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Atomic Quantum Simulation of $U(N)$ and $SU(N)$ Non-Abelian Lattice Gauge Theories

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Using ultracold alkaline-earth atoms in optical lattices, we construct a quantum simulator for $U(N)$ and $SU(N)$ lattice gauge theories with fermionic matter based on quantum link models. These systems share qualitative features with QCD, including chiral symmetry breaking and restoration at non-zero temperature or baryon density. Unlike classical simulations, a quantum simulator does not suffer from sign problems and can address the corresponding chiral dynamics in real time.

Introduction. Non-Abelian gauge fields play a central role in the dynamics of the Standard Model of particle physics. In particular, the strong $SU(3)$ gauge interactions between quarks and gluons in Quantum Chromodynamics (QCD) give rise to the spontaneous breakdown of the chiral symmetry of the light quarks. Heavy-ion collisions produce a high-temperature quark-gluon plasma in which chiral symmetry is restored. The deep interior of neutron stars contains high-density nuclear matter or even quark matter, which may be a baryonic superfluid or a color superconductor [1]. Unfortunately, due to severe sign problems, the real-time evolution of heavy-ion collisions or the phase structure of dense QCD matter is inaccessible to first principles classical simulation methods. In condensed matter physics strongly coupled gauge theories play a prominent role in strongly correlated systems. In particular, the non-Abelian $SU(2)$ variant of quantum spin liquids has long been debated as a possible connection between the doped Mott insulator and the high-T_c superconducting phase in cuprates [2]. The challenge of solving such problems motivates the development of quantum simulators for non-Abelian lattice gauge theories. Recently, quantum simulators have been constructed for Abelian $U(1)$ gauge theories with [3–5] and without coupling to matter fields [6, 7]. Here, we construct a quantum simulator of $U(N)$ and $SU(N)$ strongly coupled lattice gauge theories in $(1+1)$, $(2+1)$, and $(3+1)$D using ultracold alkaline-earth (AE) atoms in an optical lattice. On the one hand, our approach is based on quantum link models (QLMs) [8–10], which allow the exact embodiment of non-Abelian gauge interactions in ultracold matter. On the other hand, we utilize fundamental symmetries of matter, such as the $SU(2I+1)$ invariance of interactions between fermionic AE isotopes such as ^{87}Sr or ^{173}Yb [11–20]. While still being far from a quantum simulator for full QCD, simpler model systems share several qualitative features, including confinement, chiral symmetry breaking (χSB), and its restoration (χSR) [1]. They provide a unique environment to investigate important dynamical questions which are out of reach for classical simulation.

The non-perturbative physics of non-Abelian gauge theories is traditionally addressed in the context of Wilson's lattice gauge theory [21], in which the gluon field is represented by parallel transporter matrices residing on the links connecting neighboring lattice points of a 4D space-time lattice. Since Wilson's classical link variables take values in the continuous gauge group $SU(N)$, the corresponding Hilbert space is infinite-dimensional even for a single link. The elements of a quantum link matrix are non-commuting operators, similar to the components of a quantum spin. As a result, QLMs have a finite-dimensional Hilbert space, and therefore provide an attractive framework for the construction of quantum simulators for dynamical Abelian and non-Abelian gauge theories. In the continuum limit of QLMs, which is naturally realized via dimensional reduction, one recovers QCD with chiral quarks as domain wall fermions

