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An Integrable Nonlocal Nonlinear Schrödinger Equation

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A new integrable nonlocal nonlinear Schrödinger equation is introduced. It possesses a Lax pair and an infinite number of conservation laws and is $PT$-symmetric. The inverse scattering transform and scattering data with suitable symmetries are discussed. A method to find pure soliton solutions is given. An explicit breathing one soliton solution is found. Key properties are discussed and contrasted with the classical NLS equation.

Introduction. In the study of nonlinear wave propagation exactly solvable models play exceptional role. There are many physically important integrable equations. Examples include small amplitude waves in shallow water where the Korteweg-de-Vries (KdV) equation [1] and its multidimensional analog, the Kudomtsev-Petviashvili equation [2] arise; in generic weakly nonlinear dispersive systems, in the quasi-monochromatic limit the integrable cubic nonlinear Schrödinger equation [3] is applicable. Furthermore, in nonlinear optics the integrable cubic nonlinear Schrödinger equation is a key equation describing optical wave propagation in Kerr media [4, 5]. Indeed there are many physically significant integrable systems [6] which apply to diverse problems in fluid mechanics, electromagnetics, gravitational waves, elasticity, fundamental physics and lattice dynamics, to name but a few.

Generally speaking integrability is established once an infinite number of constants of motion or an infinite number of conservation laws are obtained. However considerably more information about the solution can be obtained if the inverse scattering transform (IST) can be carried out [7]. Corresponding to rapidly decaying initial data, IST provides a linearization and a class of explicit solutions--i.e. solitons. The method associates a compatible pair of linear equations (i.e. a Lax pair) with the integrable nonlinear equation. One of the equations, the scattering problem, is used to determine suitably analytic integrable nonlinear equation. One of the equations, the linear term $q^*(−x, t)$ is replaced by $q^*(x, t)$. Indeed we note that both equation (1) and the classical NLS share the symmetry that when $x → −x$, $t → −t$ and a complex conjugate is taken, then the equation remains invariant. Thus, the new nonlocal equation is $PT$ symmetric [11] which, in the case of classical optics, amounts to the invariance of the so-called self-induced potential cf. [12] $V(x, t) = q(x, t)q^*(−x, t)$ under the combined action of parity and time reversal symmetry. Finally, wave propagation in $PT$ symmetric coupled waveguides/photonic lattices has been experimentally observed in classical optics [13–15].

Linear pair and the nonlocal NLS equation. We begin our analysis by considering the following scattering problem [8, 16]

\[
v_x = \begin{pmatrix} -ik & q(x, t) \\ r(x, t) & ik \end{pmatrix} v, \tag{2}
\]

\[
v_t = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v, \tag{3}
\]

where $v$ is a two-component vector, $v(x, t) = (v_1(x, t), v_2(x, t))^T$, $q(x, t)$ and $r(x, t)$ vanish rapidly as $x → ±∞$. $k$ is a spectral parameter and $A = 2ik^2 + iq(x, t)r(x, t)$, $B = -2kq(x, t) - iq_x(x, t)$, $C = -2kr(x, t) + ir_x(x, t)$. The compatibility condition of system (2) and (3) i.e., $v_{xt} = v_{tx}$ yields

\[
iq_t(x, t) = q_{xx}(x, t) - 2r(x, t)q^2(x, t), \tag{4}
\]

\[-ir_t(x, t) = r_{xx}(x, t) - 2q(x, t)r^2(x, t). \tag{5}\]

In this Letter, the following nonlocal nonlinear Schrödinger equation is introduced and investigated in detail

\[
iq_t(x, t) = q_{xx}(x, t) ± 2q(x, t)q^*(−x, t)q(x, t), \tag{1}\]

where $*$ denotes complex conjugation and $q(x, t)$ is a complex valued function of the real variables $x$ and $t$. Eq. (1) admits a linear (Lax) pair formulation and possesses an infinite number of conservation laws, hence it is an integrable system. Via the inverse scattering transform, corresponding to rapidly decaying initial data, one can linearize the equation and obtain solutions to Eq. (1) including pure solitons solutions. Some of the important properties of the nonlocal NLS equation are contrasted with the classical NLS equation where the nonlocal nonlinear term $q^*(−x, t)$ is replaced by $q^*(x, t)$. Indeed we note that both equation (1) and the classical NLS share the symmetry that when $x → −x$, $t → −t$ and a complex conjugate is taken, then the equation remains invariant. Thus, the new nonlocal equation is $PT$ symmetric [11] which, in the case of classical optics, amounts to the invariance of the so-called self-induced potential cf. [12] $V(x, t) = q(x, t)q^*(−x, t)$ under the combined action of parity and time reversal symmetry. Finally, wave propagation in $PT$ symmetric coupled waveguides/photonic lattices has been experimentally observed in classical optics [13–15].
Equation (1) is then obtained from system (4) and (5) under the symmetry reduction
\begin{equation}
  r(x,t) = \mp q^*(x,t) .
\end{equation}

