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Hong Qin and Ronald C. Davidson

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A class of generalized Kapchinskij-Vladimirskij solutions and associated envelope equations for high-intensity charged particle beams

Hong Qin[1,2] and Ronald C. Davidson[1]

[1]*Plasma Physics Laboratory, Princeton University, Princeton, NJ 08543*

[2]*Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

Abstract

A class of generalized Kapchinskij-Vladimirskij solutions of the Vlasov-Maxwell equations and the associated envelope equations for high-intensity beams in an uncoupled lattice is derived. It includes the classical Kapchinskij-Vladimirskij solution as a special case. For a given lattice, the distribution functions and the envelope equations are specified by ten free parameters. The class of solutions derived captures a wider range of dynamical envelope behavior for high-intensity beams, and thus provides a new theoretical tool to investigate the dynamics of high-intensity beams.

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For high-intensity charged particle beams in an uncoupled periodic transverse focusing lattice, the beam envelope dynamics described by the envelope equations is an important research topic for optimizing beam quality and controlling beam instability. The most comprehensive self-consistent description of high-intensity beam dynamics, including both collective transverse dynamics [1–3] and longitudinal dynamics[4], is a kinetic description using the Vlasov-Maxwell (VM) equations [5]. In 1959, Kapchinskij and Vladimirskij [5, 6] derived the envelope equations as a rigorous solution of the VM equations for a special distribution function, which is now called the Kapchinskij-Vladimirskij (KV) distribution. Since then, the envelope equations have become a very important theoretical tool for investigating the transverse dynamics of high-intensity beams in uncoupled focusing lattices [1–3, 7–15].

In this Letter, we derive a class of generalized Kapchinskij-Vladimirskij solutions of the VM equations and the associated nonlinear envelope equations for high-intensity beams in an uncoupled transverse focusing lattice. The new class of distribution functions and the associated envelope equations include the classical KV distribution function and the associated envelope equations as a special case. In the classical KV solution, for a prescribed focusing lattice and line density of the beam, the distribution function and associated envelope equations are specified by two free parameters (excluding the initial conditions), *i.e.*, the transverse emittances ε_x and ε_y. The (x, y)-projection of the KV distribution is a upright ellipse with constant density inside. The dimensions of the ellipse $a(s)$ and $b(s)$ are time dependent and determined by the envelope equations (8). In the generalized solutions described in this Letter, for a given uncoupled focusing lattice and line density, the distribution functions and associated envelope equations are specified by ten free parameters, defined by a constant 4×4 symmetric and positive definite matrices ξ, called emittance matrix in this Letter. The (x, y)-projection of the distribution is an ellipse with constant density inside as in the classical KV solution. However, the beam ellipse is allowed to rotating around the beam centroid in addition to the pulsating dynamics of the transverse dimensions. This extra degree of freedom is specified by the time-dependent tilt angle $\theta(s)$ of the ellipse. The generalized distribution function and the associated envelope equations are given by Eqs. (17) and (16). The classical KV solution is a special case of the generalized solutions presented here when the emittance matrix is chosen to be $\xi = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_1 \end{pmatrix}$, where

$\xi_1 = \begin{pmatrix} 1/\varepsilon_x & 0 \\ 0 & 1/\varepsilon_y \end{pmatrix}$ is a 2×2 matrix corresponding to the emittance ε_x and ε_y in the two transverse directions for the classical KV solution.

When the beam ellipse is tilted, the space-charge force couples the $x-$dynamics and $y-$dynamics of a single particle. Therefore, to construct the generalized KV solution for high-intensity beams in an uncoupled lattice, we will first generalize the classical KV solution to an arbitrary coupled lattice, where the coupled dynamics can be induced by both the external focusing lattice and the self-potential. This most generalized solution is then restricted to the case of uncoupled external focusing lattice, allowing coupled focusing force to be induced only by the self-force. In this most generalized solution, the ten free parameters are specified by the emittance matrix ξ, which is a large-scale generalization of the previous self-consistent solution of high-intensity beams in a coupled focusing lattice with only one free parameter, *i.e.*, one scalar emittance [16]. The case of one scalar emittance corresponds to a beam with the same normalized emittance in the two transverse directions, which obviously does not include many beam configurations for practical applications. However, generalized solutions with a general emittance maxtrix ξ are technically more difficult to treat. Specifically, the difficulty is associated with the calculation of the velocity integral [see Eq. (20)] for the general case. It turns out that this difficulty can be overcome by the technique of Cholesky decomposition for a symmetric, positive definite matrix [see Eq. (19)]. Using this technique, we are able to obtain an envelope equation for the most general case. We note that Barnard and Losic [12] developed a set of moment equations from the Vlasov-Maxwell equations to described beam dynamics with angular momentum in a coupled focusing lattice. The moment formulation and the envelope formulation given in this letter are equivalent to each other. In Ref. [12], a uniform density beam was assumed, without showing an underlying self-consistent distribution function explicitly, appealing to the general work of Sacherer [1] to justify the assumption. An explicit derivation of the underlying distribution function is given in this letter.

