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Entanglement and Particle Identity: A Unifying Approach

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It has been known for some years that entanglement entropy obtained from partial trace does not provide the correct entanglement measure when applied to systems of identical particles. Several criteria have been proposed that have the drawback of being different according to whether one is dealing with fermions, bosons or distinguishable particles. In this paper, we give a precise and mathematically natural answer to this problem. Our approach is based on the use of the more general idea of restriction of states to subalgebras. It leads to a novel approach to entanglement, suitable to be used in general quantum systems and specially in systems of identical particles. This settles some recent controversy regarding entanglement for identical particles. The prospects for applications of our criteria are wide-ranging, from spin chains in condensed matter to entropy of black holes.

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INTRODUCTION

The study of subsystems of a quantum system is of paramount importance in many branches of physics. In quantum information it enters in the analysis of local operations performed by different parties of a multipartite system. In statistical physics it enters in the very definition of the different ensembles since this involves considering a given physical system as embedded in a bigger one. In the physics of black holes, the distinction between accessible and inaccessible regions of space-time plays a crucial role for the study of black hole entropy. Indeed as pointed out in [1], the coupling from outside to inside the horizon is very strong, while the reverse coupling is nonexistent! In all these situations partial trace is the preferred tool to extract physical properties of the given subsystems. Nevertheless, it is well-known that in some cases of great physical interest like systems of identical particles the use of partial trace leads to contradictory results.

In this Letter we provide a resolution of such contradictions which turns out to be of general application. We show that, by treating observables and states on an equal footing, a generalized notion of entanglement emerges. A relevant consequence is that the entanglement measure that naturally arises in this algebraic approach is shown to be easily computed. Our approach thus opens up a wide range of applications, from condensed matter systems, like spin chains and anyonic models, to black hole physics.

For bipartite systems contradictory results due to partial trace are explicitly seen to appear in the computation of entanglement measure for identical particles systems. In spite of the numerous efforts to achieve a satisfactory understanding of entanglement for systems of identical particles, there is no general agreement on the appropriate generalization of concepts valid for non-identical constituents [2–8]. That is because many concepts are usually only discussed in the context of quantum systems for which the Hilbert space $\mathcal{H}$ is a simple tensor product with no additional structure like, for example, $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. In this case, the partial trace $\rho_A = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$ for $|\psi\rangle \in \mathcal{H}$ to obtain the reduced density matrix has a good physical meaning: it corresponds to observations only on the subsystem A.

In contrast, the Hilbert space of a system of N identical bosons (fermions) is given by the symmetric (antisymmetric) N-fold tensor product of the single-particle spaces. The consequence is that any multi-particle state contains *intrinsic* correlations between subsystems due to quantum indistinguishability. This, in turn, forces a departure from the straightforward application of entanglement-related concepts like singular value decomposition (SVD), Schmidt rank or entanglement entropy.

We propose here an approach to the study of entanglement where the notion of *partial trace* is replaced by the more appropriate notion of *restriction of a state to a subalgebra*[9]. This approach is based on the well established GNS construction[10]. It allows us to meaningfully treat entanglement of identical and non-identical particles on an equal footing, without the need to resort to different criteria according to the case under study.

The usefulness of our approach will be displayed in three explicit simple examples (for more examples see [11]). In particular we obtain a vanishing von Neumann entropy of a fermionic or a bosonic state containing the least possible amount of correlations. We believe that this settles an issue that has caused a lot of confusion regarding the use of von Neumann entropy as a measure of entanglement for identical particles [7, 12, 13].

THE GNS CONSTRUCTION

A general quantum system is usually described in terms of a Hilbert space $\mathcal{H}$ and linear operators acting thereon. Physical *observables* correspond to self-adjoint operators ($\mathcal{O} \equiv \mathcal{O}^\dagger : \mathcal{H} \to \mathcal{H}$). The probabilistic character of the theory is based on the notion of *state*, from which probabilities and expectation values can be computed. Generically, a state is described in terms of a *density matrix* $\rho : \mathcal{H} \to \mathcal{H}$, a linear map satisfying $\mathrm{Tr}\,\rho = 1$ (normalization), $\rho^\dagger = \rho$ (self-adjointness) and $\rho \geq 0$ (positivity). For *pure* states, the additional condition $\rho^2 = \rho$ is required, so that ρ is of the form $|\psi\rangle\langle\psi|$ for some normalized vector $|\psi\rangle \in \mathcal{H}$.

