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Lifshitz Gravity for Lifshitz Holography

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and work only in the low-energy bulk gravity approximation. This is equivalent to the large N limit in the dual field theory. In this low-energy limit, HL gravity is dominated by the most relevant terms compatible with the gauge symmetries, and its effective action is

$$S = \frac{1}{2\kappa^2} \int dt d^D x dr \sqrt{g} N \left\{ K_{ab} K^{ab} - \lambda K^2 + \beta(R - 2\Lambda) + \frac{\alpha^2 \nabla_a N \nabla^a N}{2N^2} \right\}. \quad (4)$$

(Here $K_{ab} = \frac{1}{2N}(\partial_t g_{ab} - \nabla_a N_b - \nabla_b N_a)$ is the extrinsic curvature of the foliation, R the scalar curvature of g_{ab} , and $K = g^{ab} K_{ab}$.) The novelty compared to pure Einstein gravity is in the three couplings β , λ and α , which in Einstein gravity are fixed to $\lambda = \beta = 1$ and $\alpha = 0$. Note that turning on the α coupling is important for the consistency of nonprojectable HL gravity [6, 7]: Taking the naive $\alpha \rightarrow 0$ limit in (4) would lead to a non-closure of the constraint algebra.

When $\Lambda = 0$, the flat spacetime \mathbf{R}^{D+2} is a solution of (4). The propagating graviton modes consist of the transverse-traceless tensor polarizations with dispersion relation $\omega^2 = \beta k^2$ (here $k \equiv \sqrt{k_a k_a}$ is the magnitude of the spatial momentum), plus an extra scalar graviton polarization, with dispersion

$$\omega^2 = \frac{\beta(1-\lambda)}{[1-(D+1)\lambda]} \left[1 + D \left(\frac{2\beta}{\alpha^2} - 1 \right) \right] k^2. \quad (5)$$

The requirement of stability and perturbative unitarity around flat spacetime constrains the couplings to be in the range $\beta > 0$,

$$\alpha^2 \leq \frac{2\beta D}{D-1}, \quad (6)$$

and

$$\lambda \geq 1 \quad \text{or} \quad \lambda \leq 1/(D+1). \quad (7)$$

Turning on the cosmological constant $\Lambda < 0$, we find that the Lifshitz geometry (2) is a vacuum solution of HL gravity with low-energy effective action (4), if

$$\Lambda = -\frac{(D+z-1)(D+z)}{2} \quad (8)$$

and

$$\alpha^2 = \frac{2\beta(z-1)}{z}. \quad (9)$$

This simple interpretation of Lifshitz spacetimes as vacuum solutions of HL gravity suggests that the latter is the natural minimal holographic model of holographic duality for Lifshitz-type field theories.

Further evidence for the universality of this minimal model of Lifshitz holography comes from the analysis of

anisotropic Weyl anomalies in holographic renormalization, initiated in [8] (and also later in [9]). Just like their relativistic counterparts, anisotropic Weyl anomalies contain a lot of universal information about the system, and serve as useful probes of the duality. Consider again some Lifshitz-type QFT on \mathbf{R}^{D+1} with coordinates (t, x^i) and a general background metric

$$ds^2 = -\hat{N}^2 dt^2 + \hat{g}_{ij}(dx^i + \hat{N}^i dt)(dx^j + \hat{N}^j dt).$$

It is useful to think of this theory as residing at the spacetime boundary $r \rightarrow \infty$ of the $D+2$ dimensional asymptotically Lifshitz spacetime, with \hat{g}_{ij} , \hat{N}_i and \hat{N} being the components of the appropriately defined (anisotropic conformal class of the) boundary metric [8]. Generally, we will put hats on quantities defined at the boundary, to distinguish them from their bulk counterparts. Define anisotropic Weyl transformations, generated by a spacetime dependent $\sigma(t, x^i)$:

$$\delta \hat{g}_{ij} = 2\sigma \hat{g}_{ij}, \quad \delta \hat{N}_i = 2\sigma \hat{N}_i, \quad \delta \hat{N} = z\sigma \hat{N}. \quad (10)$$