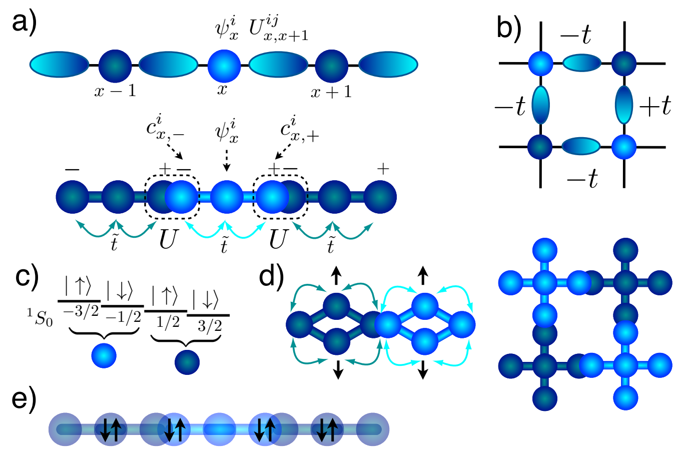

FIG. 1. [Color online] a) (upper panel) $U(N)$ QLM in (1+1)D with quark fields ψ_x^i on lattice sites and gauge fields $U_{x,x+1}^{ij}$ on links; (lower panel) hopping of AE atoms between quark and rishon sites of the same shading. b) Implementation of the QLM in rishon representation with fermionic atoms in $(2+1)$D. c) Encoding of the color degrees of freedom for $N=2$ $(\uparrow,\downarrow)$ in Zeeman states of a fermionic AE atom with $I=3/2$. d) Lattice structure to avoid the interaction in fermionic matter sites using a species-dependent optical lattice (for an alternative method using site-dependent optical Feshbach resonances see the main text). e) Initial state loaded in the optical lattice with a staggered distribution of doubly occupied sites for a $U(2)$ QLM with $\mathcal{N}=2$.

[22, 23]. A pedagogical introduction to QLMs, together with an extensive explanation of the corresponding terminology, is contained in the supplementary information (SI) [24].

Quantum Link Models. The hopping of electrons between lattice sites x and y in an external magnetic background field $\vec{B} = \vec{\nabla} \times \vec{A}$ is described by $\psi_x^\dagger u_{xy} \psi_y$, where $u_{xy} = \exp(i \int_x^y d\vec{l} \cdot \vec{A}) \in U(1)$ is the phase picked up in this process [25]. In particle physics, gauge fields appear as dynamical quantum degrees of freedom, not just as classical background fields. Here we consider $U(N)$ and $SU(N)$ lattice gauge theories without approaching the continuum limit, using so-called staggered fermions, which are represented by creation and annihilation operators $\psi_x^{i\dagger}$ and ψ_x^i, that obey standard anti-commutation relations. Here $i \in \{1, 2, \dots, N\}$ represents the non-Abelian color index of a quark. The fundamental gauge degrees of freedom representing the gluon field are $N \times N$ matrices U_{xy} (with elements U_{xy}^{ij}) associated with the link between nearest-neighbor points x and y (c.f. Fig.1a). The hopping of a quark, which exchanges color with the gluon field, is then described by $\psi_x^\dagger U_{xy} \psi_y = \psi_x^{i\dagger} U_{xy}^{ij} \psi_y^j$. This term is invariant against gauge transformations, ${}^\Omega\psi_x = \Omega_x \psi_x$, ${}^\Omega\psi_x^\dagger = \psi_x^\dagger \Omega_x^\dagger, {}^\Omega U_{xy} = \Omega_x U_{xy} \Omega_y^\dagger$, with $\Omega_x \in U(N)$. The $SU(N)$ gauge transformations and the additional $U(1)$ gauge transformation contained in $U(N)$ are generated by

$$G_x^a = \psi_x^{i\dagger} \lambda_{ij}^a \psi_x^j + \sum_k \left(L_{x,x+\hat{k}}^a + R_{x-\hat{k},x}^a \right),$$
$$G_x = \psi_x^{i\dagger} \psi_x^j - \sum_k \left(E_{x,x+\hat{k}} - E_{x-\hat{k},x} \right), \quad (1)$$