Since the expectation value of an observable $\mathcal{O}$ is defined by $\langle\mathcal{O}\rangle_\rho = \mathrm{Tr}(\rho\,\mathcal{O})$, we can equivalently regard ρ as a *linear functional* $\mathcal{O} \mapsto \langle\mathcal{O}\rangle_\rho$ from the space of operators to $\mathbb{C}$. Moreover, since the space of all (bounded) operators on $\mathcal{H}$ forms an algebra $\mathcal{L}(\mathcal{H})$, it is possible to give a formulation of quantum physics which does not *a priori* make use of Hilbert spaces. Such a formulation was initially envisaged by von Neumann. The formulation due to Gel'fand and Naimark and further developed by Segal (GNS construction) led to the notion of an "abstract algebra of physical observables", or C^*-algebra. This construction (explained below) has played a very important role in quantum field theory [10] and statistical mechanics [14]. We propose to show that this approach is also very well-suited to deal with the problem described in the introduction.

We thus consider an abstract algebra $\mathcal{A}$ (playing the role of $\mathcal{L}(\mathcal{H})$ above) that represents the physical observables. Since these observables are (not yet) acting on any Hilbert space, an abstract notion for the adjoint of an operator is required. This is provided by an operation ("involution") $\alpha \mapsto \alpha^*$. The algebra is assumed to contain an identity $\mathbb{1}_\mathcal{A}$ and to be closed under products, linear combinations and under the involution. In this context, a *state* is defined as a linear functional $\omega : \mathcal{A} \to \mathbb{C}$. Again, since there is no Hilbert space, no density matrix appears at this stage. But from the interpretation of $\omega(\alpha) \equiv \langle\alpha\rangle_\omega$ as the expectation value of α, the conditions of normalization $\omega(\mathbb{1}_\mathcal{A}) = 1$, reality $\omega(\alpha^*) = \overline{\omega(\alpha)}$ and positivity $\omega(\alpha^*\alpha) \geq 0$ (for any $\alpha \in \mathcal{A}$) are physically motivated properties that any state ω must, by definition, satisfy.

Given a quantum system defined by an algebra $\mathcal{A}$ and a state ω, how do we recover the usual Hilbert space on which the algebra elements act as linear operators? Since $\mathcal{A}$ is an algebra, it is in particular a vector space, denoted here as $\hat{\mathcal{A}}$. Elements $\alpha \in \mathcal{A}$ regarded as elements of the vector space $\hat{\mathcal{A}}$ are written as $|\alpha\rangle$. Then, $\beta \in \mathcal{A}$ will act on $|\alpha\rangle \in \hat{\mathcal{A}}$ as a linear operator by $\beta|\alpha\rangle := |\beta\alpha\rangle$. A similar construction occurs when we study the *regular representation* of a group through its action on its group algebra [15].

In order for the vector space $\hat{\mathcal{A}}$ to become a Hilbert space, an inner product is required. If we set $\langle\alpha|\beta\rangle = \omega(\alpha^*\beta)$, we obtain almost all properties of an inner product. In fact, reality and positivity can be used to show that $\langle\beta|\alpha\rangle = \overline{\langle\alpha|\beta\rangle}$ and also that $\langle\alpha|\alpha\rangle \geq 0$. But it can happen that $\langle\alpha|\alpha\rangle = 0$ for some non-zero elements α. That is, there could be a null space $\widehat{\mathcal{N}}_\omega$ of zero norm vectors: $\widehat{\mathcal{N}}_\omega = \{|\alpha\rangle \in \hat{\mathcal{A}} \,|\, \omega(\alpha^*\alpha) = 0\}$. The solution to this problem is obtained by considering the *quotient vector space* $\hat{\mathcal{A}}/\widehat{\mathcal{N}}_\omega$. Its elements are *equivalence classes* $|[\alpha]\rangle$, with $|[\alpha]\rangle$ equivalent to $|[\beta]\rangle$ precisely when $\alpha - \beta \in \widehat{\mathcal{N}}_\omega$. In particular, if $\alpha \in \widehat{\mathcal{N}}_\omega$, then $|[\alpha]\rangle = 0$. The space $\hat{\mathcal{A}}/\widehat{\mathcal{N}}_\omega$ has now a well-defined scalar product given by

