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An Integrable Nonlocal Nonlinear Schrödinger Equation

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A new integrable nonlocal nonlinear Schrödinger equation is introduced. It possesses a Lax pair and an infinite number of conservation laws and is PT-symmetric. The inverse scattering transform and scattering data with suitable symmetries are discussed. A method to find pure soliton solutions is given. An explicit breathing one soliton solution is found. Key properties are discussed and contrasted with the classical NLS equation.

Introduction. In the study of nonlinear wave propagation exactly solvable models play exceptional role. There are many physically important integrable equations. Examples include small amplitude waves in shallow water where the Korteweg-deVries (KdV) equation [1] and its multidimensional analog, the Kadomtsev-Petviashvili equation [2] arise; in generic weakly nonlinear dispersive systems, in the quasi-monochromatic limit the integrable cubic nonlinear Schrödinger equation [3] is applicable. Furthermore, in nonlinear optics the integrable cubic nonlinear Schrödinger equation is a key equation describing optical wave propagation in Kerr media [4, 5]. Indeed there are many physically significant integrable systems [6] which apply to diverse problems in fluid mechanics, electromagnetics, gravitational waves, elasticity, fundamental physics and lattice dynamics, to name but a few.

Generally speaking integrability is established once an infinite number of constants of motion or an infinite number of conservation laws are obtained. However considerably more information about the solution can be obtained if the inverse scattering transform (IST) can be carried out [7]. Corresponding to rapidly decaying initial data, IST provides a linearization and a class of explicit solutions–i.e. solitons. The method associates a compatible pair of linear equations (i.e. a Lax pair) with the integrable nonlinear equation. One of the equations, the scattering problem, is used to determine suitably analytic eigenfunctions and transform the initial data to appropriate scattering data. The other linear equation serves to determine the evolution of the scattering data. Using the analytic behavior of the eigenfunctions an inverse scattering problem, or linear Riemann-Hilbert (RH) problem, is constructed. With the time dependence of the scattering data one can find the solution of the nonlinear evolution equation from the inverse or RH problem. There are many books describing the IST method cf. [8–10].

In this Letter, the following nonlocal nonlinear Schrödinger equation is introduced and investigated in detail

$$iq_t(x,t) = q_{xx}(x,t) \pm 2q(x,t)q^*(-x,t)q(x,t) \ , \qquad (1)$$

where $*$ denotes complex conjugation and $q(x,t)$ is a complex valued function of the real variables x and t. Eq. (1) admits a linear (Lax) pair formulation and possesses an infinite number of conservation laws, hence it is an integrable system. Via the inverse scattering transform, corresponding to rapidly decaying initial data, one can linearize the equation and obtain solutions to Eq. (1) including pure solitons solutions. Some of the important properties of the nonlocal NLS equation are contrasted with the classical NLS equation where the nonlocal nonlinear term $q^*(-x,t)$ is replaced by $q^*(x,t)$. Indeed we note that both equation (1) and the classical NLS share the symmetry that when $x \to -x, \ t \to -t$ and a complex conjugate is taken, then the equation remains invariant. Thus, the new nonlocal equation is PT symmetric [11] which, in the case of classical optics, amounts to the invariance of the so-called self-induced potential cf. [12] $V(x,t) = q(x,t)q^*(-x,t)$ under the combined action of parity and time reversal symmetry. Finally, wave propagation in PT symmetric coupled waveguides/photonic lattices has been experimentally observed in classical optics [13–15].

