Fig. 3.

Non-Abelian statistics. In the absence of overlap between the bound states $\psi_L(x)$ and $\psi_R(x)$, the zero-energy level is two-fold degenerate, so it can potentially be used for braiding and ultimately for topological quantum computation. Taking into account that the system is charge-neutral, we focus on the states that have equal density distributions on both ends,

$$\psi_\pm(x) = \psi_L(x) \pm e^{i\alpha} \psi_R(x), \quad (16)$$

where α is an arbitrary phase (see also below). The creation operator F_\pm^\dagger corresponding to the state $\psi_\pm(x)$ is given by $F_\pm = \int dx \psi_\pm(x) \cdot \Psi \equiv \frac{1}{\sqrt{2}}(F_L \pm e^{i\alpha} F_R)$, where F_L (F_R) is the left (right) endstate annihilation operator.

During the braiding process, which corresponds to exchanging the right and left endstates, the parity, defined by $P_{\pm} = 1 - F_{\pm}^{\dagger}F_{\pm}$, should be conserved [29]. Thus, the old states have to transform into new ones under braiding as

$$F_L \rightarrow F_R, \quad F_R \rightarrow e^{-2i\alpha}F_L, \quad (17)$$

or, equivalently, as $F_{\pm} \rightarrow \pm e^{-i\alpha}F_{\pm}$. The corresponding unitary operator U_{LR} that implements this braiding rule is found to be $U_{LR} = e^{-i\alpha(n_++n_-)+i\pi n_-}$, where $n_{\pm} = F_{\pm}^{\dagger}F_{\pm}$. Indeed, it is easy to show that $U_{LR}^{\dagger}F_{\pm}U_{LR} = \pm e^{-i\alpha}F_{\pm}$, and thus $U_{LR}^{\dagger}F_LU_{LR} = F_R$, and $U_{LR}^{\dagger}F_RU_{LR} = e^{-2i\alpha}F_L$. In terms of the $F_{R/L}$ operators we have

$$U_{LR} = e^{-i(\alpha-\pi/2)(n_R+n_L)-i\pi(e^{i\alpha}F_L^{\dagger}F_R+h.c.)/2}, \quad (18)$$

where $n_{L/R} = F_{L/R}^{\dagger}F_{L/R}$. Next, let us assume a network of such double ladders similar to the one proposed for Majorana fermions [29, 33, 34]. The braiding operations can be performed by exchanging bound states localized at different double ladders. This could be achieved by tuning locally $t_{y,z}$ and/or $\delta\mu$, which shifts the boundary in real space between topological and non-topological phases [see Eq. (12)]. Using U_{LR} given explicitly by Eq. (18), one can show that two braiding operations do not commute, $[U_{ij}, U_{jk}] \neq 0$, where the labels i, j, k denote three states. Thus, we see that our bound states obey non-Abelian braiding statistics. In particular, it is interesting to consider the case of $\alpha = \pm\pi/2$, since these states, $\psi_{s/a}(x)$ [see Eq. (15)], can be easily prepared by lifting the degeneracy temporarily (via tuning the wave function overlap), resulting in filling either the symmetric or the antisymmetric energy levels after some dephasing time. Moreover, we can determine the phase α [see Eq. (16)] by projective measurements on the states $\psi_{s/a}$. For example, the probability to measure the symmetric state is given by $|\langle\psi_s|\psi_+\rangle|^2 = \sin^2(\alpha/2 + \pi/4)$. All this taken together opens up the possibility to use these bound states for topological quantum computation along the lines proposed for Ising anyons [35, 36].

Single-ladder model. An alternative representation of the double-ladder model with spinless particles is a single-ladder model with spinful particles, see Fig. 4. In this model the $\sigma = -1$ (lower) and the $\sigma = 1$ (upper) ladders are identified with the spin-down and spin-up components, resp. A uniform magnetic field $\bar{\mathbf{B}}_1$ is applied perpendicular to the ladder, leading to both orbital and spin effects. The Zeeman energy acts as a chemical potential with opposite signs for opposite spin directions, $\mu_{\sigma} = \sigma g\mu_B\bar{B}_1$, where g is the g -factor, and μ_B is the Bohr magneton. A spatially periodic field \mathbf{B}_2 of period $2a_x$ couples opposite spins, so the effective hopping is given by $t_z = g\mu_B\bar{B}_2$. Such a periodic field can be produced by nanomagnets [37] or, equivalently, by Rashba spin orbit interaction and a uniform magnetic field [38].

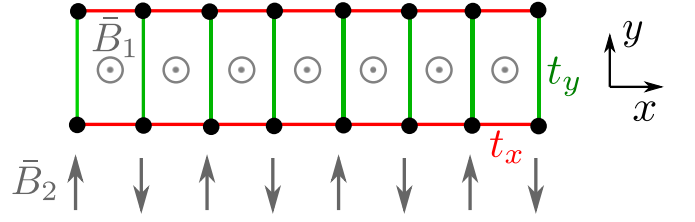


FIG. 4. Single-ladder model. A uniform magnetic field $\bar{\mathbf{B}}_1$ is applied perpendicular to the ladder. A spatially periodic magnetic field \mathbf{B}_2 with period $2a_x$ is applied in the xy -plane.