$$\langle[\alpha]|[\beta]\rangle = \omega(\alpha^*\beta), \tag{1}$$

independently of the choice of α from $[\alpha]$ and with no non-zero vectors of zero norm. Its closure is the GNS Hilbert space $\mathcal{H}_\omega$. In this way, one obtains a representation π_ω of $\mathcal{A}$ on $\mathcal{H}_\omega$ by linear operators [10, 16]: $\pi_\omega(\alpha)|[\beta]\rangle = |[\alpha\beta]\rangle$.

Partial Trace as Restriction - Consider a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with a density matrix ρ. The description of $\mathcal{H}_A$ as a *subsystem* involves the reduced density matrix ρ_A, obtained through partial tracing over B. Using the language of algebras and states, we observe that the algebra corresponding to the joint system AB is given by $\mathcal{A} = \mathcal{L}(\mathcal{H})$. Expectation values are computed using the state ω_ρ induced by ρ, $\langle\mathcal{O}\rangle_\rho \equiv \omega_\rho(\mathcal{O}) \equiv \mathrm{Tr}_\mathcal{H}(\rho\,\mathcal{O})$. Corresponding to subsystem A, we can consider the subalgebra $\mathcal{A}_0$ of "local" operators of the form $K \otimes \mathbb{1}_B$, for K an observable on $\mathcal{H}_A$. We can then define a state $\omega_{\rho,0} : \mathcal{A}_0 \to \mathbb{C}$ which is the *restriction* $\omega_\rho|_{\mathcal{A}_0}$ of ω_ρ to $\mathcal{A}_0$ defined by $\omega_{\rho,0}(\alpha) = \omega_\rho(\alpha)$ if $\alpha \in \mathcal{A}_0$.

Now we observe that the reduced density matrix ρ_A, obtained by partial tracing, gives rise to a state on subsystem A that is precisely the restriction of ω_ρ to $\mathcal{A}_0$:

$$\omega_{\rho,0}(K \otimes \mathbb{1}_B) \equiv \mathrm{Tr}_{\mathcal{H}_A}(\rho_A\,K). \tag{2}$$

Hence, partial trace and restriction give the same answer in this case. The importance of this observation lies in the fact that when $\mathcal{H}$ is not of the form of a 'simple tensor product', partial trace is not a suitable operation. In contrast, if the system is described in terms of a state ω_ρ on an algebra $\mathcal{A}$, it is still sensible to describe a subsystem in terms of a corresponding subalgebra $\mathcal{A}_0$ and of the restriction $\omega_{\rho,0}$ of ω_ρ to $\mathcal{A}_0$. The GNS theory is well-suited for the study of $\omega_{\rho,0}$ for general algebras $\mathcal{A}_0 \subseteq \mathcal{A}$.

von Neumann Entropy - The representation π_ω is in general reducible. This means that $\mathcal{H}_\omega$ can be decomposed into a direct sum of irreducible spaces: $\mathcal{H}_\omega = \bigoplus_i \mathcal{H}_i$, where $\pi_\omega(\alpha)\mathcal{H}_i \subseteq \mathcal{H}_i$ for all $\alpha \in \mathcal{A}$. Let $P_i : \mathcal{H}_\omega \to \mathcal{H}_i$ be the corresponding orthogonal projectors. These projectors can be used to construct a density matrix ρ_ω on the GNS space $\mathcal{H}_\omega$ that yields the same expectation values as the original state ω. The von Neumann entropy of ρ_ω can then be evaluated in the standard way. The construction of ρ_ω goes as follows.