Importantly, the symmetry reduction (6) is new and leads to a new class of nonlocal integrable PDEs including a nonlocal NLS hierarchy. This is a special and remarkably simple reduction of the more general AKNS system [7] which has not been previously found.

**Infinite number of conserved quantities and conservation laws.** The infinite number of conserved quantities of (1) can be derived as follows. We assume that \( q(x,t) \) decays rapidly at infinity. Then solutions of the scattering problem (2) can be defined. Indeed we define four eigenfunctions which satisfy the following boundary conditions
\begin{equation}
  \phi \sim \left( \begin{array}{c}
  1 \\
  0 
  \end{array} \right) e^{-ikx} , \quad \overline{\phi} \sim \left( \begin{array}{c}
  0 \\
  1 
  \end{array} \right) e^{ikx} , \quad \text{as } x \to -\infty
\end{equation}
\begin{equation}
  \psi \sim \left( \begin{array}{c}
  0 \\
  1 
  \end{array} \right) e^{ikx} , \quad \overline{\psi} \sim \left( \begin{array}{c}
  1 \\
  0 
  \end{array} \right) e^{-ikx} , \quad \text{as } x \to +\infty .
\end{equation}

Note that \( \overline{\phi} \) is not the complex conjugate of \( \phi \). We use \( \phi^* \) to denote complex conjugation of \( \phi \). If \( (\phi(x,t), \overline{\phi}(x,t)) \) is the solution to (2) that satisfies the above boundary conditions then, for Im \( k \geq 0 \), \( \phi_1(x,t) k is analytic and approaches 1 as \( x \to \pm \infty \). Substituting \( \phi(x,t) = \exp[-ikx + \phi(x,t)] \) into (2) we find (after eliminating \( \phi_2 \)) that the function \( \mu(x,t) = \varphi_x(x,t) \) satisfies the Riccati equation
\begin{equation}
  q \frac{\partial}{\partial t} \left( \frac{\mu}{q} \right) + \mu^2 - qr - 2ik\mu = 0 .
\end{equation}

For Im \( k > 0 \), \( \lim_{|k|\to\infty} \varphi(x,k) = 0 \). Substituting the expansion \( \mu(x,k) = \sum_{n=0}^{\infty} \mu_n(x,t)/(2ik)^{n+1} \) into (8) to find \( \mu_0 = -qr, \mu_1 = -qr_x \) and a recursion relation for any \( n \geq 1 \) cf.[8]. From (2) it follows that the scattering data \( a(k) \equiv \lim_{x \to -\infty} \varphi_1(x,t) e^{ikx} \) is time-independent. Since \( \varphi(x,t) \) vanishes as \( x \to -\infty \) we conclude that \( C_n \equiv \int_{-\infty}^{+\infty} \mu_n(x,t) dx \) are time-independent and constitute an infinite number of constants of motion. The first few global conservation laws are listed below (here \( \sigma = \mp 1 \)):
\begin{equation}
  C_0 = \int_{-\infty}^{+\infty} q(x,t) q^*(-x,t) dx ,
\end{equation}
\begin{equation}
  C_1 = \int_{-\infty}^{+\infty} \left[ q_x(x,t) q^*(-x,t) + q(x,t) q_x^*(-x,t) \right] dx ,
\end{equation}
\begin{equation}
  C_2 = \int_{-\infty}^{+\infty} \left[ q_x(x,t) q_x^*(-x,t) - \sigma q^2(x,t) q^2(-x,t) \right] dx .
\end{equation}

In the context of \( PT \) symmetric classical optics, the quantity \( C_0 \) is referred to as the “quasipower”. We also note that equation (1) is an integrable Hamiltonian system with Hamiltonian given by \( C_0 \). The local conservation laws (both densities and fluxes) can be derived from the linear pair. They are given by \( \partial_t \mu_n(x,t) + i \partial_x F_n(x,t) = 0 \) where the fluxes are \( F_n(x,t) = \frac{q_n(x,t)}{q(x,t)} \mu_n(x,t) - \mu_{n+1}(x,t), n = 0,1,2, \ldots \).

