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Yuan-Ming Lu and Ziqiang Wang
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Majorana fermions in spin-singlet nodal superconductors with coexisting non-collinear magnetic order

Yuan-Ming Lu\textsuperscript{1,2} and Ziqiang Wang\textsuperscript{3}

\textsuperscript{1}Department of Physics, University of California, Berkeley, CA 94720
\textsuperscript{2}Materials Science Division, Lawrence Berkeley National Laboratories, Berkeley, CA 94720
\textsuperscript{3}Department of Physics, Boston College, Chestnut Hill, Massachusetts 02467

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Realizations of Majorana fermions in solid state materials have attracted great interests recently in connection to topological order and quantum information processing. We propose a novel way to create Majorana fermions in superconductors. We show that an incipient non-collinear magnetic order turns a spin-singlet superconductor with nodes into a topological superconductor with a stable Majorana bound state in the vortex core; at a topologically-stable magnetic point defect; and on the edge. We argue that such an exotic non-Abelian phase can be realized in extended $t$-$J$ models on the triangular and square lattices. It is promising to search for Majorana fermions in correlated electron materials where nodal superconductivity and magnetism are two common caricatures.

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A Majorana fermion is an electrically neutral fermion whose antiparticle is itself [1]. In recent years, Majorana fermions have attracted growing attention in condensed matter physics [2–4]. Specifically, Majorana fermions can be realized as zero-energy bound states in the vortex core or on the edge of certain two-dimensional superconductors. Instead of the usual Bose or Fermi statistics, these vortices obey non-Abelian statistics [5–10] as a manifestation of topological order [11, 12]. Due to this remarkable feature, Majorana bound states (MBSs) can be utilized for topologically-protected qubits in fault-tolerant quantum computation [2, 13, 14]. Several systems have been proposed to realize MBSs, such as even-denominator fractional quantum Hall states [5, 6, 8, 15], $p + i p$ superconductors [8–10] and superfluids [16, 17], superconductor-topological insulator interfaces [18–21], s-wave Rashba superconductor [22–24] and spin-orbit-coupled nodal superconductors [25].

In this work, we present a novel realization of MBSs in spin-singlet superconductors with nodal excitations. We show that when a coexisting non-collinear magnetic order (NCMO) develops with a wavevector connecting two nodes at opposite momenta, there will be one MBS in each vortex core and on the edge of such topological superconductors. Moreover, each stable point defect of the NCMO also hosts a MBS. We demonstrate our proposal with two explicit examples. The first one is a nodal $d + id$ superconductor [26] coexisting with $1 \times 3$ (or $3 \times 3$) coplanar magnetic order on the triangular lattice. We argue that this state is likely to be realized in a doped $t$-$J$ model on the triangular lattice and is relevant for the sodium cobaltate superconductors $Na_xCoO_2 \cdot yH_2O$ near $x = 1/3$ [26, 27]. The second example is a $d_{x^2-y^2}$ superconductor with coexisting $Q = (Q_0, Q_0)$ NCMO on the square lattice which may be realized in a doped $t$-$J_1$-$J_2$-$J_3$ model on the square lattice. Many strongly correlated materials, from high-$T_c$ cuprates to heavy-fermion compounds, exhibits the $d$-wave superconductivity [28, 29] in proximity to [30] or coexisting with [29, 31] magnetic orders. Our findings suggest that Majorana fermions may exist in correlated electron materials with magnetic frustration and nodal superconductivity.

We begin with a general discussion. The low-energy excitations of a nodal superconductor are massless Dirac fermions with linear dispersion. Our basic idea is to create a topological superconductor by adding a proper mass to the nodal fermions. Consider the spin-singlet case with $n$ pairs of isolated nodes located at crystal momenta $\pm q_{\ell}, \ell = 1, \ldots, n$. Expanding around the nodes, the low-energy BCS Hamiltonian describing the quasi-particle excitations has a generic form in the Nambu basis $\Psi_{\ell k} \equiv (c_{q_{\ell}+k, \uparrow} c_{q_{\ell}-k, \downarrow}; c_{q_{\ell}+k, \downarrow} c_{q_{\ell}-k, \uparrow})^T$ for each pair of nodes at opposite momenta:

\[ H_{\text{eff}} = \sum_{\ell, k} \psi_{\ell k}^\dagger h_{\ell k} \psi_{\ell k}, \quad h_{\ell k} = (t_{\ell k} \cdot \vec{v}) \sigma_z, \quad (1) \]

and $\psi_{\ell k} = (\text{Re} \Delta_{q_{\ell}+k} \xi_{q_{\ell}+k} + \text{Im} \Delta_{q_{\ell}+k} \xi_{q_{\ell}+k}) = \sum_{\alpha} k_{\alpha} v_{\alpha}^\ell + \mathcal{O}(|k|^2)$ with $\alpha = x, y$. Here $\Delta_k$ is the pairing gap function and $\xi_k$ the kinetic energy. $\vec{v}$ and $\vec{\sigma}$ are Pauli matrices operating in the particle-hole (Nambu) basis $H_{\ell k} = (k_{1x} + k_{2y}) \sigma_x + (k_{1y} - k_{2x}) \sigma_y$ for each pair of nodes at opposite momenta. Let’s focus on the $\ell$-th pair of nodes at $\pm q_{\ell}$. Clearly, a gap will be generated by adding a generic mass term of the form $\tau_0 (\sigma_x \text{Re} M - \sigma_y \text{Im} M)$ to the Hamiltonian,

\[ H_{\ell k}^U = (k_{1x} + k_{2y}) \sigma_x + \tau_0 (\sigma_x \text{Re} M - \sigma_y \text{Im} M), \quad (2) \]

where $M$ is a complex order parameter. This is nothing but the effective Hamiltonian for proximity induced $s$-wave superconductivity on the surface of a 3D topological insulator [18], with $M$ playing the role of the superconducting (SC) order parameter. The latter is known to
contain a single MBS in the vortex core. In the present context of singlet nodal superconductors, the physical origin of the local order \( M \) turns out to be a non-collinear (coplanar) magnetic order (NCMO) described by

\[
\mathcal{H}_{cp} = \sum_{\mathbf{r}} \left[ M(S^x_{\mathbf{r}} + iS^y_{\mathbf{r}})e^{i\mathbf{q}_0 \cdot \mathbf{r}} + h.c. \right] = \sum_{\ell k} M^\ell c^\dagger_{\ell a + k + \mathbf{q}_0 \uparrow} c_{\ell a + k - \mathbf{q}_0 \downarrow} + h.c., \tag{3}
\]

where \( S^a_{\mathbf{r}} = \sum_{\alpha \beta} \epsilon_{\alpha \beta} c^\dagger_{\alpha \mathbf{r}} c_{\beta \mathbf{r}} / 2, a = x, y, z \) are the spin operators at site \( \mathbf{r} \). When the ordering wavevector \( \mathbf{q}_0 \) connects the nodes at \( \pm \mathbf{q}_0 \), the magnetic scattering generates precisely the mass term in Eq. (2). Since \( SO(3) \) spin-rotational symmetry is completely broken, such a NCMO bears a topologically stable point defect [32] characterized by the nontrivial homotopy \( \pi_1(SO(3)) = \mathbb{Z}_2 \).

The SC vortex in Fu-Kane model [18] maps exactly to the trivial weak-pairing phase [8] by the mass gap, while the odd combination is driven into the topologically non-Abelian pfaffian state by interlayer tunneling [8, 33, 34]. The existence of a single MBS in the quantum Hall system, where the Abelian \((331)\) state can be driven to a non-Abelian state by interlayer tunneling [8, 33, 34]. The existence of a single MBS in the SC vortex core is thus implied by the vortex-boundary correspondence [8]. Note that the existence of a single MBS will not be affected by the other nodal fermions [25]. They are spin-flip scattered either to finite energy away from the Fermi level or, in special cases, to different nodes not connected by pairing; they either remain gapless or are gapped out in the magnetic sector.

We stress that it is crucial to require the magnetic order to be non-collinear: a collinear spin order, such as \( c_{\alpha a \mathbf{k} \uparrow} c_{\beta a \mathbf{k} \downarrow} \), the odd combination is driven into the topologically non-trivial weak-pairing phase [8] by the mass gap, while the even one to the trivial strong-pairing phase. The situation is analogous to a doubled-layer \( \nu = 1/2 \) fractional quantum Hall system, where the Abelian \((331)\) state can be driven to a non-Abelian pfaffian state by interlayer tunneling [8, 33, 34]. The existence of a single MBS in the SC vortex core is thus implied by the vortex-boundary correspondence [8]. Note that the existence of a single MBS will not be affected by the other nodal fermions [25]. They are spin-flip scattered either to finite energy away from the Fermi level or, in special cases, to different nodes not connected by pairing; they either remain gapless or are gapped out in the magnetic sector.