First, we observe that the identity $\mathbb{1}_\mathcal{A}$ of $\mathcal{A}$ satisfies $\mathbb{1}_\mathcal{A}\alpha = \alpha$ for all $\alpha \in \mathcal{A}$, as well as $\mathbb{1}_\mathcal{A}^* = \mathbb{1}_\mathcal{A}$. This, together with (1), implies $\omega(\alpha) = \langle[\mathbb{1}_\mathcal{A}]|[\alpha]\rangle$. Since the linear operator $\pi_\omega(\alpha)$ is defined by $\pi_\omega(\alpha)|[\beta]\rangle = |[\alpha\beta]\rangle$, we know that $|[\alpha]\rangle = \pi_\omega(\alpha)|[\mathbb{1}_\mathcal{A}]\rangle$. It follows that $\omega(\alpha) = \langle[\mathbb{1}_\mathcal{A}]|\pi_\omega(\alpha)|[\mathbb{1}_\mathcal{A}]\rangle$. Using $|[\mathbb{1}_\mathcal{A}]\rangle = \sum_i P_i|[\mathbb{1}_\mathcal{A}]\rangle$, $\pi_\omega(\alpha) = \sum_i P_i\pi_\omega(\alpha)P_i$ and from the orthogonality of the projectors, one obtains $\omega(\alpha) = \mathrm{Tr}_{\mathcal{H}_\omega}(\rho_\omega\,\pi_\omega(\alpha))$, where $\rho_\omega = \sum_i P_i\,|[\mathbb{1}_\mathcal{A}]\rangle\langle[\mathbb{1}_\mathcal{A}]|P_i$. The von Neumann entropy of ρ_ω is then given by $S(\rho_\omega) = -\sum_i \mu_i \log_2 \mu_i$, where $\mu_i = \|P_i\,|[\mathbb{1}_\mathcal{A}]\rangle\|^2$.

The crucial fact is that ω is *pure* if and only if the representation π_ω is *irreducible*. In particular, the von Neumann entropy of ω, $S(\omega) \equiv S(\rho_\omega)$, is zero if and only if $\mathcal{H}_\omega$ is irreducible. *This property depends on* both *the algebra $\mathcal{A}$ and the state ω.*

Consider now a subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ of $\mathcal{A}$. Let ω_0 denote the *restriction* to $\mathcal{A}_0$ of a pure state ω on $\mathcal{A}$ [9]. We can apply the GNS construction to the pair $(\mathcal{A}_0, \omega_0)$ and use the von Neumann entropy of ω_0 to study the entanglement emergent from restriction.

Bipartite Entanglement from GNS - We now illustrate how to apply the GNS construction to entanglement. Consider $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \equiv \mathbb{C}^2 \otimes \mathbb{C}^2$. The algebra of the full system is $\mathcal{A} = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. Let us consider the normalized state vector $(0 < \lambda < 1)$: $|\psi_\lambda\rangle = \sqrt{\lambda}|+,-\rangle + \sqrt{(1-\lambda)}|-,+\rangle$, with corresponding state ω on the algebra $\mathcal{A}$: $\omega(\mathcal{O}) = \langle\psi_\lambda|\mathcal{O}|\psi_\lambda\rangle$, $\mathcal{O} \in \mathcal{A}$.

Entanglement of $|\psi_\lambda\rangle$ is to be understood in terms of correlations between "local" measurements performed separately on subsystems A and B. Measurements performed on A correspond to the restriction $\omega_0 = \omega\mid_{\mathcal{A}_0}$ of ω to the subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ generated by elements of the form $\alpha \otimes \mathbb{1}_2$, with $\alpha \in M_2(\mathbb{C})$. We obtain $\omega_0(\alpha\otimes\mathbb{1}_2) = \langle\psi_\lambda|\alpha\otimes\mathbb{1}_2|\psi_\lambda\rangle = \lambda\langle+|\alpha|+\rangle + (1-\lambda)\langle-|\alpha|-\rangle$. In accordance with (2), we have $\omega_0(\alpha\otimes\mathbb{1}_2) = \mathrm{Tr}_{\mathbb{C}^2}(\rho_A\alpha)$, where $\rho_A = \mathrm{Tr}_B\,|\psi_\lambda\rangle\langle\psi_\lambda|$, namely,

$$\rho_A = \begin{pmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}. \tag{3}$$