These represent a local generalization of the rigid scaling transformations (1). A QFT which is classically invariant under (10) can develop an anisotropic Weyl anomaly at the quantum level, with the effective action transforming as (see Appendix C of [8] for details):

$$\delta S_{\text{eff}}[\hat{g}_{ij}, \hat{N}_j, \hat{N}] = \int dt d^D x \sqrt{\hat{g}} \hat{N} \sigma(t, x^i) \mathcal{A}(t, x^i).$$

The independent terms that can appear in $\mathcal{A}(t, x^i)$ are local functionals of the metric, invariant under foliation preserving diffeomorphisms. They can be classified by solving a cohomological problem [8] designed to automatically incorporate the Wess-Zumino consistency conditions on the anomaly. However, the multiplicative coefficients with which these terms contribute to the anomaly (and which we will refer to as ‘‘central charges’’) must be calculated for a given theory on a case-by-case basis. Perhaps the simplest nontrivial case is $D = 2$ and $z = 2$. In this case, the anomaly is [8]

$$\mathcal{A} = c_K \left(\hat{K}_{ij} \hat{K}^{ij} - \frac{1}{2} \hat{K}^2 \right) + c_V \left(\hat{R} - \frac{\hat{\nabla}^i \hat{N} \hat{\nabla}_i \hat{N}}{\hat{N}^2} + \frac{\hat{\Delta} \hat{N}}{\hat{N}} \right)^2, \quad (11)$$

with two independent central charges, c_K and c_V . (Here \hat{K}_{ij} , $\hat{\nabla}_i$ and \hat{R} are the extrinsic curvature, connection and the scalar curvature constructed from \hat{g}_{ij}). As noted in [8], \mathcal{A} in (11) takes the form of the Lagrangian for $z = 2$ conformal HL gravity [2, 3] in $2+1$ dimensions. Moreover, while the first term in (11) satisfies the so-called ‘‘detailed balance condition,’’ (*i.e.*, it is related to the square of the variation of another local functional W , see [2, 3]), the other does not.

For QFTs with holographic gravity duals, we can calculate the anomaly by performing holographic renormalization of the bulk theory [10–12] (see [13–15] for reviews). The relativistic Weyl anomaly was calculated

this way in [10]. Holographic renormalization for the relativistic bulk theory (3) in asymptotically Lifshitz spacetimes was developed and applied to the anisotropic Weyl anomalies in [8], following the earlier work of [16–19]. This procedure relies substantially on the notion of anisotropic conformal infinity developed in [20]. In the low-energy gravity approximation, $S_{\text{eff}}[\hat{g}_{ij}, \hat{N}_i, \hat{N}]$ is calculated by evaluating the on-shell gravity action with the appropriate fall-off conditions on the metric field as $r \rightarrow \infty$. This on-shell action is divergent due to infinite volume, and needs to be renormalized. We regulate it by cutting r off at $r_\epsilon = 1/\epsilon$, and expand the on-shell action asymptotically to reveal the structure of its divergences. For the special case of $D = z$, this expansion gives [8] (modulo terms that vanish as $\epsilon \rightarrow 0$):

$$\int dt d^D x \sqrt{\hat{g}} \hat{N} \left\{ \sum_{n=0}^{D-1} \frac{\mathcal{L}^{(2n)}}{\epsilon^{2(D-n)}} - \tilde{\mathcal{L}}^{(2D)} \log \epsilon + \mathcal{L}^{(2D)} \right\}.$$

(The $\log \epsilon$ term is present only for special values of D and z [8], including the case $D = z = 2$ of interest here.) The divergent terms are then cancelled by local counterterms, and $S_{\text{eff}} = \int dt d^D x \sqrt{\hat{g}} \hat{N} \mathcal{L}^{(2D)}$.