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clockwise-propagating chiral fermions on the edge; each is charge neutral but carries spin $\frac{1}{2}$ [26]. When the FS encloses the six gap nodes ($x < x_c$), $W = 4$ and there are four spin-carrying chiral fermions on the edge.

A NCMO described by $\mathcal{H}_{cp}$ in Eq. (3), with $2q_0$ pointing from $N_i+3$ to $N_i$ (Fig. 1a), produces a mass gap for the nodal fermions as discussed above. The magnetic order corresponds to the $1 \times 3$ coplanar pattern shown in Fig. 1(b). The nodal chiral superconductor is thus turned into a non-Abelian topological superconductor with a winding number $W = +1$. Similar to a spinless $p + i p$ superconductor [8], it supports a single MBS in the vortex core and on the sample edge. To demonstrate the latter, we calculate explicitly the edge spectrum of $\mathcal{H} + \mathcal{H}_{cp}$ on a cylinder with two parallel edges along the $\vec{a}_2$ direction. The solutions of the BdG equations [38] are shown in Fig. 2(a) for $\Delta_2 = 150$ meV and $M = 200$ meV. The bulk excitations are completely gapped since the scattering wave vector $2q_0$ not only connects the Nambu pair $(N_5, N_2)$ but also $(N_3, N_4)$ and $(N_1, N_6)$ in the magnetic sector. There is a single branch of gapless Majorana mode crossing $k = 0$ that is localized at the edges. We also performed direct calculations of the SC vortex and the magnetic defect spectrum on 90 $\times$ 30 periodic lattices in the presence of a vortex-antivortex pair and a pair of stable point defects respectively [38]. The results are shown in Fig. 2(b) and (c). Clearly, a single zero-energy MBS emerges with a density profile localized in the SC vortex core and at the magnetic defect. Note that since the topological superconductor is in the gapped phase, its stability is protected against perturbations that are not strong enough to destroy the gap and create a quantum phase transition into a different state. As a result, the non-Abelian topological phase supporting MBS proposed here is not limited to very particular parameters and will remain stable when, e.g., small variations in doping around $x_c$ cause the FS to deviate from the gap nodes, or a small NN pairing component induced by a subdominant NN exchange $J_1$ causes the gap nodes to shift and the magnetic ordering wave vector not to connect precisely the pair of nodes at opposite momenta.

To see how the $1 \times 3$ NCMO can arise microscopically, we consider the 2nd NN antiferromagnetic Heisenberg model, i.e. the $J_2$ model, on the triangular lattice. The classical ground state is well-known to have $3 \times 3$ NCMO [39, 40]: on each of the three sublattices connected by 2nd-NN bonds the spins exhibit 120 degree coplanar order. There is a large ground state degeneracy due to the relative spin orientations. This $3 \times 3$ NCMO already induces a single MBS crossing $k = 0$ in the edge spectrum [38]. Quantum fluctuations would lift the degeneracy through the order-due-to-disorder mechanism [41–43], and the true quantum ground state has the $1 \times 3$ order. We have carried out a Schwinger-boson large-$S$ expansion study [38] of the spin-$S$ Heisenberg $J_2$ model and found that when $S$ is larger than a critical value $S_c \approx 0.17$ (such as for spin-$1/2$), the system develops the $1 \times 3$ NCMO shown in Fig. 1(b). This suggests that if the residual interactions between the nodal fermions are dominated by the 2nd NN Heisenberg exchange $J_2$, the $1 \times 3$ NCMO is likely to develop with $2q_0$ connecting the gap nodes at opposite momenta as shown in Fig. 1. More intriguingly, to the extent that $J_2$ would favor a 2nd NN $d + id$ resonance valence bond pairing state, it is likely that both the nodal chiral superconductor and the NCMO can emerge from the same exchange interaction in a doped $t$-$J_2$ model.