Our starting point is the Vlasov-Maxwell equations that govern the nonlinear evolution of the distribution function f and the normalized self-field potential ψ,

$$\frac{\partial f}{\partial s} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - (\nabla\psi + \kappa_{qx} x \boldsymbol{e}_x + \kappa_{qx} y \boldsymbol{e}_y) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0\,, \tag{1}$$

$$\nabla^2\psi = \frac{-2\pi K_b}{N_b} \int f dv_x dv_y\,. \tag{2}$$

Here, the normalized self-field potential is defined by $\psi = q_b\phi/\gamma_b^3 m\beta_b^2 c^2$, where ϕ is the space-charge potential, $\beta_b c$ is the directed beam velocity in the longitudinal direction, $\gamma_b = (1-\beta_b^2)^{-1/2}$ is the relativistic mass factor, $s = \beta_b ct$ is an effective time variable normalized by $1/\beta_b c$, $K_b = 2N_b q_b^2/\gamma_b^3 m\beta_b^2 c^2$ is the beam self-field perveance, and $N_b = \int f dxdydv_x dv_y$ is the line density. Particle motion in the beam frame is assumed to be non-relativistic, and (x, y) is the transverse displacement of a beam particle, $\mathbf{v} = d\mathbf{x}/ds = (v_x, v_y)$ is the normalized transverse velocity in the beam frame, and κ_{qx} and κ_{qy} are the focusing coefficients for the uncoupled quadrupole lattice. The $-\nabla\psi$ term in Eq. (1) describes the self-field force due to the self-electric and self-magnetic fields of the beam, and it is nonlinearly coupled to the distribution function f through Eq. (2). Equations (1) and (2) form a integro-differential equation system, and it is in general difficult to find analytical solutions.

Kapchinskij and Vladimirskij [5, 6] discovered a remarkable solution of the VM equations (1) and (2), which is now called the KV distribution. The solution is constructed from the well-known Courant-Snyder (CS) invariants [17] for a linear focusing lattice

$$I_x = \frac{x^2}{w_x^2} + \left(w_x\dot{x} - x\dot{w}_x\right)^2 \ , \ I_y = \frac{y^2}{w_y^2} + \left(w_y\dot{y} - y\dot{w}_y\right)^2 . \tag{3}$$

Here, ε_x and ε_y are the constant transverse emittances, and w_x and w_y are the envelope functions satisfying the envelope equations,

$$\ddot{w}_x + (\kappa_{qx} + \kappa_{sx})w_x = w_x^{-3} \ , \ \ddot{w}_y + (\kappa_{qy} + \kappa_{sy})w_y = w_y^{-3} . \tag{4}$$

In Eq. (4), the self-field force are assumed *a prior* to be uncoupled and proportional to the displacement with the defocusing coefficient κ_{sx} and κ_{sy}, *i.e.*, $-\nabla\psi = -\kappa_{sx}x\boldsymbol{e}_x - \kappa_{sy}y\boldsymbol{e}_y$. The coefficients κ_{sx} and κ_{sy} will be determined self-consistently from the distribution function, which is required to satisfy the Vlasov equation (1) and simultaneously generate a linear self-field force in order for the CS invariants to be valid. A distribution function that satisfies both conditions is the KV distribution given by

$$f_{KV} = \frac{N_b}{\pi\varepsilon_x\varepsilon_y}\delta\left(\frac{I_x}{\varepsilon_x} + \frac{I_y}{\varepsilon_y} - 1\right) , \tag{5}$$

which obviously satisfies the Vlasov equation (1) because it is a function of the invariants of the particle dynamics. Here, the constants ε_x and ε_y are the transverse emittances. The density profile projected by the distribution function f_{KV} in the transverse configuration space is