To calculate these divergent terms, we use the Hamiltonian form of holographic renormalization, as developed for relativistic AdS/CFT in [21, 22] and extended to the asymptotically Lifshitz case in [8] following [17]. In this formulation, the on-shell bulk action is determined as a functional of the boundary metric because it satisfies the Hamilton-Jacobi (HJ) equation for the radial evolution along r . The operator of radial evolution $\delta_{\mathcal{D}}$ is given by the generator of anisotropic dilatations on the boundary, and the HJ equation yields a recursive relation between the counterterms $\mathcal{L}^{(m)}$ of adjacent scaling dimensions m . One of these recursive relations implies [8]

$$\delta_{\mathcal{D}} \mathcal{L}^{(2D)} = -2D \mathcal{L}^{(2D)} + \tilde{\mathcal{L}}^{(2D)}. \quad (12)$$

Interpreted from the boundary point of view, this means that when $\tilde{\mathcal{L}}^{(2D)} \neq 0$, S_{eff} scales anomalously under the $z = D$ anisotropic Weyl transformations, and $\tilde{\mathcal{L}}^{(2D)}$ is the anisotropic Weyl anomaly.

For the special case of the relativistic model (3) with $D = z = 2$, the divergent terms were calculated in [8], where we obtained for the anisotropic Weyl anomaly

$$\mathcal{A} \equiv \tilde{\mathcal{L}}^{(4)} = \frac{1}{16\pi G_N} \left(\hat{K}_{ij} \hat{K}^{ij} - \frac{1}{2} \hat{K}^2 \right). \quad (13)$$

Thus, the anomaly in this relativistic model turns out to have $c_V = 0$, or in other words, satisfies the detailed balance condition. Why is it so? The conclusive answer was found in [8]: The relation implying that the anomaly in the relativistic model (3) should satisfy detailed balance follows from the holographic recursion relation between the counterterms $\mathcal{L}^{(2)}$ and $\tilde{\mathcal{L}}^{(4)}$, with $\mathcal{L}^{(2)}$ effectively playing the role of the local functional W (see [8] for details).

Looking for holographic duals of more general QFTs with both central charges independently nonzero is an interesting challenge. Before we embark on this pursuit, we should first check that QFTs whose central charges c_K and c_V are both nonzero indeed exist. Examples of strongly coupled Lifshitz field theories are very scarce to say the least, but our point can be made by considering the theory of the free $z = 2$ Lifshitz scalar Φ . When Φ is minimally coupled to background HL gravity,

$$S_\Phi = \int dt d^2 x \sqrt{\hat{g}} \left\{ \frac{1}{\hat{N}} \left(\partial_t \Phi - \hat{N}^i \hat{\nabla}_i \Phi \right)^2 - \hat{N} \left(\hat{\Delta} \Phi \right)^2 \right\},$$

this theory is classically invariant under (10) (with $\delta\Phi = 0$), but develops an anisotropic Weyl anomaly at the quantum level. This anomaly was calculated in [9], and it turns out to have $c_V = 0$. One could perhaps speculate that $c_V = 0$ might be a universal property of all consistent QFTs, hence eliminating the need for finding gravity duals with $c_V \neq 0$. A simple counterexample comes from coupling Φ to background gravity non-minimally, adding

$$- e^2 \int dt d^D x \sqrt{\hat{g}} \hat{N} \left\{ \hat{R} - \frac{\nabla^i \hat{N} \nabla_i \hat{N}}{\hat{N}^2} + \frac{\hat{\Delta} \hat{N}}{\hat{N}} \right\}^2 \Phi^2$$

to S_Φ . Even with this non-minimal coupling, this theory stays classically invariant under the anisotropic Weyl transformations (again with $\delta\Phi = 0$), and develops a quantum anomaly. We calculated this anisotropic Weyl anomaly using the ζ -function regularization, and found $c_K = 1/(32\pi)$ and $c_V = -e^2/(8\pi)$.

Having demonstrated the existence of QFTs with $c_V \neq 0$, we can now ask how to reproduce this second central charge in a holographic gravity dual. One could look for relativistic models more complicated than (3). Instead, we will show that minimal HL gravity already accounts for both of the independent central charges c_K and c_V in the anisotropic Weyl anomaly. In order to show that, we have performed holographic renormalization of Lifshitz spacetimes in pure HL gravity. Since the technicalities are quite involved (as they were in the relativistic model (3) studied in [8]), here we only present our main results; all technical details will appear in [23].