where $\hat{k}$ is a unit-vector in the k-direction, λ^a ($a \in \{1, 2, \dots, N^2 - 1\}$) are the $SU(N)$ Gell-Mann matrices, and f_{abc} are the $SU(N)$ structure constants, such that $[G_x^a, G_y^b] = 2i\delta_{xy} f_{abc} G_x^c$. The operators L_{xy}^a and R_{xy}^a represent $SU(N)$ electric field operators associated with the left and right end of a link $\langle xy \rangle$, while E_{xy} represents the Abelian $U(1)$ electric field operator. Physical states $|\Psi\rangle$ obey the $SU(N)$ Gauss law $G_x^a|\Psi\rangle = 0$, while in a $U(N)$ gauge theory also $G_x|\Psi\rangle = 0$. The operators U, L^a, R^a, and E associated with the same link obey

$$[L^a, L^b] = 2if_{abc} L^c, \ [R^a, R^b] = 2if_{abc} R^c,$$
$$[L^a, R^b] = [E, L^a] = [E, R^a] = 0,$$
$$[L^a, U] = -\lambda^a U, \ [R^a, U] = U\lambda^a, \ [E, U] = U, \quad (2)$$

while operators associated with different links commute.

In Wilson's lattice gauge theory, U is an element of the gauge group. In a $U(N)$ gauge theory, $\det U = \exp(i\varphi) \in U(1)$ represents a $U(1)$ link variable, canonically conjugate to the electric flux operator $E = -i\partial_\varphi$. In an $SU(N)$ gauge theory $U \in SU(N)$ and L^a, R^a take appropriate derivatives with respect to the matrix elements U^{ij}. The resulting Hilbert space per link is then unavoidably infinite-dimensional. In order to represent the commutation relations of the gauge algebra of Eq.(2) in a finite-dimensional Hilbert space, QLMs give up the commutativity of the matrix elements U^{ij} without compromising gauge invariance. The real and imaginary parts of the matrix elements U^{ij} of the $N \times N$ quantum link matrix are represented by $2N^2$ Hermitean operators. Together with the electric field operators L^a, R^a, and E these are $2N^2 + 2(N^2 - 1) + 1 = (2N)^2 - 1$ generators which form the embedding algebra $SU(2N)$. While $U(1)$ quantum links can be represented by quantum spins embedded in an $SU(2)$ algebra, $U(N)$ or $SU(N)$ QLMs can be realized with different representations of $SU(2N)$. A useful representation is based on fermionic rishon constituents [23]

$$L^a = c_+^{i\dagger} \lambda_{ij}^a c_+^j, \ R^a = c_-^{i\dagger} \lambda_{ij}^a c_-^j, \ E = \frac{1}{2}(c_-^{i\dagger} c_-^i - c_+^{i\dagger} c_+^i),$$
$$U^{ij} = c_+^i c_-^{j\dagger}, \ \mathcal{N} = c_-^{i\dagger} c_-^i + c_+^{i\dagger} c_+^i. \quad (3)$$

The rishon creation and annihilation operators, $c_\pm^{i\dagger}$ and $c_\pm^i$, are associated with the left and right ends of a link (c.f. Fig.1a) and obey standard anti-commutation relations. Our construction of a quantum simulator for $U(1)$ gauge theories used Schwinger bosons to represent quantum links [3]. Here it is natural to replace Schwinger bosons by rishon fermions. $\mathcal{N}$ counts the number of rishons on a link.

The Hamiltonian of a $(d+1)$D $U(N)$ QLM with staggered fermions takes the form

$$H = -t \sum_{\langle xy \rangle} \left(s_{xy} \psi_x^{i\dagger} U_{xy}^{ij} \psi_y^j + \text{h.c.} \right) + m \sum_x s_x \psi_x^{i\dagger} \psi_x^i$$
$$= -t \sum_{\langle xy \rangle} \left(s_{xy} Q_{x,+k}^\dagger Q_{y,-k} + \text{h.c.} \right) + m \sum_x s_x M_x, \quad (4)$$