The first few local conservation laws are
\begin{equation}
  \partial_t [q(x,t) q^*(-x,t)] + i \partial_x [q(x,t) q_x^*(-x,t) + q^*(-x,t) q_x(x,t)] = 0 ,
\end{equation}
\begin{equation}
  \partial_t [\overline{q}(x,t) q^*(x,t)] + i \partial_x [\overline{q}(x,t) q_x^*(x,t) + q(x,t) q_x^*(x,t)]
  - \sigma q^2(x,t) q^2(-x,t) = 0 .
\end{equation}

**Direct scattering problem.** We define the functions \( M(x,k) = e^{ikx} \varphi(x,k), \overline{M}(x,k) = e^{-ikx} \overline{\varphi}(x,k) \) and \( N(x,k) = e^{ikx} \psi(x,k), \overline{N}(x,k) = e^{ikx} \overline{\psi}(x,k) \) satisfying constant boundary conditions induced from (7). One can then obtain an integral representations for the above functions and show that \( M(x,k), N(x,k) \) are analytic functions in the upper half complex plane whereas \( \overline{M}(x,k), \overline{N}(x,k) \) are analytic functions in the lower half complex plane [16]. The solutions \( \varphi(x,k) \) and \( \overline{\varphi}(x,k) \) of the scattering problem (2) with the boundary conditions (7) are linearly independent. This follows from the fact that the Wronskian, \( W(u,v) \equiv u_1 v_2 - u_2 v_1 \) of any two solutions \( u \) and \( v \) to (2) is independent of \( x \). Similar arguments hold for \( \psi(x,k) \) and \( \overline{\psi}(x,k) \). Therefore because the scattering problem (2) is a second order linear ODE, the pairs \( \{ \phi, \overline{\phi} \} \) and \( \{ \psi, \overline{\psi} \} \) are linearly dependent and one can express one set of basis in terms of the other:
\begin{equation}
  \Phi(x,k) = S(k) \Psi(x,k) ,
\end{equation}
where \( \Phi(x,k) \equiv (\phi(x,k), \overline{\phi}(x,k)), \Psi(x,k) \equiv (\psi(x,k), \overline{\psi}(x,k)) \) and \( S(k) \) is the scattering matrix
\begin{equation}
  S(k) = \begin{pmatrix}
  a(k) & b(k) \\
  b(k) & \overline{a}(k)
  \end{pmatrix} .
\end{equation}

Then the scattering data are expressed as \( a(k) = W(\phi(x,k), \overline{\phi}(x,k)), \overline{a}(k) = W(\overline{\psi}(x,k), \overline{\phi}(x,k)), b(k) = W(\overline{\psi}(x,k), \phi(x,k)), \overline{b}(k) = W(\overline{\phi}(x,k), \psi(x,k)) \). Moreover, it can be shown that \( a(k), \overline{a}(k) \) are respectively analytic functions in the upper/lower half complex plane. In general, \( b(k), \overline{b}(k) \) need not be analytic anywhere. As stated above, the nonlocal NLS equation (1) is a special case of the system (4) and (5) under the symmetry reduction \( r(x,t) = \mp q^*(-x,t) \). This symmetry in the potential induces a symmetry in the eigenfunctions that in turn imposes a symmetry in the scattering data. Indeed, if \( (\phi_1(x,k), \phi_2(x,k))^T \) satisfies Eq. (2) and the symmetry (6) holds, then \( (\phi_2^*(-x,-k^*), \pm \phi_1^*(-x,-k^*))^T \) also satisfies the scattering problem (2). Similar symmetry result holds for \( \overline{a}(x,k) \). Therefore, because the solutions of the
scattering problem (9) are uniquely determined by their respective boundary conditions (7) we obtain the important symmetry relations valid for \( r(x) = \mp q^*(-x) \)

\[
N(x, k) = \Lambda M^*(-x, -k^*),
\]

\[
N(x, k) = \Lambda^{-1} \overline{M}^*(-x, -k^*),
\]