Linear pair and the nonlocal NLS equation. We begin our analysis by considering the following scattering problem [8, 16]

$$v_x = \begin{pmatrix} -ik & q(x,t) \\ r(x,t) & ik \end{pmatrix} v \ , \qquad (2)$$

$$v_t = \begin{pmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & -\mathsf{A} \end{pmatrix} v \ , \qquad (3)$$

where v is a two-component vector, $v(x,t) = (v_1(x,t), v_2(x,t))^T$, $q(x,t)$ and $r(x,t)$ vanish rapidly as $x \to \pm\infty$, k is a spectral parameter and $\mathsf{A} = 2ik^2 + iq(x,t)r(x,t)$, $\mathsf{B} = -2kq(x,t) - iq_x(x,t)$, $\mathsf{C} = -2kr(x,t) + ir_x(x,t)$. The compatibility condition of system (2) and (3) i.e., $v_{xt} = v_{tx}$ yields

$$iq_t(x,t) = q_{xx}(x,t) - 2r(x,t)q^2(x,t) \ , \qquad (4)$$

$$-ir_t(x,t) = r_{xx}(x,t) - 2q(x,t)r^2(x,t) \ . \qquad (5)$$

Equation (1) is then obtained from system (4) and (5) under the symmetry reduction

$$r(x,t) = \mp q^*(-x,t) \ . \tag{6}$$

Importantly, the symmetry reduction (6) is *new* and leads to a new class of nonlocal integrable PDEs including a nonlocal NLS hierarchy. This is a special and remarkably simple reduction of the more general AKNS system [7] which has not been previously found.

Infinite number of conserved quantities and conservation laws. The infinite number of conserved quantities of (1) can be derived as follows. We assume that $q(x,t)$ decays rapidly at infinity. Then solutions of the scattering problem (2) can be defined. Indeed we define four eigenfunctions which satisfy the following boundary conditions

$$\begin{aligned} \phi &\sim \begin{pmatrix}1\\0\end{pmatrix} e^{-ikx} \ , \ \ \overline{\phi} \sim \begin{pmatrix}0\\1\end{pmatrix} e^{ikx} \ , \ \ \text{as } x \to -\infty \\ \psi &\sim \begin{pmatrix}0\\1\end{pmatrix} e^{ikx} \ , \ \ \overline{\psi} \sim \begin{pmatrix}1\\0\end{pmatrix} e^{-ikx} \ , \ \ \text{as } x \to +\infty \ . \end{aligned} \tag{7}$$

Note that $\overline{\phi}$ is *not* the complex conjugate of ϕ. We use ϕ^* to denote complex conjugation of ϕ. If $\phi(x,t) = (\phi_1(x,t), \phi_2(x,t))^T$ is the solution to (2) that satisfies the above boundary conditions then, for Im $k \geq 0$, $\phi_1(x,t)e^{ikx}$ is analytic and approaches 1 as $x \to \pm\infty$. Substituting $\phi_1(x,t) = \exp\left[-ikx + \varphi(x,t)\right]$ into (2) we find (after eliminating ϕ_2) that the function $\mu(x,t) \equiv \varphi_x(x,t)$ satisfies the Riccati equation

$$q\frac{\partial}{\partial x}\left(\frac{\mu}{q}\right) + \mu^2 - qr - 2ik\mu = 0 \ . \tag{8}$$

For Im$k > 0$, $\lim_{|k|\to\infty} \varphi(x,k) = 0$. Substituting the expansion $\mu(x,k) = \sum_{n=0}^{\infty} \mu_n(x,t)/(2ik)^{n+1}$ into (8) to find $\mu_0 = -qr, \mu_1 = -qr_x$ and a recursion relation for any $n \geq 1$ cf.[8]. From (2) it follows that the scattering data $a(k) \equiv \lim_{x\to+\infty} \phi_1(x,t)e^{ikx}$ is time-independent. Since $\varphi(x,t)$ vanishes as $x \to -\infty$ we conclude that $\mathcal{C}_n \equiv \int_{-\infty}^{+\infty} \mu_n(x,t)dx$ are time-independent and constitute an infinite number of constants of motion. The first few global conservation laws are listed below (here $\sigma = \mp 1$):

$$\mathcal{C}_0 = \int_{-\infty}^{+\infty} q(x,t)q^*(-x,t)dx \ ,$$

$$\mathcal{C}_1 = \int_{-\infty}^{+\infty} \left[q_x(x,t)q^*(-x,t) + q(x,t)q_x^*(-x,t)\right] dx \ ,$$

$$\mathcal{C}_2 = \int_{-\infty}^{+\infty} \left[q_x(x,t)q_x^*(-x,t) - \sigma q^2(x,t){q^*}^2(-x,t)\right] dx \ .$$