where $s_x = (-1)^{x_1 + \dots + x_d}$ and $s_{xy} = (-1)^{x_1 + \dots + x_{k-1}}$, with $y = x + \hat{k}$. t is the strength of the hopping term, and m is the mass. The summation convention is implicit in the color indices. We have also introduced the $U(N)$ gauge invariant "meson" and "constituent quark" operators $M_x = \psi_x^{i\dagger} \psi_x^i$ and $Q_{x,\pm k} = c_{x,\pm k}^{i\dagger} \psi_x^i$. Together with the "glueball" operators $\Phi_{x,\pm k,\pm l} = c_{x,\pm k}^{i\dagger} c_{x,\pm l}^i$, they form a site-based $U(2d+1)$ algebra. The rishon number is conserved locally on each link. The $U(N)$ model has no baryons, since the $U(1)$ baryon number symmetry is gauged. In order to obtain charge conjugation invariance C and to reduce the gauge symmetry to $SU(N)$, one must work with $\mathcal{N}_{xy} = N$ rishons per link. Adding the term $\gamma \sum_{\langle xy \rangle} (\det U_{xy} + \text{h.c.})$ to the Hamiltonian, explicitly breaks the $U(N)$ gauge symmetry down to a local $SU(N)$ and a global $U(1)$ baryon number symmetry generated by $B = \sum_x \left(\psi_x^{i\dagger} \psi_x^i - \frac{N}{2} \right)$. The symmetries of various model systems are summarized in Table 1 of the SI. All models have a $\mathbb{Z}(2)$ chiral symmetry, which is spontaneously broken at a critical temperature T_c, and may get restored at non-zero baryon density n_B. It would be natural to add electric and magnetic field energy terms $\frac{g^2}{2}\sum_{\langle xy\rangle}\left(L^a_{xy}L^a_{xy}+R^a_{xy}R^a_{xy}\right)$, $\frac{g'^2}{2}\sum_{\langle xy\rangle}E^2_{xy}$, and $\frac{1}{4g^2}\sum_{\langle wxyz\rangle}\left(U_{wx}U_{xy}U_{yz}U_{zw}+\text{h.c.}\right)$, where $\langle wxyz\rangle$ denotes an elementary plaquette with g^2 and g'^2 as the coupling constants. At strong coupling these terms are inessential for qualitative features of the dynamics at finite temperature or baryon density, and are thus not yet included in our implementation.

Atomic quantum simulation of $U(N)$ QLMs. An illustration of the QLM and its rishon representation for (1+1)D and (2+1)D is provided in Fig.1. Quark fields ψ^i_x reside on the lattice sites x, while the rishons $c^i_{x,\pm k}$ are on "link-sites" $(x,\pm k)$ at the left (right) end of the links exiting (entering) the point x (c.f. Fig.1a lower panel). The key step in our physical implementation is to interpret the *lattice with quark and rishon sites* in Figs.1a,b as a *physical optical lattice for fermionic atoms*. Hence, an atom on site x of the optical lattice represents a quark ψ^i_x, while hopping of this atom to a link-site $(x,\pm k)$ converts it to a rishon $c^i_{x,\pm k}$. The color index i is encoded in internal atomic states.

The basic building blocks in our atomic setup are the tunnel-coupled triple-wells in (1+1)D (Fig.1a) or the cross-shaped vertices in (2+1)D (Fig.1b). The corresponding hopping dynamics of the atoms is described by the Hamiltonian $h_{x,k}=\tilde{t}(s_{xy}Q_{x,+k}+Q_{x,-k}+\text{h.c.})$. Physically, the overlap of the Wannier wave functions can be used to implement the usual tunneling [26]. In case different phases are needed to simulate staggered fermions in the lattice, Raman assisted tunneling [25] or shaken optical lattices [27, 28] can be applied. In order to obtain the desired quark-rishon dynamics, we introduce the microscopic atomic Hamiltonian

$$\widetilde{H}=U\sum_{\langle xy\rangle}(\mathcal{N}_{xy}-n)^2+\sum_{x,k}h_{x,k}+m\sum_x s_x M_x. \quad (5)$$