$$n\left(x, y, s\right) = \int dv_x dy_y f_{KV} = \begin{cases} N_b/\pi ab, \; 0 \le x^2/a^2 + y^2/b^2 < 1\,, \\ \quad 0, \; 1 < x^2/a^2 + y^2/b^2\,. \end{cases} \tag{6}$$

where $a \equiv \sqrt{\varepsilon_x} w_x$, $b \equiv \sqrt{\varepsilon_y} w_y$. This density profile in the (x, y)-plane corresponds to a constant-density beam with elliptical cross-section and pulsating transverse dimensions a and b. The associated normalized self-field inside the beam, determined from Eq. (2), is given by

$$\psi = \frac{-K_b}{a+b}\left(\frac{x^2}{a} + \frac{y^2}{b}\right), \; 0 \le x^2/a^2 + y^2/b^2 < 1, \tag{7}$$

which indeed generates a linear defocusing force with coefficients $\kappa_{sx} = -2K_b/a(a+b)$, and $\kappa_{sy} = -2K_b/b(a+b)$. The KV solution reduces the VM equations to the envelope equations given by Eq. (4) in terms of w_x and w_y, or equivalently, in terms of a and b as

$$\ddot{a} + \kappa_{qx} a - \frac{2K_b}{(a+b)} = \frac{\varepsilon_x^2}{a^3}\,, \; \ddot{b} + \kappa_{qy} b - \frac{2K_b}{(a+b)} = \frac{\varepsilon_y^2}{b^3}\,. \tag{8}$$

The envelope equations have become an indispensable tool for our understanding of the dynamical behavior of high-intensity beams.

We now show how to construct a class of more general solutions of the VM equations and the associated envelope equations, which include the classical KV solution as a special case. It turns out that the class of distribution functions that generate a linear space-charged force and satisfy the Vlasov equation is much wider than the classical KV distribution given by Eq. (5). As mentioned earlier, the generalized solution projects to a rotating, pulsating elliptical beam which induces coupled dynamics in the transverse direction. Our strategy is to allow the external focusing lattice to be coupled as well. In this case, the Vlasov equation can be written as

$$\frac{\partial f}{\partial s} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \left(\nabla\psi + \kappa_q \mathbf{x}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0\,, \tag{9}$$

where

$$\kappa_q = \begin{pmatrix} \kappa_{qx} & \kappa_{qxy} \\ \kappa_{qyx} & \kappa_{qy} \end{pmatrix} \tag{10}$$

is the matrix of coupling coefficients, κ_{qx} and κ_{qy} are the focusing coefficients for the lattice, and $\kappa_{qxy} = \kappa_{qyx}$ are the the coupling coefficients, which can be produced, for example, by a skew-quadrupole component of the lattice. Every component of κ_q is a function of s. The generalized KV distribution that solves the Vlasov-Maxwell system (9) and (2) projects to a rotating, pulsating beam with elliptical cross-section in transverse configuration space with constant density inside the beam. Both the transverse dimensions a and b, and the tilt angle θ are functions of $s = \beta_b ct$, in contrast with the pulsating upright elliptical beam cross-section for the classical KV solution.

The rotating, pulsating beam with elliptical cross-section in transverse configuration space, and constant density inside the beam, generates a coupled linear space-charge force of the form

$$-\nabla\psi = -\kappa_s \mathbf{x}\,,\ \kappa_s = \begin{pmatrix} \kappa_{sx} & \kappa_{sxy} \\ \kappa_{syx} & \kappa_{sy} \end{pmatrix}\,, \tag{11}$$

where $\kappa_{sxy} = \kappa_{syx}$, which allows us to apply the generalized Courant-Snyder (CS) theory [18–20] for the coupled transverse dynamics. The exact form of κ_s will be determined self-consistently [see Eq. (21)]. We now use the generalized CS invariant to construct a generalized KV solution of the Vlasov equation (9), which also projects to a rotating, pulsating elliptical beam with constant density inside the beam. In this manner, a self-consistent solution of the Vlasov-Maxwell equations (9) and (2) is found for high-intensity beams in a coupled transverse focusing lattice. For a charged particle subject to the coupled linear focusing force and the coupled linear space-charge force

$$-\nabla\psi - \kappa_q \mathbf{x} = -\kappa \mathbf{x}\,,\ \kappa = \kappa_q + \kappa_s\,, \tag{12}$$