In the context of PT symmetric classical optics, the quantity $\mathcal{C}_0$ is reffered to as the "quasipower". We also note that equation (1) is an integrable Hamiltonian system with Hamiltonian given by $\mathcal{C}_2$.The local conservation laws (both densities and fluxes) can also be derived from the linear pair. They are given by $\partial_t\mu_n(x,t) + i\partial_x F_n(x,t) = 0$ where the fluxes are $F_n(x,t) = \frac{q_x(x,t)}{q(x,t)}\mu_n(x,t) - \mu_{n+1}(x,t), \ n = 0,1,2,\cdots$. The first two local conservation laws are

$$\begin{aligned} &\partial_t \left[q(x,t)q^*(-x,t)\right] \\ &+ i\partial_x \left[q(x,t)q_x^*(-x,t) + q^*(-x,t)q_x(x,t)\right] = 0 \ , \end{aligned}$$

$$\begin{aligned} &\partial_t \left[q(x,t)q_x^*(-x,t)\right] \\ &+ \ i\partial_x[q_x^*(-x,t)q_x(x,t) + q(x,t)q_{xx}^*(-x,t) \\ &\quad - \sigma q^2(x,t){q^*}^2(-x,t)] = 0 \ . \end{aligned}$$

Direct scattering problem. We define the functions $M(x,k) = e^{ikx}\phi(x,k)$, $\overline{M}(x,k) = e^{-ikx}\overline{\phi}(x,k)$ and $N(x,k) = e^{-ikx}\psi(x,k)$, $\overline{N}(x,k) = e^{ikx}\overline{\psi}(x,k)$ satisfying constant boundary conditions induced from (7). One can then obtain an integral representations for the above functions and show that $M(x,k), N(x,k)$ are analytic functions in the upper half complex plane whereas $\overline{M}(x,k), \overline{N}(x,k)$ are analytic functions in the lower half complex plane [16]. The solutions $\phi(x,k)$ and $\overline{\phi}(x,k)$ of the scattering problem (2) with the boundary conditions (7) are linearly independent. This follows from the fact that the Wronskian, $W(u,v) \equiv u_1v_2 - u_2v_1$ of any two solutions u and v to (2) is independent of x. Similar arguments hold for $\psi(x,k)$ and $\overline{\psi}(x,k)$. Therefore because the scattering problem (2) is a second order linear ODE, the pairs $\{\phi, \overline{\phi}\}$ and $\{\psi, \overline{\psi}\}$ are linearly dependent and one can express one set of basis in terms of the other:

$$\Phi(x,k) = S(k)\Psi(x,k) \ , \tag{9}$$

where $\Phi(x,k) \equiv (\phi(x,k), \ \overline{\phi}(x,k))$, $\Psi(x,k) \equiv (\overline{\psi}(x,k), \ \psi(x,k))$ and $S(k)$ is the scattering matrix

$$S(k) = \begin{pmatrix} a(k) & b(k) \\ \overline{b}(k) & \overline{a}(k) \end{pmatrix} \ . \tag{10}$$

Then the scattering data are expressed as $a(k) = W(\phi(x,k), \psi(x,k))$, $\overline{a}(k) = W(\overline{\psi}(x,k), \overline{\phi}(x,k))$, $b(k) = W(\overline{\psi}(x,k), \phi(x,k))$, $\overline{b}(k) = W(\overline{\phi}(x,k), \psi(x,k))$. Moreover, it can be shown that $a(k), \overline{a}(k)$ are respectively analytic functions in the upper/lower half complex plane. In general, $b(k), \overline{b}(k)$ need not be analytic anywhere. As stated above, the nonlocal NLS equation (1) is a special case of the system (4) and (5) under the symmetry reduction $r(x,t) = \mp q^*(-x,t)$. This symmetry in the potential induces a symmetry in the eigenfunctions that in turn imposes a symmetry in the scattering data. Indeed, if $(\phi_1(x,k), \phi_2(x,k))^T$ satisfies Eq. (2) and the symmetry (6) *holds*, then $(\phi_2^*(-x,-k^*), \pm\phi_1^*(-x,-k^*))^T$ also satisfies the scattering problem (2). Similar symmetry result holds for $\overline{\phi}(x,k)$. Therefore, because the solutions of the scattering problem (9) are uniquely determined by their respective boundary conditions (7) we obtain the important symmetry relations *valid* for $r(x) = \mp q^*(-x)$