The first term enforces the constraint of $\mathcal{N}_{xy}=n$ rishons per link, with $U\gg\tilde{t}$. In a physical setup, this is implemented as a strong repulsion between atoms occupying rishon-sites, indicated in Fig.1a by the overlapping link-sites, and by a potential off-sets in the rishon sites. Details on the lattice structure are discussed in the SI. The second term represents atomic hopping, while the last term realizes the staggered fermion mass with a superlattice. In second order perturbation theory in the tunnel-coupling, the above Hamiltonian induces the hopping term of Eq.(4) with $t=\tilde{t}^2/U$. Fig.2a illustrates the matter-gauge interaction. We note that an additional term $t\sum_{x,\pm k}Q^\dagger_{x,\pm k}Q_{x,\pm k}$ is also generated. This is no problem, because this term is invariant under all relevant symmetries. It is also possible to add a 4-fermion term $V\sum_x M_x^2$.

With the Hamiltonian of Eq.(5) we have reduced the realization of $U(N)$ QLMs to a lattice dynamics of interacting fermions. This is enabled by the factorization of the quantum link variables into rishons. We emphasize that the building blocks in $\widetilde{H}$ are *gauge invariant* "meson" and "constituent quark" operators, which allows a gauge invariant implementation of the dynamics. This is in contrast to previous work, where Gauss' law was enforced by an energy constraint in the microscopic dynamics. The essential symmetries of $\widetilde{H}$ to be respected by the implementation are: (i) the color-independent hopping of fermions and rishons, and (ii) the color-independent interaction between rishons to ensure the *local* particle number conservation on each link. Indeed these symmetries are accurately respected in setups with AE atoms [11, 13, 20].

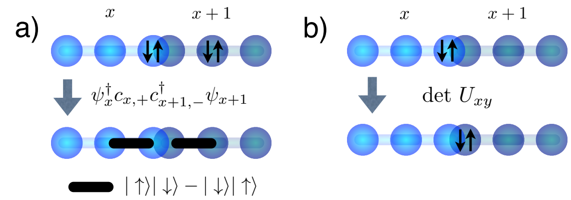

FIG. 2. [Color online] Dynamical processes in $U(2)$ QLMs with $\mathcal{N}=2$. a) Matter-gauge interaction as correlated hopping of quarks and rishons. Starting with a configuration of site-singlets, the matter-gauge interaction converts them into nearest-neighbor singlets, keeping the rishon number per link constant. b) The determinant term corresponds to two-body hopping of both rishons on the link.

For a given nuclear spin I, the electronic ground state ${}^1S_0$ of fermionic AE atoms has $2I+1$ Zeeman levels $m_I=-I,\ldots,+I$. We encode the color degrees of freedom for the even (odd) building blocks (triple-wells in (1+1)D and cross-shaped vertices in (2+1)D, represented by the light (dark) shading in Fig.1) in the N lowest (highest) m_I levels (c.f. Fig.1c). For example, to implement a $U(2)$ QLM, we choose positive nuclear spin states $m_I=3/2,1/2$ on the even and negative nuclear spin states $m_I=-3/2,-1/2$ on the odd building blocks. The AE atoms have the unique property that their scattering is almost exactly $SU(2I+1)$-symmetric, i.e., all pairs of states have the same scattering length [11, 13, 20]. This guarantees the symmetry of the U term in Eq.(5). The m_I-dependent hopping illustrated in Fig.1a can be realized in optical lattices with an appropriate choice of laser frequencies and polarizations [29, 30], or with optical potentials obtained by holographic techniques [31, 32]. Finally, the repulsion U, which only affects the rishon- but not the quark-sites, can be realized with optical Feshbach resonances of AE atoms allowing spatially dependent on-site interactions [33–39]. An alternative setup uses m_I-dependent optical lattices with overlapping sites for the interacting, and spatially separated sites for the non-interacting fermions (c.f. Fig.1d).