Now we perform the GNS construction based on the algebra $\mathcal{A}_0 \cong M_2(\mathbb{C})$ and the state ω_0. These are the data needed to describe subsystem A. For $\alpha \in M_2(\mathbb{C})$, we have $\omega_0(\alpha) = \lambda\alpha_{11} + (1-\lambda)\alpha_{22}$. Now we consider $\mathcal{A}_0$ as a *vector space*. This is just the assertion that $M_2(\mathbb{C})$ is, by itself, a vector space. From the explicit form of ω_0, one readily concludes that, as long as $0 < \lambda < 1$, there are no null states. This means that the GNS space $\mathcal{H}_{\omega_0}$ is just the *four dimensional* space of 2×2 matrices, endowed with the inner product $\langle\alpha|\beta\rangle = \omega_0(\alpha^\dagger\beta)$. We can consider a basis of four 2×2 matrices defined as $e_{ij} = |i\rangle\langle j|$ for $i,j \in \{1,2\}$, where $|1\rangle \equiv |+\rangle$ and $|2\rangle \equiv |-\rangle$. Then, for example, $\langle e_{11}|e_{11}\rangle = \lambda$ and $\langle e_{22}|e_{22}\rangle = 1-\lambda$. With an appropriate normalization and ordering of this basis, one checks that the operator corresponding to $\alpha \in \mathcal{A}_0$ is the 4×4 matrix $\pi_{\omega_0}(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$, showing in an explicit way that the representation is reducible (the GNS-space splits as $\mathcal{H}_{\omega_0} = \mathbb{C}^2 \oplus \mathbb{C}^2$). Following the prescription described above one obtains, for the density matrix, $\rho_{\omega_0} = \mathrm{diag}\{\lambda, 0, 0, 1-\lambda\}$. The identity $\omega_0(\alpha) = \mathrm{Tr}_{\mathcal{H}_{\omega,0}}(\rho_{\omega_0}\pi_{\omega_0}(\alpha))$ is readily checked.

From the explicit form of ρ_{ω_0} we conclude that the entropy of the *restricted* state is $S(\omega_0) = -\lambda\log_2\lambda - (1-\lambda)\log_2(1-\lambda)$. This is precisely the entropy of the *reduced* density matrix ρ_A obtained by partial tracing. Recalling that a (pure) state of the full system is entangled with respect to a bipartition into subsystems if and only if $S(\rho_A) > 0$, we have thus verified that our method reproduces the standard results in the case of bipartite systems. This is in fact a general result:

For bipartite systems of the form $\mathcal{H}_A\otimes\mathcal{H}_B$ (pure case), the GNS construction yields a vanishing entropy for the restricted state precisely when the original state of the full-system is separable. Moreover, in the case of entangled states, the entropy computed via the GNS construction coincides with the von Neumann entropy of the reduced density matrix computed via partial trace and can therefore be used as an entanglement measure.

We remark that, in the pure case, entanglement can also be characterized by the impossibility of writing the state ω as a product state. That is, if ω is of the form $\omega(\alpha\otimes\beta) = \omega_A(\alpha)\omega_B(\beta)$ for α (β) any observable on subsystem A (B) and ω_A, ω_B states on the corresponding subsystems, then ω is a *product*, or *separable* state, and it is not entangled. This observation makes clear that entanglement for *mixed* states can also be studied using our approach: If a mixed state ω_{m} can be written as a convex combination of product states, then it is called separable, otherwise it is called entangled.

SYSTEMS OF IDENTICAL PARTICLES

In the case of identical particles, the Hilbert space of the system is not anymore of the tensor product form. Therefore, the treatment of subsystems using partial trace becomes problematic. In contrast, in our approach all that is needed to describe a subsystem is the specification of a subalgebra corresponding to the subsystem. Then, the restriction of the original state to the subalgebra provides a physically motivated generalization of the concept of partial trace, the latter not being sensible anymore. Applying the GNS construction to the restricted state, we can study the entropy emerging from the restriction and use it as a generalized measure of entanglement.