$$\begin{aligned} N(x,k) &= \Lambda M^*(-x,-k^*), \\ \overline{N}(x,k) &= \Lambda^{-1}\overline{M}^*(-x,-k^*), \end{aligned} \tag{11}$$

where Λ is a 2×2 matrix with zeros on the main diagonal and $1, \pm 1$ on the lower and upper diagonal respectively. From the Wronskian representations for the scattering data it follows $a(k) = a^*(-k^*)$,$\overline{a}(k) = \overline{a}^*(-k^*)$ and $\overline{b}(k) = \mp b^*(-k^*)$. These relations imply that if k_j is a zero (eigenvalue) of $a(k)$ then $-k_j^*$ is a zero of $a(k)$. Similarly, if $\overline{k}_j$ is a zero of $\overline{a}(k)$ so is $-\overline{k}_j^*$. In what follows, we assume that the eigenvalues $k_\ell, \overline{k}_\ell$ are only on the imaginary axis.

Inverse scattering problem: Left-right RH approach. The inverse problem consists of constructing the potential functions $r(x,t)$ and $q(x,t)$ from the scattering data (reflection coefficients) $\rho(k,t) = e^{-4ik^2t}b(k,0)/a(k,0)$ and $\overline{\rho}(k,t) = e^{4ik^2t}\overline{b}(k,0)/\overline{a}(k,0)$ defined on $\mathrm{Im}k = 0$ as well as the eigenvalues $k_j, \overline{k}_j$ and norming constants (in x) $C_j(t), \overline{C}_j(t)$. Hereafter, for simplicity of notation, we suppress the time dependence. Using the RH approach, from equation (9) one can find equations governing the eigenfunctions $N(x,k), \overline{N}(x,k)$

$$\begin{aligned} \overline{N}(x,k) &= \begin{pmatrix}1\\0\end{pmatrix} + \sum_{j=1}^{J} \frac{C_j e^{2ik_jx} N(x,k_j)}{k-k_j} \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\zeta)e^{2i\zeta x}N(x,\zeta)}{\zeta-(k-i0)}d\zeta \ , \end{aligned} \tag{12}$$

$$\begin{aligned} N(x,k) &= \begin{pmatrix}0\\1\end{pmatrix} + \sum_{j=1}^{\overline{J}} \frac{\overline{C}_j e^{-2i\overline{k}_jx} \overline{N}(x,\overline{k}_j)}{k-\overline{k}_j} \\ &- \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\overline{\rho}(\zeta)e^{-2i\zeta x}\overline{N}(x,\zeta)}{\zeta-(k+i0)}d\zeta \ . \end{aligned} \tag{13}$$

The time evolution of the norming constants are given by $C_j(t) = C_j(0)e^{-4ik_j^2t}$, $\overline{C}_j(t) = \overline{C}_j(0)e^{4i\overline{k}_j^2t}$. To close the system we substitute $k = \overline{k}_\ell$ and $k = k_\ell$ in (12) and (13) respectively and obtain a linear algebraic integral system of equations that solve the inverse problem for the eigenfunctions $N(x,k)$ and $\overline{N}(x,k)$. To account for the symmetry condition (6), we view system (9) as a left scattering problem; we supplement it with the right scattering problem

$$\Psi(x,k) = \mathcal{S}(k)\Phi(x,k) \ , \tag{14}$$

where

$$\mathcal{S}(k) = \begin{pmatrix} \overline{\alpha}(k) & \overline{\beta}(k) \\ \beta(k) & \alpha(k) \end{pmatrix} \ . \tag{15}$$