SU(N) lattice gauge theories. We now reduce the gauge symmetry from $U(N)$ to $SU(N)$ by activating the $\det U_{xy}$ term. For definiteness, we investigate the $N = \mathcal{N} = 2$ case, for which $\det U_{xy} = 2c^1_{x,+k}c^{1\dagger}_{y,-k}c^2_{x,+k}c^{2\dagger}_{y,-k}$. This corresponds to two-particle tunneling between the overlapping rishon-sites. As indicated in Fig.2b we assume in our AE setup partially overlapping rishon-sites implying a different overlap of the Wannier functions. This generates a repulsive interaction energy, which differs by ΔU between rishons on the same and on different link-sites, thus breaking the $SU(2I+1)$ symmetry. The two-particle transfer is now implemented as a Raman process with a Rabi frequency Ω and some large detuning δ, so that single particle transitions are strongly suppressed, while a two-particle transfer can be an energy conserving process enabled by energy exchange between the atoms (see also Ref. [40]). The resulting coefficient of the $\det U_{xy}$ term is $\gamma = 2\Omega^2\Delta U/\delta^2$, which should be larger than the typical temperature scale in cold atoms experiments.

Initial conditions, loading the optical lattice, and imperfections. We now discuss how to load the lattice with the gauge invariant state illustrated in Fig.1e. This state has local color-singlet pairs of atoms on alternating quark and rishon sites. It is an eigenstate of the Hamiltonian

$$H_{\text{strong}} = m\sum_x s_x \psi^{i\dagger}_x\psi^i_x + \gamma\sum_{\langle xy\rangle}(\det U_{xy} + \text{h.c.}), \quad (6)$$

which is induced by $\widetilde{H}$ in the limit $U \to \infty$. The $\det U_{xy}$ term favors the state $|\uparrow\downarrow, 0\rangle - |0, \uparrow\downarrow\rangle$ where $|\uparrow\downarrow, 0\rangle = c^{\uparrow\dagger}_{x,+}c^{\downarrow\dagger}_{x,+}|0,0\rangle$ and $|0,\uparrow\downarrow\rangle = c^{\uparrow\dagger}_{y,-}c^{\downarrow\dagger}_{y,-}|0,0\rangle$. The preparation of the initial state requires to adiabatically ramp up the optical lattice on an ultracold cloud of atoms which are internally in a 50% mixture of the states $m_I = 3/2, 1/2$. This leads to a band insulator with two atoms of positive nuclear spin on the dark-shaded sites in Fig.1. Then, an on-site Raman two-body process will generate the desired state of Fig.1e after a coherent transfer of the rishon population from the dark- to the light-shaded rishon-sites.

We have investigated the effect of imperfections in the microscopic Hamiltonian on gauge invariance by performing exact diagonalization of small system size $U(2)$ QLMs. The results, extensively discussed in the SI, show how the system preserves gauge invariance even in the presence of relatively large imperfections, of order 10% of the original parameters. Moreover, we emphasize that the low-energy, many-body properties of the system are expected to be robust in the presence of small gauge variant terms (see for instance [41]).

Exact diagonalization results. We have performed exact diagonalization studies of the $(1+1)$D $U(2)$ model with $\mathcal{N} = 1$ rishon per link. Figure 3a shows the splitting between two almost degenerate vacuum states, which decreases exponentially with the system size L, thus indicating $\mathbb{Z}(2)$ χSB. Figure 3b shows the real-time evolution of the chiral order parameter profile $(\overline{\psi}\psi)_x = s_x\langle\psi^{i\dagger}_x\psi^i_x - \frac{N}{2}\rangle$, starting from an initial chirally restored "fireball" embedded in the chirally broken vacuum. This dynamics can be observed by initializing the system in a product state of Mott double wells, and subsequently lower the lattice potential. This mimics the expanding quark-gluon plasma generated in a heavy-ion collision, and can be probed in an experimental setup by just monitoring the time-dependence of the particle density, similarly to Ref. [42].