Stability against perturbations. We next address the stability of the topological phase and the bound states against local perturbations. In general, we find numerically that local fluctuations that preserve the symmetry between the upper and lower ladders are not harmful. This includes correlated fluctuations of chemical potentials ($\Delta\mu_1 = -\Delta\mu_{\bar{1}}$), magnetic fluxes, and hopping matrix elements. If the symmetry is not preserved, the perturbations act like a level-detuning, and the bound states separate in energy independent of wave function overlap but proportional to their occupation probability at the site of fluctuation. We emphasize that the single-ladder model is protected against such symmetry-breaking terms, except against local chemical potential fluctuations that break charge neutrality, i.e. $\Delta\mu_1 = \Delta\mu_{\bar{1}}$. However, we note that chains without charge impurities are stable against such fluctuations, as local differences in μ would lead to charge redistribution, restoring a uniform chemical potential in the chain. Another problem can arise from flux fluctuations. They can decrease the Fourier components at $k_{F,\sigma}$ of the backscattering terms and thereby reduce the gaps. As a result, the system can move out of the topological phase. However, these flux fluctuations become irrelevant deep inside the topological phase.

Conclusions. We have uncovered model systems of striking simplicity that allow for a topological phase with degenerate bound states that are fractionally charged and obey non-Abelian braiding statistics. We have shown that these exotic quantum states are rather robust against a large class of perturbations. Quite remarkably, our models demonstrate that non-Abelian states can exist in single-particle systems, without any correlations and in the complete absence of superconductivity or BCS-like pairing. This should open the path for novel implementations of topological matter in realistic systems. One promising candidate system that suggests itself for implementations of such tight-binding ladders are optical lattices with ultralong dephasing times [39], because they allow for a high degree of control and, in particular, possess charge stability of the type invoked here.

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- [1] F. Wilczek, Nat. Phys. **5**, 614 (2009).
- [2] N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
- [3] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
- [4] L. Fu and C. L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).
- [5] Y. Tanaka, T. Yokoyama, and N. Nagaosa, Phys. Rev. Lett. **103**, 107002 (2009).
- [6] S. Sasaki, M. Kriener, K. Segawa, K. Yada, Y. Tanaka, M. Sato, and Y. Ando, Phys. Rev. Lett. **107**, 217001 (2011).
- [7] M. Sato and S. Fujimoto, Phys. Rev. B **79**, 094504 (2009).
- [8] L. Jiang, T. Kitagawa, J. Alicea, A. Akhmerov, D. Pekker, G. Refael, J. I. Cirac, E. Demler, M. D. Lukin, and P. Zoller, Phys. Rev. Lett. **106**, 220402 (2011).
- [9] A. C. Potter and P. A. Lee, Phys. Rev. B **83**, 094525 (2011).
- [10] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. **105**, 077001 (2010).
- [11] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. **105**, 177002 (2010).
- [12] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, L. P. Kouwenhoven, Science, **336**, 1003 (2012).
- [13] M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, and H. Q. Xu, Nano Lett. **12**, 6414 (2012).
- [14] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, H. Shtrikman, Nat. Phys. **8**, 887-895 (2012).
- [15] J. Alicea, Phys. Rev. B **81**, 125318 (2010).
- [16] J. Klinovaja, S. Gangadharaiah, and D. Loss, Phys. Rev. Lett. **108**, 196804 (2012); J. Klinovaja, M. J. Schmidt, B. Braunecker, and D. Loss, Phys. Rev. Lett. **106**, 156809 (2011).
- [17] J. Klinovaja, G. J. Ferreira, and D. Loss, Phys. Rev. B **86**, 235416 (2012).
- [18] J. Klinovaja and D. Loss, Phys. Rev. X **3**, 011008 (2013).
- [19] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).
- [20] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. **42**, 1698 (1979).
- [21] S. Kivelson and J. R. Schrieffer, Phys. Rev. B **25**, 6447, (1982).
- [22] R. Rajaraman and J. S. Bell, Phys. Lett. **116B**, 151 (1982).
- [23] C. Hou, C. Chamon, and C. Mudry, Phys. Rev. Lett. **98**, 186809 (2007).
- [24] B. Seradjeh, J. E. Moore, and M. Franz, Phys. Rev. Lett. **103**, 066402 (2009).
- [25] L. Santos, Y. Nishida, C. Chamon, and C. Mudry, Phys. Rev. B **83**, 104522 (2011).
- [26] S. Gangadharaiah, L. Trifunovic, and D. Loss, Phys. Rev. Lett. **108**, 136803 (2012).
- [27] J. Klinovaja, P. Stano, and D. Loss, Phys. Rev. Lett. **109**, 236801 (2012).
- [28] J. Alicea, Rep. Prog. Phys. **75**, 076501 (2012).
- [29] B. I. Halperin, Y. Oreg, A. Stern, G. Refael, J. Alicea, and F. von Oppen, Phys. Rev. B **85**, 144501 (2012).
- [30] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, New J. Phys. **12**, 065010 (2010).
- [31] J. Klinovaja and D. Loss, Phys. Rev. B **86**, 085408 (2012).
- [32] Here, we assume that the chains are composed of an odd number of sites. The even number case is obtained by replacing $(-1)^n$ by $(-1)^{n+1}$ in $\Psi_R(x)$.
- [33] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, Nat. Phys. **7**, 412 (2011).
- [34] F. L. Pedrocchi, S. Chesi, S. Gangadharaiah, and D. Loss, Phys. Rev. B **86**, 205412 (2012).
- [35] S. Bravyi, Phys. Rev. A **73**, 042313 (2006).
- [36] M. Freedman, C. Nayak, and K. Walker, Phys. Rev. B **73**, 245307 (2006).
- [37] B. Karmakar, D. Venturelli, L. Chirolli, F. Taddei, V. Giovannetti, R. Fazio, S. Roddaro, G. Biasiol, L. Sorba, V. Pellegrini, and F. Beltram, Phys. Rev. Lett. **107**, 236804 (2011).
- [38] B. Braunecker, G. I. Japaridze, J. Klinovaja, and D. Loss, Phys. Rev. B **82**, 045127 (2010).
- [39] M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen, and U. Sen, Adv. Phys. **56**, 243 (2007).