In the same way as for the left RH above, we can formulate the corresponding RH problem on the right and find the following linear integral equations which govern the functions $M(x,k), \overline{M}(x,k)$:

$$\begin{aligned} M(x,k) &= \begin{pmatrix}1\\0\end{pmatrix} + \sum_{\ell=1}^{\overline{J}} \frac{\overline{B}_\ell e^{2i\overline{k}_\ell x} \overline{M}(x,\overline{k}_\ell)}{k-\overline{k}_\ell} \\ &- \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\overline{R}(\zeta)e^{2i\zeta x}\overline{M}(x,\zeta)}{\zeta-(k+i0)}d\zeta \ , \end{aligned} \tag{16}$$

$$\begin{aligned} \overline{M}(x,k) &= \begin{pmatrix}0\\1\end{pmatrix} + \sum_{\ell=1}^{J} \frac{B_\ell e^{-2ik_\ell x} M(x,k_\ell)}{k-k_\ell} \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{R(\zeta)e^{-2i\zeta x}M(x,\zeta)}{\zeta-(k-i0)}d\zeta \ , \end{aligned} \tag{17}$$

where $R(k) = \beta(k)/\alpha(k)$ and $\overline{R}(k) = \overline{\beta}(k)/\overline{\alpha}(k)$ are the reflection coefficients. The time evolution of the norming constants (in x) are given by $B_\ell(t) = B_\ell(0)e^{4ik_\ell^2t}$, $\overline{B}_\ell(t) = \overline{B}_\ell(0)e^{-4i\overline{k}_\ell^2t}$. From the symmetry relation (11) it follows that $B_\ell = \mp C_\ell^*$ and $\overline{B}_\ell = \mp\overline{C}_\ell^*$. Using the relation between the two scattering matrices, i.e., $\mathcal{S}(k) = S^{-1}(k)$ we find $R^*(-k) = \pm\rho(k)$ and $\overline{R}^*(-k) = \pm\overline{\rho}(k), k$ is real. To close the system we substitute $k = k_j$ and $k = \overline{k}_j$ in (16) and (17) respectively and obtain a linear algebraic integral system of equations that solve the inverse problem for the eigenfunctions $M(x,k), \overline{M}(x,k)$. It is also interesting to note that at the eigenvalues $k_\ell, \overline{k}_\ell$ the eigenfunctions satisfy the relation

$$N_2(x,k_\ell)N_2^*(-x,k_\ell) = N_1(x,k_\ell)N_1^*(-x,k_\ell) \ , \tag{18}$$

$$\overline{M}_2(x,\overline{k}_\ell)\overline{M}_2^*(-x,\overline{k}_\ell) = \overline{M}_1(x,\overline{k}_\ell)\overline{M}_1^*(-x,\overline{k}_\ell) \ . \tag{19}$$

Recovery of the potentials. To reconstruct the potentials $q(x)$, $r(x)$ we compare the asymptotic expansions of Eq. (12) and (13) to that of $\overline{M}(x,k), \overline{N}(x,k), M(x,k), N(x,k)$ at large k and use the symmetry relation (11) between the eigenfunctions to find

$$\begin{aligned} r(x) &= -2i\sum_{j=1}^{J} C_j e^{2ik_jx} N_2(x,k_j) \\ &+ \frac{1}{\pi}\int_{-\infty}^{+\infty} \rho(\zeta)e^{2i\zeta x}N_2(x,\zeta)d\zeta \ , \end{aligned} \tag{20}$$

$$\begin{aligned} q(x) &= \mp 2i\sum_{\ell=1}^{J} C_\ell^* e^{2ik_\ell^* x} N_2^*(-x,k_\ell) \\ &\mp \frac{1}{\pi}\int_{-\infty}^{\infty} \rho^*(\zeta)e^{2i\zeta x}N_2^*(-x,\zeta)d\zeta \ . \end{aligned} \tag{21}$$

From equations (20) and (21) it is now obvious that the symmetry $r(x) = \mp q^*(-x)$ is automatically preserved.