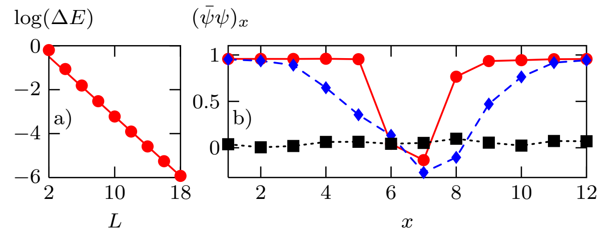

FIG. 3. [Color online] a) Size L dependence of the energy splitting between the lowest energy eigenstates of a $U(2)$ QLM with $m = 0$ and $V = -6t$. b) Real-time evolution of the order parameter profile $(\overline{\psi}\psi)_x(\tau)$ for $L = 12$, mimicking the expansion of a hot quark-gluon plasma. Here, circles (thin line), diamonds (dashed line) and squares (dotted line) correspond to $\tau/t = 0, 1, 10$ respectively.

Conclusions. We have proposed an implementation of a quantum simulator for non-Abelian $U(N)$ and $SU(N)$ gauge theories for staggered fermions with ultracold atoms. The proposal builds on the unique properties of quantum link models with rishons representing the gauge fields: this allows a formulation in terms of a Fermi-Hubbard model, which can be realized with multi-component alkaline-earth atoms in optical lattices, and where atomic physics provides both the control fields and measurement tools for studying the equilibrium and non-equilibrium dynamics and spectroscopy. Extending such investigations towards QCD requires the incorporation of multi-component Dirac fermions with the appropriate chiral symmetries, and of additional link and plaquette terms for electric and magnetic field energies [43].

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Note: after the submission of this manuscript, two related works on SU(2) gauge theories in cold atomic systems have appeared on the arXiv preprint server [44, 45].