Let $\mathcal{H}^{(1)} = \mathbb{C}^d$ be the Hilbert space of a one-particle system. The k-particle Hilbert space $\mathcal{H}^{(k)}$ for bosons (fermions) is the symmetrized (antisymmetrized) k-fold tensor product of $\mathcal{H}^{(1)}$. To any one-particle observable $A^{(1)}$ on $\mathcal{H}^{(1)}$, we can associate the operator $A^{(k)} := (A^{(1)} \otimes \mathbb{1}_d \cdots \otimes \mathbb{1}_d) + (\mathbb{1}_d \otimes A^{(1)} \otimes \cdots \otimes \mathbb{1}_d) + \cdots + (\mathbb{1}_d \otimes \cdots \otimes \mathbb{1}_d \otimes A^{(1)})$ on $\mathcal{H}^{(k)}$. The operator $A^{(k)}$ preserves the symmetries of $\mathcal{H}^{(k)}$. The map $A^{(1)} \longrightarrow A^{(k)}$ allows us to study subalgebras of one-particle observables. These constructions are most conveniently expressed in terms of a *coproduct* Δ [15]. In fact, an approach based on Hopf algebras [15] has the advantage that para- and braid-statistics can be *automatically* included. In what follows we use the simple coproduct $\Delta(g) = g \otimes g$, $g \in U(d)$, linearly extended to all of $\mathbb{C}U(d)$. It gives the formula for $A^{(k)}$ at the Lie algebra level. Physically, the existence of such a coproduct is very important. It allows us to homomorphically represent one-particle observables in the k-particle sector. In the examples considered below, observables on such identical-particle systems can also be described in terms of creation/annihilation operators.

In the following examples we will concentrate, for the sake of clarity, on systems of two fermions and two bosons (more examples will be presented in a forthcoming paper). However, our methods can be easily generalized to study many-particle entanglement.

Two Fermions - Consider, as in [4], a one-particle space describing fermions with two *external* degrees of freedom (e.g. *'left'* and *'right'*) and two *internal* degrees of freedom (e.g. *'spin 1/2'*). They are described by fermionic creation/annihilation operators $a_\lambda^{(\dagger)}, b_\lambda^{(\dagger)}$, with a standing for *'left'*, b for *'right'* and $\lambda = 1, 2$ for spin up and down, respectively. The single-particle space is therefore $\mathcal{H}^{(1)} = \mathbb{C}^4$. The two-fermion space is given by $\mathcal{H}^{(2)} = \bigwedge^2 \mathbb{C}^3 \subset \mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$ ($\bigwedge$ denoting antisymmetrization). $\mathcal{H}^{(2)}$ is generated from the "vacuum" $|\Omega\rangle$ using pairs of creation operators. An orthonormal basis is given by the vectors $a_1^\dagger a_2^\dagger |\Omega\rangle$, $b_1^\dagger b_2^\dagger |\Omega\rangle$ and $a_\lambda^\dagger b_{\lambda'}^\dagger |\Omega\rangle$, with $\lambda, \lambda' \in \{1, 2\}$. The two-particle algebra $\mathcal{A}$ of observables is thus isomorphic to the matrix algebra $M_6(\mathbb{C})$.