the generalized CS invariant is given by [18–20]

$$I_\xi = \mathbf{z}^T Q^T P^T \xi P Q \mathbf{z}, \tag{13}$$

where $\mathbf{z} \equiv (x, y, v_x, v_y)^T$, ξ is the constant 4×4 emittance matrix, which is symmetric and positive definite, and superscript "T" denotes transpose. Here, P and Q are 4×4 matrices determined by a 2×2 envelope matrix $w = \begin{pmatrix} w_1 & w_2 \\ w_3 & w_4 \end{pmatrix}$ as follows

$$Q = \begin{pmatrix} \left(w^{-1}\right)^T & 0 \\ -\dot{w} & w \end{pmatrix} \tag{14}$$

$$\dot{P} = P\dot{\phi}\,,\ \dot{\phi} \equiv \begin{pmatrix} 0 & -\left(w^{-1}\right)^T w^{-1} \\ \left(w^{-1}\right)^T w^{-1} & 0 \end{pmatrix}\,. \tag{15}$$

The 2×2 envelope matrix w is determined from the matrix envelope equation

$$\ddot{w} + w\kappa = \left(w^{-1}\right)^T w^{-1} \left(w^{-1}\right)^T\,. \tag{16}$$

The P matrix defined by w is a rotation in the 4D phase space, *i.e.*, $P \in SO(4)$, which is a generalization of the phase advance to higher dimension, and $\dot{\phi}$ is the corresponding generating angular momentum, *i.e.*, $\dot{\phi} \in so(4)$. The matrix product $Q^T P^T \xi P Q$ can be viewed as the beam matrix. It is symmetric and positive definite because ξ is symmetric and positive definite. Its determinate is constant, *i.e.*, $|Q^T P^T \xi P Q| = |\xi|$, due to the fact that P and Q are symplectic. This fact has also been verified numerically in the numerical example given near the end of this letter.

Since I_ξ is an invariant of the particle dynamics, any function of I_ξ is a solution of the Vlasov equation (9). However, in order to solve for the Vlasov-Maxwell equations (9) and (2), the distribution function must generate the coupled linear space-charge force of the form in Eq. (11). For this purpose, we select the distribution function to be the following generalized distribution

$$f = \frac{N_b \sqrt{|\xi|}}{\pi} \delta \left(I_\xi - 1 \right) . \tag{17}$$

Here, N_b is the line-density which is a constant. To be consistent with the assumption that the space-charge force is linear, it is necessary to verify that this distribution function indeed generates a linear space-charge force. The number density in configuration space is $n\left(x, y, s\right) = \int dv_x dv_y f$. The velocity integral here is much more difficult to calculate than in the classical KV case, because I_ξ depends on the phase advance matrix P. The special technique required here is the Cholesky decomposition. For a symmetric, positive definite matrix M, it is always possible to uniquely decompose it into the form

$$M = L^T L \,,$$

where L is a lower triangular matrix. This is the Cholesky decomposition. In the present case, the matrix product $Q^T P^T \xi P Q$ is symmetric and positive definite and its Cholesky decomposition is

$$Q^T P^T \xi P Q = L^T L \,, \tag{18}$$

$$L = \begin{pmatrix} R^{T/2} w^{-T} & 0 \\ D^{-1/2} C w^T - D^{T/2} w & D^{T/2} w \end{pmatrix} \tag{19}$$

where superscript “1/2” denotes the square-root operation of a matrix. For a symmetric and positive definite matrix M, its square-root is defined by $M^{1/2} M^{T/2} = M$. The matrices D,

C, and R in Eq. (19) are related to the phase advance matrix P and the emittance matrix ξ as follows,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv P^T \xi P \,,$$

$$R \equiv A - BD^{-1}C \,.$$

The R matrix is known as the Schur complement of D.