where \( \Lambda \) is a \( 2 \times 2 \) matrix with zeros on the main diagonal and \( 1, \pm 1 \) on the lower and upper diagonal respectively. From the Wronskian representations for the scattering data it follows \( a(k) = a^*(-k^*) \), \( \overline{\alpha}(k) = \overline{\alpha}(k^*) \) and \( \overline{\beta}(k) = \overline{\beta}(k^*) \). These relations imply that if \( k_j \) is a zero (eigenvalue) of \( a(k) \) then \(-k_j^*\) is a zero of \( a(k) \). Similarly, if \( \overline{k}_j \) is a zero of \( \overline{\alpha}(k) \) so is \( -\overline{k}_j \). In what follows, we assume that the eigenvalues \( k_j, \overline{k}_j \) are only on the imaginary axis.

**Inverse scattering problem: Left-right RH approach.** The inverse problem consists of constructing the potential functions \( r(x, t) \) and \( q(x, t) \) from the scattering data (reflection coefficients) \( \rho(k, t) = e^{-4ik^2t} / a(k, 0) \) and \( \overline{\alpha}(k, t) = e^{4ik^2t} / \overline{\alpha}(k, 0) \) defined on \( \text{Im} k = 0 \) as well as the eigenvalues \( k_j, \overline{k}_j \) and norming constants (in \( x \)) \( C_j(t), \overline{C}_j(t) \). Hereafter, for simplicity of notation, we suppress the time dependence. Using the RH approach, from equation (9) one can find equations governing the eigenfunctions \( N(x, k), \overline{N}(x, k) \)

\[
N(x, k) = \begin{pmatrix} 1 & \sum_{j=1}^J C_j e^{2ik_j x} \frac{N(x, k_j)}{k - k_j} \\ 0 & 1 \end{pmatrix} + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\rho(\zeta) e^{2i\zeta x} N(x, \zeta)}{\zeta - (k - i0)} d\zeta,
\]

\[
N(x, k) = \begin{pmatrix} 0 & 1 \end{pmatrix} + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\overline{\alpha}(\zeta) e^{-2i\zeta x} \overline{N}(x, \zeta)}{\zeta - (k - i0)} d\zeta.
\]

The time evolution of the norming constants are given by \( \overline{C}_j(t) = C_j(0) e^{-4ik_j^2 t} \), \( \overline{C}_j(t) = C_j(0) e^{4ik_j^2 t} \). To close the system we substitute \( k = \overline{k}_j \) and \( k = k_j \) in (12) and (13) respectively and obtain a linear algebraic integral system of equations that solve the inverse problem for the eigenfunctions \( N(x, k), \overline{N}(x, k) \). To account for the symmetry condition (6), we view system (9) as a left scattering problem; we supplement it with the right scattering problem

\[
\Psi(x, k) = S(k) \Phi(x, k),
\]

where

\[
S(k) = \begin{pmatrix} \overline{\alpha}(k) & \overline{\beta}(k) \\ \beta(k) & \alpha(k) \end{pmatrix}.
\]

In the same way as for the left RH above, we can formulate the corresponding RH problem on the right and find the following linear integral equations which govern the functions \( M(x, k), \overline{M}(x, k) \):

\[
M(x, k) = \begin{pmatrix} 1 & \sum_{\ell=1}^J B_\ell e^{2ik_\ell x} M(x, k_\ell) \\ 0 & 1 \end{pmatrix} + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\overline{R}(\zeta) e^{2i\zeta x} \overline{M}(x, \zeta)}{\zeta - (k - i0)} d\zeta,
\]

\[
\overline{M}(x, k) = \begin{pmatrix} 0 & 1 \end{pmatrix} + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{R(\zeta) e^{-2i\zeta x} M(x, k_\ell)}{\zeta - (k - i0)} d\zeta.
\]

Recovery of the potentials. To reconstruct the potentials \( q(x) \), \( r(x) \) we compare the asymptotic expansions of Eq. (12) and (13) to that of \( M(x, k), \overline{N}(x, k), M(x, k), N(x, k) \) at large \( k \) and use the symmetry relation (11) between the eigenfunctions to find

\[
r(x) = -2i \sum_{j=1}^J C_j e^{2ik_j x} N_2(x, k_j)
\]

\[
+ \frac{1}{\pi} \int_{-\infty}^{+\infty} \rho(\zeta) e^{2i\zeta x} N_2(x, \zeta) d\zeta,
\]

\[
q(x) = \mp 2i \sum_{\ell=1}^J C_\ell^* e^{2ik_\ell x} N_2^*(x, k_\ell)
\]

\[
\mp \frac{1}{\pi} \int_{-\infty}^{+\infty} \rho^*(\zeta) e^{2i\zeta x} N_2^*(x, \zeta) d\zeta.
\]