For $|\psi_\theta\rangle = (\cos\theta a_1^\dagger b_2^\dagger + \sin\theta a_2^\dagger b_1^\dagger)|\Omega\rangle$, the corresponding state ω_θ is given by $\omega_\theta(\alpha) = \langle\psi_\theta|\alpha|\psi_\theta\rangle$ for $\alpha \in \mathcal{A}$. We now choose the subalgebra $\mathcal{A}_0$ to be given by the *one-particle observables corresponding to measurements at the left location.* It is generated by $\mathbb{1}_\mathcal{A}$, $n_{12} = a_1^\dagger a_1 a_2^\dagger a_2$, $N_a = a_1^\dagger a_1 + a_2^\dagger a_2$ and $T_{i=1,2,3} = (1/2)\ a_\lambda^\dagger (\sigma_i)^{\lambda\lambda'} a_{\lambda'}$. Now we consider the restriction of ω_θ to $\mathcal{A}_0$ and study the GNS representation corresponding to this choice. For $0 < \theta < \pi/2$, the null space turns out to be spanned by $|n_{12}\rangle$ and $|\mathbb{1}_\mathcal{A} - N_a\rangle$. Therefore, the GNS-space $\mathcal{H}_\theta$ is four-dimensional and spanned by $|[\mathbb{1}_\mathcal{A}]\rangle$ and $\{|[T_i]\rangle\}_{i=1,2,3}$. One may show that $\mathcal{H}_\theta = \mathcal{H}_1 \oplus \mathcal{H}_2$, with $\mathcal{H}_1$ spanned by $|[T_1 + iT_2]\rangle = |[a_1^\dagger a_2]\rangle$ and $|[a_2^\dagger a_2]\rangle$, and $\mathcal{H}_2$ spanned by $|[a_1^\dagger a_1]\rangle$ and $|[T_1 - iT_2]\rangle = |[a_2^\dagger a_1]\rangle$. The two representations are isomorphic. Moreover, from the decomposition $|[\mathbb{1}_\mathcal{A}]\rangle = |[a_2^\dagger a_2]\rangle + |[a_1^\dagger a_1]\rangle$ of $|[\mathbb{1}_\mathcal{A}]\rangle$ into these irreducible subspaces, we obtain the entropy $S(\theta) = -\cos^2\theta \log_2 \cos^2\theta - \sin^2\theta \log_2 \sin^2\theta$.

For $\theta = 0$, the null space is spanned by $|n_{12}\rangle$, $|\mathbb{1}_\mathcal{A} - a_1^\dagger a_1\rangle$, $|a_2^\dagger a_2\rangle, |a_1^\dagger a_2\rangle$. The GNS-space $\mathcal{H}_0$ is $\mathbb{C}^2$ and isomorphic to the above $\mathcal{H}_2$. Similarly, for $\theta = \pi/2$ we find that the GNS-space is isomorphic to the above $\mathcal{H}_1$. Both GNS-spaces are irreducible, so that the corresponding $\omega_{0,0}$ and $\omega_{\frac{\pi}{2},0}$ are pure states with zero entropy.

This result should be contrasted with the entropy $S = \log_2 2$ obtained via partial trace for states with Slater rank one such as $\omega_{0,0}$, $\omega_{\frac{\pi}{2},0}$ above (cf. [5, 7] and references therein), that correspond to simple Slater determinants and, therefore, should not be regarded as entangled states.

Two Bosons - Consider the one-particle space $\mathcal{H}^{(1)} = \mathbb{C}^3$ with an orthonormal basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$. The two-boson space $\mathcal{H}^{(2)}$ is the space of symmetrized vectors in $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)}$. It corresponds to the decomposition $3 \otimes 3 = 6 \oplus \bar{3}$ of $SU(3)$. An orthonormal basis for $\mathcal{H}^{(2)}$ is given by vectors $\{|e_i \vee e_j\rangle\}_{i,j\in\{1,2,3\}}$ where $\vee$ denotes symmetrization (and the vectors are normalized). The two-boson algebra of observables $\mathcal{A}^{(2)}$ is thus isomorphic to $M_6(\mathbb{C})$.

For the particular choice $|\psi_{\theta,\phi}\rangle = \sin\theta\cos\phi|e_1 \vee e_2\rangle + \sin\theta\sin\phi|e_1 \vee e_3\rangle + \cos\theta|e_3 \vee e_3\rangle$, the corresponding state is $\omega_{\theta,\phi}$ defined by $\omega_{\theta,\phi}(\alpha) = \langle\psi_{\theta,\phi}|\alpha|\psi_{\theta,\phi}\rangle$ for any $\alpha \in \mathcal{A}$. For the sake of concreteness, we choose $\mathcal{A}_0$ to be given by those one-particle observables pertaining *only* to the one-particle states $|e_1\rangle$ and $|e_2\rangle$.