We find modified recursion relations for the divergent terms in the regulated action. In the special case $D = z = 2$, we solved these recursion relations and found that the logarithmic term $\tilde{\mathcal{L}}^{(4)}$ is equal to

$$\frac{1}{2\kappa^2} \left(\hat{K}_{ij} \hat{K}^{ij} - \frac{1}{2} \hat{K}^2 \right) + \frac{\beta}{48\kappa^2} \left(\hat{R} - \frac{\hat{\nabla}^i \hat{N} \hat{\nabla}_i \hat{N}}{\hat{N}^2} + \frac{\hat{\Delta} \hat{N}}{\hat{N}} \right)^2.$$

This is the anisotropic Weyl anomaly in our minimal model of Lifshitz holography with vacuum HL gravity. It is indeed of the most general form, with the two independent central charges given in terms of two low-energy couplings in minimal HL gravity: $c_K = 1/(2\kappa^2)$ and $c_V = \beta/(48\kappa^2)$.

The remaining coupling λ does not appear in the anomaly, but it still plays an important physical role. Just as in the case of flat spacetime, λ enters into the dispersion relation of the extra polarization of the graviton in the bulk. For example in the radial gauge $g_{ir} = 0$, $g_{rr} = 1/r^2$, the extra graviton mode ϕ is found as the linearized fluctuation of the radial component of the shift vector, $N_r = \phi/r$. Returning to the case of general D and z , the linearized equations of motion imply the asymptotic behavior near infinity $\phi(r) \sim r^{D\pm}$, with

$$D_{\pm} = \frac{1}{2} \left\{ z - D \pm \sqrt{(z + D)^2 + \frac{4D(1 - z)}{1 - \lambda}} \right\}. \quad (14)$$

Standard rules of holographic duality will map D_{\pm} to the scaling dimensions Δ_{\pm} of the operator dual to the extra graviton. Unitarity of the dual field theory requires that the scaling dimensions be real, implying (for $z > 1$)

$$\lambda \geq 1 \quad \text{or} \quad \lambda \leq \lambda_U \equiv \frac{(D - z)^2 + 4D}{(D + z)^2}. \quad (15)$$

This unitarity bound represents an intriguing analog of the Breitenlohner-Freedman bound familiar from relativistic holography: In HL gravity, the unitarity bound (15) allows the coupling λ to dip into the region between $1/(D + 1)$ and 1, which according to (7) would be forbidden around flat spacetime. In the particularly interesting case of $D = z$, we get $\lambda_U = 1/D$, which opens up the previously forbidden regime $1/(D + 1) \leq \lambda \leq 1/D$.

Now that we have seen that HL gravity provides candidate holographic duals for QFTs with anisotropic Lifshitz scaling, is it possible to apply HL gravity also to QFTs with isotropic $z = 1$ scaling? Interestingly, the limit $z \rightarrow 1$ corresponds to $\alpha \rightarrow 0$, the “unhealthy reduction” [6] of nonprojectable HL gravity, and may therefore be difficult to make sense of. This is perhaps to be expected: $z = 1$ QFTs with such gravity duals would likely exhibit isotropic dilatation symmetry without full relativistic conformal symmetry, a phenomenon whose examples are few and far between. Further study of our holographic duality in the $\alpha \rightarrow 0$ limit may shed new light on this rare class of QFTs.

Finally, throughout this paper we have used the effective low-energy limit of HL gravity, dominated by the terms of the lowest dimension in the action. We have been agnostic about how the model is completed at high energies. This completion may come from additional degrees of freedom, perhaps via an embedding into string theory; or it can be via a self-completion of HL gravity, due to highly anisotropic scaling at short distances. This latter possibility would be particularly interesting, as it could open a new door away from the large N limit and small bulk curvature. As this paper was being finalized, complementary results about another form of nonrelativistic holography with HL gravity were presented in

[24, 25]. Our results, and those of [24, 25], thus provide further evidence for the picture proposed originally in [8], that the natural arena for nonrelativistic holography is nonrelativistic HL gravity. It remains to be seen whether – as suggested in [8] – the nonrelativistic field theories whose holographic duals happen to be relativistic indeed represent only a minority among all theories with gravity duals.

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