From equations (20) and (21) it is now obvious that the symmetry \( r(x) = \mp q^*(-x) \) is automatically preserved.
Soliton solutions. In the case where the scattering data only comprise eigenvalues off the real axis and \( \rho(k) = 0, \overline{\sigma}(k) = 0 \) for all \( k \), the inverse scattering system (12), (13) with (16), (17) subject to the symmetry relations (11) reduces to finite-dimensional linear algebraic equations for \( N(x, k_j) \) and \( \overline{M}(x, k_j) \). Recall that, when \( r(x) = -q^*(-x) \), the eigenvalues appear in pairs \( \{ k_j, -k_j^* \} \) and \( \{ k_j^*, -k_j \} \). Thus, the one-soliton solution to the focusing nonlocal NLS equation (1) which is obtained from (24) by substituting \( q(x) = x \) is given by

\[
q(x) = -\frac{2(q_1 + q_2)}{1 + e^{i(\theta_1 + \pi)} e^{4q_1 q_2}} e^{-2|q_1| x} \quad .
\]

The family of solutions (22) is characterized by four independent parameters that breathe in time and eventually develops a singularity in finite time \( t = t_s \) at \( x = 0 \) with

\[
t_s = \frac{2n + 1}{4(q_k^2 - q_1^2)} \quad , \quad n \in \mathbb{Z} \quad .
\]

As pointed out earlier, the classical NLS equation is recovered when the nonlocal nonlinear term \( q^*(-x) \) is replaced by \( q^*(x) \). In this regard, the one soliton solution for the classical NLS equation is obtained from (22) by letting \( q_1 = \theta_1 \) and \( q_2 = -\theta_1 \) and \( C_1(0) = 0 \), i.e., \( \theta_1 + \theta_2 = 0 \).

A nonlocal NLS hierarchy. In this section, we discuss a class of nonlocal nonlinear evolution equations associated with the AKNS scattering problem (2) that are integrable and solvable by IST. Following closely the derivation outlined in [8] we obtain

\[
\sigma_3 u_{t} (x, t) = i \omega (2L) u(x, t) \quad ,
\]

where, for example, \( \omega(\zeta) = c\zeta^{2n}, n \) is a positive integer, \( c \) is constant and

\[
L = \frac{1}{2} \left( \partial_x + 2r I q - 2r I q - \partial_x - 2q I q \right) \quad ,
\]

where \( I_q = \int_{-\infty}^{\infty} dy \) is an integral operator. \( u(x, t) \equiv (r(x, t), q(x, t))^T \) and \( \sigma_3 \equiv diag(1, -1) \) is a \( 2 \times 2 \) diagonal matrix and \( r(x, t) = \pm q^*(-x, t) \). An example is the nonlocal NLS equation (1) which is obtained from (24) by choosing \( \omega(\zeta) = -\zeta^2 \). Another example is the following nonlocal PDE obtained from (24) using the dispersion relation \( \omega(\zeta) = -\zeta^2 \) and taking \( r(x, t) = \pm q^*(-x, t) \):

\[
\sigma_3 u_{t} = -q_{xxxx} - 2q\theta - 6(q q_x)_x + 6q^2 r^2 \quad ,
\]

\[
\theta = q_{xx} + r q_{xx} - q_x r_x \quad .
\]
where the nonlinear term is now evaluated at $x$ which has an elliptic function solution. Elliptic functions are known to be of Painlevé type; i.e. their solution has movable poles, but no movable branch points.

**Conclusion.** A nonlocal nonlinear Schrödinger equation is found from a new and simple reduction of the well-known AKNS system. It has a Lax pair and an infinite number of conservation laws. The inverse scattering transform (IST) for decaying data is developed and a one breathing soliton solution is found. The IST requires different scattering data symmetries than the classical NLS equation. A nonlocal NLS hierarchy as well as novel nonlocal Painlevé type equations are also derived.

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