With the help of the Cholesky decomposition, we can introduce a coordinate transformation in the velocity space

$$\boldsymbol{V}^T = (V_x, V_y) = (D^{-1/2}Cw^T - D^{T/2})\boldsymbol{x} + D^{T/2}w\boldsymbol{v},$$

where $\boldsymbol{x} \equiv (x, y)^T$ and $\boldsymbol{v} \equiv (v_x, v_y)^T$. The Jacobian of the coordinate transformation is

$$dv_x dv_y = \frac{1}{|D^{T/2}w|} dV_x dV_y \,.$$

The velocity integral in Eq. (2) can then be carried out in closed form as

$$\begin{aligned} n\left(x, y, s\right) &= \int dv_x dv_y f = \int dV_x dV_y \frac{N_b \sqrt{|\xi|}}{\pi |D^{T/2}w|} \delta\left(\boldsymbol{x}^T w^{-1} R w^{-T} \boldsymbol{x} + V_x^2 + V_y^2 - 1\right) \\ &= \begin{cases} N_b |R^{T/2} w^{-T}|, \ 0 \le \boldsymbol{x}^T w^{-1} R w^{-T} \boldsymbol{x} < 1 \,, \\ 0, \ 1 < \boldsymbol{x}^T w^{-1} R w^{-T} \boldsymbol{x} \,. \end{cases} \end{aligned} \tag{20}$$

As expected, the beam density profile in the (x, y)-plane is indeed a tilted ellipse with constant density inside. The beam ellipse is given by $\boldsymbol{x}^T w^{-1} R w^{-T} \boldsymbol{x} < 1$, whose area is $|w^{-1} R w^{-T}|^{-1/2} = |R^{T/2} w^{-T}|^{-1}$. The transverse dimensions $a(s)$ and $b(s)$ and the tilt angle $\theta(s)$ of the ellipse are determined by the eigenvalues (λ_1, λ_2) and eigenvectors $(\boldsymbol{v}_1, \boldsymbol{v}_2)$ of the matrix $w^{-1} R w^{-T}$ as

$$a \equiv \sqrt{1/\lambda_1} \,, \ \ b \equiv \sqrt{1/\lambda_2} \,, \ \ E = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \equiv (\boldsymbol{v}_1, \boldsymbol{v}_2) \,.$$

Here, E is the matrix defining the rotation of the ellipse relative to the upright position. Then, the self-force can be expressed as

$$-\begin{pmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \end{pmatrix} = -\kappa_s \begin{pmatrix} x \\ y \end{pmatrix} \,, \ \kappa_s = \frac{-2K_b}{a+b} E \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} E^{-1} \,. \tag{21}$$

The coupled linear space-charge coefficient κ_s is thus a function of the envelope matrix w and the constant emittance matrix ξ.

When Eq. (21) is substituted back into Eq. (12), the envelope equation (16) becomes a closed nonlinear matrix equation for the envelope matrix w. Therefore, we have succeeded in finding a class of self-consistent solutions of the Vlasov-Maxwell equations for high-intensity beams in a coupled transverse focusing lattice. The solution reduces to a nonlinear matrix ordinary differential equation for the envelope matrix w, which determines the geometry of the pulsating and rotating beam ellipse. Compared with the classical KV solution in a uncoupled lattice, the unique feature of the generalized solution is that the beam ellipse is rotating with time. This feature persists even when the external focusing lattice is the standard uncoupled lattice. In this case, the generalized solution represents a class of solutions larger than the classical KV solution. Even though the focusing lattice is uncoupled, the space-charge force can still couple the two degrees of freedom in the transverse direction. It is a pleasant surprise that an uncoupled focusing lattice can actually confine and focus a rotating and pulsating high-intensity beam in a self-consistent manner. This new family of solutions can be used as an effective beam smoothing technique for accelerator applications where smooth illumination is required, such as in the case of heavy ion fusion and medical accelerators.

Because the beam is rotating in the transverse plane, the conventional emittance ϵ_x and ϵ_y are no longer constants but periodic functions of s that take on their minima when the beam ellipse is "upright", *i.e.*, in normal form in the transverse plane. The dynamics of ϵ_x and ϵ_y can be obtained through the following equations once the envelope matrix w is solved,

$$\epsilon_x = \left\langle x^2 \right\rangle \left\langle v_x^2 \right\rangle - \left\langle x v_x \right\rangle^2 , \quad \epsilon_y = \left\langle y^2 \right\rangle \left\langle v_y^2 \right\rangle - \left\langle y v_y \right\rangle^2 , \tag{22}$$

where $\langle \chi \rangle \equiv \int \chi f \, dx dy dv_x dv_y / N_b$ is the phase space average of a function χ.