We turn to the 2nd example of a more familiar nodal $d_{x^2-y^2}$ superconductor on the square lattice described by the same pairing Hamiltonian $\mathcal{H}$ in Eq. (4), but with the dispersion $\xi_k = -2t(k_1 + k_2) + \cos(k_1 - k_2) - \mu$ for NN hopping $t$, and the NN $d_{x^2-y^2}$ pairing gap function $\Delta_1 = 2\Delta_1[\cos(k_1 + k_2) - \cos(k_1 - k_2)]$. The momentum is defined as $\mathbf{k} = k_1\vec{b}_1 + k_2\vec{b}_2$ as shown in Fig. 3. The four nodal points are located at $N_{1,3}$: $(k_1 = 0, k_2 = \pm q_0)$.
and \( N_{2.4} : (k_1 = \pm q_0, k_2 = 0) \) with \( q_0 = \arccos(-\frac{\mu}{t}) \). A NCMO described by \( \mathcal{H}_{cr} \) in Eq. (3) with ordering momentum \( Q_0 = (Q_0, Q_0) = 2q_0 = 2q_0b_2 \) gaps out the \( (c_{N_i,1}, c_{N_{i+1,1}}^\dagger) \) branch and creates a single MBS. Fig. 3 shows a specific example with \( \mu = -2\sqrt{2}t \) and \( q_0 = \pi/4 \), together with the spin configuration. In this case, the commensurate magnetic order cannot gap out all nodes since the Hamiltonian is still invariant under time reversal followed by a lattice translation [44]. As shown in Fig. 3(a), the NCMO turns the original four spin-degenerate nodes (black circles) into 6 non-degenerate ones (red diamonds). The vanished pair of nodes is gapped out by the magnetic mass (3) and enters the weak-pairing phase. The calculated edge spectra along (1,1) direction is plotted in Fig. 4 near the magnetic zone boundary, showing the zero energy MBS localized on the parallel edges. Similar results are obtained at other commensurate values of \( q_0 \) such as \( q_0 = \pi/3 \). Since the gapless bulk excitations are located at different momenta, the MBS near \( k_2 = \pi \) is expected to be stable against impurities and the mixing with bulk excitations [25]. For a generic doping, \( q_0 \) is incommensurate with the lattice and a corresponding incommensurate NCMO could produce a full gap for bulk excitations.

A remarkable feature seen in Fig. 4 is that the Majorana mode on the edge is dispersionless, i.e. it is localized and does not possess a chirality. This boundary zero-energy flat band, which begins and terminates at the reconstructed nodes of bulk excitations, is a direct consequence of the nontrivial \( \mathbb{Z}_2 \) winding number (topological index of class D in \( d = 1 \) [45]) of the momentum-space Hamiltonian around the nodes [46] in FIG. 3(a). The Majorana flat band is analogous to the Fermi arc on the 2D surface of 3D time-reversal symmetry breaking Weyl semimetals proposed for pyrochlore iridates [47]. Nevertheless, the time reversal symmetry breaking by the magnetic order (3) can induce a small imaginary part in \( \Delta_1 \), which would generate a full gap for bulk excitation and a single MBS on the edge dispersing across \( k_2 = \pi \) with a well-defined chirality [38].

It is possible to realize such a NCMO in the Heisenberg \( J_1-J_2-J_3 \) model on the square lattice. The classical ground state has NCMO with momenta \( Q_0 = Q_0b_{1,2} \) where \( \cos(Q_0) = -J_1/(2J_2 + 4J_3) \) for \( 4J_3 + 2J_2 \geq J_1 \) and \( 3 \geq J_2/2 \) [48, 49]. There are numerical evidence that the latter survives in the quantum \( S = 1/2 \) Heisenberg \( J_1-J_2-J_3 \) model in a wide parameter range [50, 51]. We thus expect that such non-Abelian magnetic \( d \)-wave superconductors may be realized in certain parameter regime of the doped \( t-J_1-J_2-J_3 \) model.

In summary, we proposed a new type of non-Abelian topological superconductors. They emerge when spin-singlet superconductors with isolated nodes coexist with NCMO at the wavevector connecting the nodes at opposite momenta. Majorana fermions arise in the vortex core and on the edge of such magnetic superconductors. Remarkably, each stable point defect of the non-collinear magnets also hosts a single MBS. Since magnetism and unconventional superconductivity are common features of strong correlation, our findings suggest searching for the MBS in correlated materials with magnetic frustration and nodal superconductivity.

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[3] M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045
(2010).