[1] K. Rajagopal and F. Wilczek, At the frontier of particle physics, ed. M. Shifman **3** (2000).
[2] P. A. Lee, N. Nagaosa, and X. G. Wen, Rev. Mod. Phys. **78**, 17 (2006).
[3] D. Banerjee, M. Dalmonte, M. Müller, E. Rico, P. Stebler, U.-J. Wiese, and P. Zoller, Phys. Rev. Lett. **109**, 175302 (2012).
[4] E. Zohar, J. Cirac, and B. Reznik, Phys. Rev. Lett. **110**, 055302 (2013).
[5] E. Kapit and E. Mueller, Phys. Rev. A **83**, 033625 (2011).
[6] E. Zohar, J. I. Cirac, and B. Reznik, Phys. Rev. Lett. **109**, 125302 (2012).
[7] L. Tagliacozzo, A. Celi, A. Zamora, and M. Lewenstein, arXiv:1205.0496 (2012).
[8] D. Horn, Phys. Lett. B **100**, 149 (1981).
[9] P. Orland and D. Rohrlich, Nucl. Phys. B **338**, 647 (1990).
[10] S. Chandrasekharan and U.-J. Wiese, Nucl. Phys. B **492**, 455 (1997).
[11] M. A. Cazalilla, A. F. Ho, and M. Ueda, New J. Phys. **11**, 103033 (2009).
[12] M. Hermele, V. Gurarie, and A. M. Rey, Phys. Rev. Lett. **103**, 135301 (2009).
[13] A. V. Gorshkov, M. Hermele, V. Gurarie, C. Xu, P. S. Julienne, J. Ye, P. Zoller, E. Demler, M. D. Lukin, and A. M. Rey, Nat. Phys. **6**, 289 (2010).
[14] T. Fukuhara, Y. Takasu, M. Kumakura, and Y. Takahashi, Phys. Rev. Lett. **98**, 30401 (2007).
[15] B. J. DeSalvo, M. Yan, P. G. Mickelson, Y. N. Martinez de Escobar, and T. C. Killian, Phys. Rev. Lett. **105**, 30402 (2010).
[16] S. Stellmer, M. K. Tey, B. Huang, R. Grimm, and F. Schreck, Phys. Rev. Lett. **103**, 200401 (2009).
[17] S. Stellmer, R. Grimm, and F. Schreck, Phys. Rev. A **84**, 043611 (2011).
[18] S. Sugawa, K. Inaba, S. Taie, R. Yamazaki, M. Yamashita, and Y. Takahashi, Nat. Phys. **7**, 642 (2011).
[19] M. D. Swallows, M. Bishof, Y. Lin, S. Blatt, M. J. Martin, A. M. Rey, and J. Ye, Science **331**, 1043 (2011).
[20] S. Taie, R. Yamazaki, S. Sugawa, and Y. Takahashi, Nat. Phys. **8**, 825 (2012).
[21] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[22] R. Brower, S. Chandrasekharan, and U.-J. Wiese, Phys. Rev. D **60**, 094502 (1999).
[23] R. Brower, S. Chandrasekharan, S. Riederer, and U.-J. Wiese, Nucl. Phys. B **693**, 149 (2004).
[24] see Supplementary Material.
[25] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, Rev. Mod. Phys. **83**, 1523 (2011).
[26] M. Lewenstein, A. Sanpera, and V. Ahufinger, *Ultracold Atoms in Optical Lattices: Simulating Quantum Many-Body Systems* (Oxford University Press, 2012).
[27] H. Lignier, C. Sias, D. Ciampini, Y. Singh, A. Zenesini, O. Morsch, and E. Arimondo, Phys. Rev. Lett. **99**, 220403 (2007).
[28] J. Struck, C. Ölschläger, R. Le Targat, P. Soltan-Panahi, A. Eckardt, M. Lewenstein, P. Windpassinger, and K. Sengstock, Science **333**, 996 (2011).
[29] A. J. Daley, M. M. Boyd, J. Ye, and P. Zoller, Phys. Rev. Lett. **101**, 170504 (2008).
[30] W. Yi, A. J. Daley, G. Pupillo, and P. Zoller, New J. Phys. **10**, 073015 (2008).
[31] W. S. Bakr, J. I. Gillen, A. Peng, S. Fölling, and M. Greiner, Nature **462**, 74 (2009).
[32] C. Weitenberg, M. Endres, J. F. Sherson, M. Cheneau, P. Schauß, T. Fukuhara, I. Bloch, and S. Kuhr, Nature **471**, 319 (2011).
[33] R. Ciuryło, E. Tiesinga, and P. S. Julienne, Phys. Rev. A **71**, 030701 (2005).
[34] P. Naidon and P. S. Julienne, Phys. Rev. A **74**, 062713 (2006).
[35] K. Enomoto, K. Kasa, M. Kitagawa, and Y. Takahashi, Phys. Rev. Lett. **101**, 203201 (2008).
[36] I. Reichenbach, P. S. Julienne, and I. H. Deutsch, Phys. Rev. A **80**, 020701 (2009).
[37] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. **82**, 1225 (2010).
[38] S. Blatt, T. L. Nicholson, B. J. Bloom, J. R. Williams, J. W. Thomsen, P. S. Julienne, and J. Ye, Phys. Rev. Lett. **107**, 73202 (2011).
[39] Notice that a finite interaction of order t at the fermionic site will not affect gauge invariance and the form of the effective Hamiltonian.
[40] C. V. Kraus, M. Dalmonte, M. A. Baranov, A. M. Läuchli, and P. Zoller, arXiv:1302.0701 (2013).
[41] D. Foerster, H. Nielsen, and M. Ninomiya, Phys. Lett. B **94**, 135 (1980).
[42] S. Trotzky, Y. Chen, A. Flesch, I. McCulloch, U. Schollwöck, J. Eisert, and I. Bloch, Nat. Phys. **8**, 325 (2012).
[43] H. P. Büchler, M. Hermele, S. D. Huber, M. P. A. Fisher, and P. Zoller, Phys. Rev. Lett. **95**, 40402 (2005).
[44] L. Tagliacozzo, A. Celi, P. Orland, and M. Lewenstein, arXiv.1211.2704.
[45] E. Zohar, J. I. Cirac, and B. Reznik, arXiv.1211.2241.