We consider the restriction $\omega_{\theta,\phi} \mid_{\mathcal{A}_0}$. The 6 representation under the $SU(2)$ action on $|e_1\rangle$ and $|e_2\rangle$, splits as $6 = 3 \oplus 2 \oplus 1$. The subalgebra $\mathcal{A}_0$ is given by block-diagonal matrices. Each block corresponds to one of the irreducible components in the decomposition $6 = 3 \oplus 2 \oplus 1$. The dimension of $\mathcal{A}_0$ is therefore $3^2 + 2^2 + 1^2 = 14$.

The construction of the corresponding GNS-representation follows the same procedure as in the previous example. The von Neumann entropy as a function of the parameters is given by $S(\theta,\phi) = -\sin^2\theta[\cos^2\phi \log_2(\sin\theta\cos\phi)^2 + \sin^2\phi \log_2(\sin\theta\sin\phi)^2] - \cos^2\theta \log_2(\cos\theta)^2$.

CONCLUSIONS

The strong point of our approach is that it provides a precise, universal, and mathematically natural way to characterize and quantify entanglement for systems of identical particles. For many years it has been known that the von Neumann entropy based on partial tracing does not give the physically correct answer when applied to systems of identical particles. Different (i.e. non-universal) criteria have been developed which strongly depend on the statistics of the particles. In contrast, our approach is conceptually clear and applies equally to any quantum system. It thus promises to resolve the controversy regarding entanglement of identical particles [7].

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[1] R. Sorkin, “Black holes and relativistic stars,” (The University of Chicago Press, 1998) Chap. 9, p. 177.
[2] S. Adhikari, B. Chakraborty, A. S. Majumdar, and S. Vaidya, Phys. Rev. A **79**, 042109 (2009).
[3] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008).
[4] K. Eckert, J. Schliemann, D. Bruss, and M. Lewenstein, Annals of Physics **299**, 88 (2002).
[5] G. C. Ghirardi and L. Marinatto, Phys. Rev. A **70**, 012109 (2004) .
[6] J. Schliemann, J. I. Cirac, M. Kuś, M. Lewenstein, and D. Loss, Phys. Rev. A **64**, 022303 (2001).
[7] M. C. Tichy, F. Mintert, and A. Buchleitner, Journal of Physics B: Atomic, Molecular and Optical Physics **44**, 192001 (2011).
[8] J. Grabowski, M. Kuś, and G. Marmo, Journal of Physics A: Mathematical and Theoretical **44**, 175302 (2011).
[9] Importance of focusing on subsystems has also been emphasized by H. Barnum, E. Knill, G. Ortiz, R. Somma, and L. Viola, Physical Review Letters **92**, 107902 (2004); T. Fritz, Arxiv preprint arXiv:1011.1247 (2010); Ł. Derkacz, M. Gwóźdź, and L. Jakóbczyk, Journal of Physics A: Mathematical and Theoretical **45**, 025302 (2012).
[10] R. Haag, *Local quantum physics* (Text and Monographs in Physics, 2nd ed., Springer, 1996).
[11] A. P. Balachandran, T. R. Govindarajan, A. R. de Queiroz, and A. F. Reyes-Lega, (2013), 1301.1300.
[12] R. Paskauskas and L. You, Phys. Rev. A **64**, 042310 (2001).
[13] H. M. Wiseman and J. A. Vaccaro, Phys. Rev. Lett. **91**, 097902 (2003).
[14] O. Bratteli, and D. Robinson, *Operator Algebras and Quantum Statistical Mechanics 1* (Text and Monographs in Physics, 2nd ed., Springer, 1987).
[15] A. P. Balachandran, S. G. Jo, and G. Marmo, *Group Theory and Hopf Algebra: Lectures for Physicists* (World Scientific, 2010).
[16] G. Landi, *An Introduction to Noncommutative Spaces and their Geometry* (vol. 51, Lecture Notes in Physics Monographs, Springer-Verlag, 2008).