We now give a numerical example of the new class of solutions in an uncoupled focusing lattice. We consider the case of a high-intensity beam with normalized self-field perveance is $K_b/\varepsilon = 0.1$ in a FODO (acronym for focusing-off-defocusing-off) focusing lattice with normalized quadrupole focusing field amplitude $\hat{\kappa}_q S \equiv q_b B_q' / \gamma_b m \beta_b c^2 = 15$ and filling factor $\eta = 0.30$, where S is the lattice period. The emittance is chosen to be $\xi = \begin{pmatrix} I & \xi_2 \\ \xi_2 & I \end{pmatrix}$ and

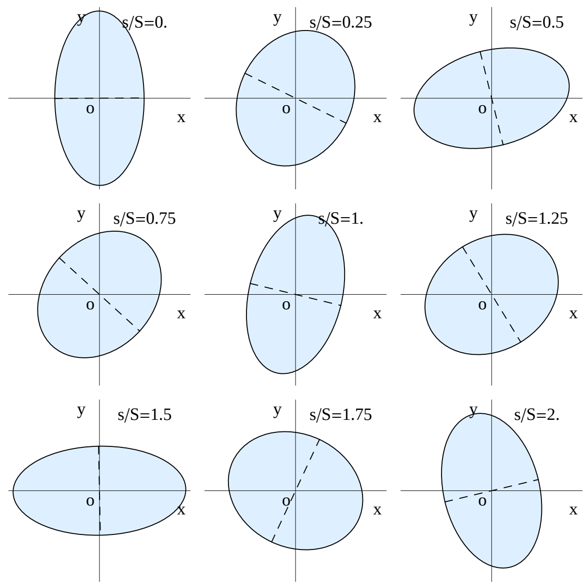

Figure 1: Beam cross section defined by $0 \le \boldsymbol{x}^T w^{-1} R w^{-T} \boldsymbol{x} < 1$ for $0 \le s/S \le 2$.

$\xi_2 = \begin{pmatrix} 0 & 0.5 \\ 0 & 0 \end{pmatrix}$. Plotted in Fig. 1 is the beam cross section as a function of time for two lattice periods. It is clear that as the beam radii pulsate with time, the beam also rotates in the transverse plane. The dynamics of beam radii and tilt angle are also plotted in Fig. 2, which indicates that the dynamics of pulsation and rotation has a period of $3S$.

In conclusion, we have derived a class of generalized KV solutions of the VM equations and the associated envelope equations for high-intensity beams in an uncoupled lattice. It includes the classical KV solution as a special case. For a given uncoupled lattice and beam line density, the distribution function and the envelope equations are specified by ten free parameters. The class of solutions derived here captures a wider range of envelope dynamics for high-intensity beams, and thus provides us with a new theoretical tool to investigate the dynamics of high-intensity beams in an uncoupled transverse focusing lattice.

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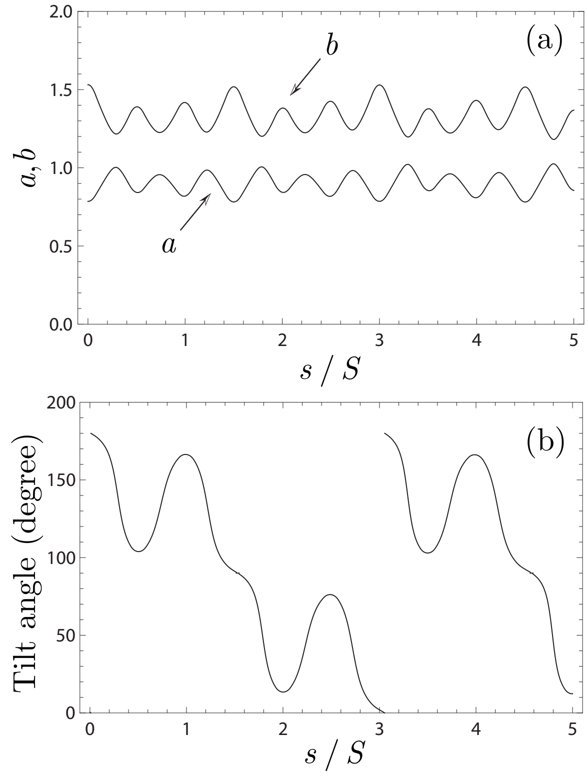

Figure 2: (a) Beam transverse dimensions and (b) tilt angle defined by $0 \leq \boldsymbol{x}^T w^{-1} R w^{-T} \boldsymbol{x} < 1$.

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