Soliton solutions. In the case where the scattering data only comprise eigenvalues off the real axis and $\rho(k) = 0$, $\overline{\rho}(k) = 0$ for all k, the inverse scattering system (12), (13) with (16), (17) subject to the symmetry relations (11) reduces to finite-dimensional linear algebraic equations for $N(x, k_j)$ and $\overline{M}(x, \overline{k}_j)$. Recall that, when $r(x) = \mp q^*(-x)$, the eigenvalues appear in pairs $\{k_j,\ -k_j^*\}$ and $\{\overline{k}_j,\ -\overline{k}_j^*\}$. Thus, the one-soliton solution to the focusing nonlocal NLS equation (1) (with a + sign), corresponds to a single pure imaginary eigenvalue ($J = \overline{J} = 1$) of the form $k_1 = i\eta_1$, $\overline{k}_1 = -i\overline{\eta}_1$, $\eta_1 \neq \overline{\eta}_1$ where $\eta_j > 0, j = 1, 2$. Using equations (18) and (19) we also find $|C_1(0)|^2 = |\overline{C}_1(0)|^2 = (\eta_1 + \overline{\eta}_1)^2$. Letting $\overline{C}_1(0) = |\overline{c}_1|e^{i(\overline{\theta}_1+\pi/2)}$ and $C_1(0) = |c_1|e^{i(\theta_1+\pi/2)}$, then the most general one-soliton solution to Eq. (1) is given by

$$q(x) = -\frac{2(\eta_1+\overline{\eta}_1)e^{i\overline{\theta}_1}e^{-4i\overline{\eta}_1^2 t}e^{-2\overline{\eta}_1 x}}{1+e^{i(\theta_1+\overline{\theta}_1)}e^{4i(\eta_1^2-\overline{\eta}_1^2)t}e^{-2(\eta_1+\overline{\eta}_1)x}} \ . \tag{22}$$

The family of solutions (22) is characterized by four independent parameters that breathes in time and eventually develops a singularity in finite time $t = t_s$ at $x = 0$ with

$$t_s = \frac{(2n+1)\pi - \theta_1 - \overline{\theta}_1}{4(\eta_1^2 - \overline{\eta}_1^2)} \ , \ \ n \in \mathbb{Z} \ . \tag{23}$$

As pointed out earlier, the classical NLS equation is recovered when the nonlocal nonlinear term $q^*(-x)$ is replaced by $q^*(x)$. In this regard, the one soliton solution for the classical NLS equation is obtained from (22) by letting $\overline{\eta}_1 = \eta_1$ and $\overline{C}_1(0) = -C_1^*(0)$, i.e., $\theta_1 + \overline{\theta}_1 = 0$.

A nonlocal NLS hierarchy. In this section, we discuss a class of nonlocal nonlinear evolution equations associated with the AKNS scattering problem (2) that are integrable and solvable by IST. Following closely the derivation outlined in [8] we obtain

$$\sigma_3 \mathsf{u}_t(x,t) = i\omega\left(-2L\right)\mathsf{u}(x,t) \ , \tag{24}$$

where, for example, $\omega(\zeta) = c\zeta^{2n}, n$ is a positive integer, c is constant and

$$L \equiv \frac{1}{2i}\begin{pmatrix} \partial_x + 2rI_+q & -2rI_+r \\ 2qI_+q & -\partial_x - 2qI_+r \end{pmatrix} \ , \tag{25}$$

where $I_+ = \int_x^{+\infty} dy$ is an integral operator. $\mathsf{u}(x,t) \equiv (r(x,t)\ ,\ q(x,t))^T$ and $\sigma_3 \equiv diag(1,-1)$ is a 2×2 diagonal matrix and $r(x,t) = \mp q^*(-x,t)$. An example is the nonlocal NLS equation (1) which is obtained from (24) by choosing $\omega(\zeta) = -\zeta^2$. Another example is the following nonlocal PDE obtained from (24) using the dispersion relation $\omega(\zeta) = \zeta^4$ and taking $r(x,t) = \mp q^*(-x,t)$:

$$iq_t = q_{xxxx} - 2q\theta - 6\left(qrq_x\right)_x + 6q^3r^2 \ , \tag{26}$$

$$\theta = qr_{xx} + rq_{xx} - q_x r_x \ . \tag{27}$$

Note also, the function $\omega(k)$ is the dispersion relation associated with the linear part of the evolution equation under the substitution $q \sim \exp(ikx - i\omega t)$.

Comparison with the classical NLS equation. In this part we briefly contrast the properties of the nonlocal NLS (1) with that of the classical (local) NLS equation:

$$iu_t = u_{xx} \pm 2|u|^2 u \ . \tag{28}$$

Three different scenarios will be addressed all of which concerning Eq. (28): (i) general and (ii) even initial conditions posed on the whole real line and (iii) general initial conditions on the semi-infinite interval ($x \geq 0$.) In [3], it was shown that (28) is integrable on the whole real line. Furthermore, it was found that the symmetries of the eigenfunctions of the associated Zakharov-Shabat scattering problem are such that the eigenfunctions in the upper half complex plane are related to those in the lower half plane. This is in sharp contrast to the nonlocal case where the eigenfunctions at the upper/lower half plane are not related. On the other hand, if one restricts the class of initial conditions to be even (in x) then one obtains extra symmetry conditions on the scattering data that resembles the one we find. This leads us to the important conclusion that soliton solutions to (1) will have a classical NLS limit so long (28) admits an *even* soliton solution. Finally, we point out that similar symmetry results were obtained in [17] for the classical NLS (28) on the semi-infinite interval.

Nonlocal Painlevé type equations. The Painlevé equations are certain class of nonlinear second-order complex ordinary differential equations that normally arise as reductions of the "soliton evolution equations" which are solvable by IST cf. [6, 18, 19]. They are particularly interesting due to their properties in the complex plane and their associated integrability properties. In this section we propose nonlocal analogues of Painlevé type equation. There are two prototypes. First, look for a self similar solution to the nonlocal NLS equation (1) of the form

$$q(x,t) = \frac{1}{(2t)^{1/2}} f(z) e^{i\nu \log t/2}, \ \ z = x/(2t)^{1/2}. \tag{29}$$

where ν is a real constant. Then, upon substituting this ansatz into (1) we find a nonlocal Painlevé type equation

$$f_{zz}(z) - \alpha f(z) - izf_z(z) + 2\sigma f^2(z) f^*(-z) = 0 \ . \tag{30}$$

where $\alpha = (\nu + i)$ and $\sigma = \mp 1$. It should be noted that substituting the same ansatz (29) into the classical NLS equation leads to the above equation (30) where the nonlinear term is now evaluated at z; this equation is of Painlevé type. Similarly, if we substitute $q(x,t) = \frac{1}{\sqrt{2}}e^{i\lambda t}f(x),\ \lambda, f \in \mathbb{R}$ into equation (1) we find

$$f_{xx}(x) + f^2(x)f(-x) - \lambda f(x) = 0 \ . \tag{31}$$

Equation (31) is a nonlocal analogue of the elliptic function associated with the classical (local) NLS equation where the the nonlinear term is now evaluated at x which has an elliptic function solution. Elliptic functions are known to be of Painlevé type; i.e. their solution has movable poles, but no movable branch points.

Conclusion. A nonlocal nonlinear Schrödinger equation is found from a new and simple reduction of the well-known AKNS system. It has a Lax pair and an infinite number of conservation laws. The inverse scattering transform (IST) for decaying data is developed and a one breathing soliton solution is found. The IST requires different scattering data symmetries than the classical NLS equation. A nonlocal NLS hierarchy as well as novel nonlocal Painlevé type equations are also derived.

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