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Thermalization of acoustic excitations in a strongly interacting one-dimensional quantum liquid

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We study inelastic decay of bosonic excitations in a Luttinger liquid. In a model with linear excitation spectrum the decay rate diverges. We show that this difficulty is resolved when the interaction between constituent particles is strong, and the excitation spectrum is nonlinear. Although at low energies the nonlinearity is weak, it regularizes the divergence in the decay rate. We develop a theoretical description of the approach of the system to thermal equilibrium. The typical relaxation rate scales as the fifth power of temperature.

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One-dimensional interacting systems [1] are fundamentally different from their higher-dimensional counterparts [2]. Regardless of the statistics of the constituent particles, elementary excitations in one dimension are believed to be bosons [1, 3, 4], the waves of density. The low-energy properties of the system are commonly described in terms of the *Luttinger liquid* [3, 4] theory of free bosons with linear spectrum $\omega_q = s|q|$ up to a certain cutoff. Here q is the wave number and s is the velocity.

Just as quasiparticles in the Fermi liquid [2], bosons in the Luttinger liquid do not represent exact eigenstates of a generic one-dimensional system. At finite energies, the corresponding effective Hamiltonian should be amended by irrelevant in the renormalization group sense perturbations [3], such as interaction between the bosons. However, a naive attempt to account for this interaction perturbatively immediately leads to difficulties.

Consider, for example, the interaction-induced decay of a boson with wave number q into two bosons with wave numbers q_1' and q_2', see Fig. 1(a). The corresponding inelastic scattering rate is given by the Fermi golden rule,

$$\tau_q^{-1} \propto \int dq_1' dq_2' \,[\ldots]\, \delta(q - q_1' - q_2')\, \delta\big(\omega_q - \omega_{q_1'} - \omega_{q_2'}\big), \quad (1)$$

where the two δ-functions express the momentum and energy conservation. When all three wave numbers have the same sign, the second δ-function reduces to $s^{-1}\delta(q - q_1' - q_2')$, and the rate (1) diverges.

One way around the failure of the perturbation theory is to abandon the effective Luttinger liquid description altogether and approach the problem from the original fermionic perspective [5, 6]. Indeed, for noninteracting fermions the spectral weight of the dynamic structure factor (Fourier transform of the density-density correlation function) at a fixed q is spread uniformly over a narrow interval of the width

$$\delta\omega_q = \hbar\rho^2 q^2/m_* \quad (2)$$

about $\omega = \omega_q$. Here m_* is the effective mass, which for free fermions coincides with the bare mass m, and q is the dimensionless (measured in units of the particle density ρ) wave number. At sufficiently small q, Eq. (2) is applicable to interacting fermions as well [5–7]. The inverse of the width, $1/\delta\omega_q$, provides a natural estimate of the lifetime of bosons in the Luttinger liquid. Since $\delta\omega_q \propto \omega_q^2$, the bosons indeed represent well-defined quasiparticles.

In this Letter we develop an alternative approach, based on the observation that divergences that plague the evaluation of the quasiparticle decay rate in the conventional Luttinger liquid theory can be cured if the boson spectrum is nonlinear, such as

$$\omega_q = s|q|\big(1 - \xi q^2\big). \quad (3)$$

Even for a weak nonlinearity $\xi q^2 \ll 1$, decay of a single boson into two is forbidden by the momentum and energy conservation laws and can only occur virtually. The simplest *real* scattering process involves two bosons both in the initial and in the final states, see Fig. 1(b), and has a finite rate.

Keeping the nonlinear correction in Eq. (3) is justified only in the limit of strong repulsion, i.e., when the Luttinger liquid parameter [1] $K = \pi\hbar\rho^2/ms$ is small. Indeed, the correction must exceed the width $\delta\omega_q$ [see Eq. (2)], which can be viewed as an uncertainty in the

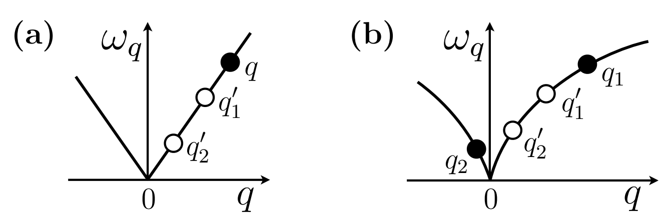

FIG. 1: (a) For bosons with a linear spectrum scattering of a single boson (filled circle) into two (open circles) has a divergent rate. (b) For bosons with a nonlinear spectrum the simplest scattering event satisfying the momentum and energy conservation laws involves two bosons both in the initial state (filled circles) and in the final state (open circles). For given q_1 and q_2, the conservation laws yield a unique set q_1', q_2', thus leading to a finite transition rate.

energy of the Luttinger liquid's boson. Using the estimate [8] $m_*/m \sim \sqrt{K}$, valid for $K \ll 1$, we arrive at the condition $\xi q \gg \sqrt{K}$.

For $K \ll 1$, Eq. (3) is applicable in a broad range of wave numbers $\sqrt{K} \ll \xi q \ll \sqrt{\xi}$, and spectrum nonlinearity has a dramatic effect on inelastic scattering. For the scattering process with two bosons (q_1 and q_2) in the initial state and two bosons (q_1' and q_2') in the final state [see Fig. 1(b)], the conservation laws $q_1+q_2 = q_1'+q_2'$ and $\omega_{q_1} + \omega_{q_2} = \omega_{q_1'} + \omega_{q_2'}$ yield a unique set q_1', q_2' for given q_1, q_2. Moreover, if q_1, q_1', and q_2' belong to the same (say, right-moving) branch of the spectrum [see Fig. 1(b)], the remaining wave number is given by $q_2 \approx -(3\xi/2)q_1q_1'q_2'$, i.e., the sign of q_2 is *opposite* to that of q_1, q_1', q_2' and the momentum transferred from the left-moving branch of the spectrum in each act of scattering is parametrically small compared with that redistributed among the three right-moving bosons. Accordingly, the process resembles decay of a single right-moving boson into two. However, unlike for bosons with strictly linear spectrum, the mere presence of the left-moving boson with a very small momentum, as required by the conservation laws, is sufficient to regularize the divergences.

We describe our strongly interacting system by the Hamiltonian

$$H = \sum_l \frac{p_l^2}{2m} + \frac{1}{2}\sum_{l\neq l'} V(x_l - x_{l'}), \tag{4}$$

where p_l and x_l are, respectively, the momentum and position of the lth particle ($l = 1, \ldots, N$), and $V(x)$ is the interaction potential. In the strong repulsion limit (i.e., for $d^2V/dx^2\big|_{x=1/\rho} \gg \hbar^2\rho^4/m$, which is equivalent to $K \ll 1$) the particles, regardless of their statistics, form at low energies a periodic chain, the so-called *Wigner crystal* (see [9] for a review). Although in one dimension quantum fluctuations destroy the true long-range order [10], the interparticle distance remains close to $1/\rho$.

Similar to ordinary crystals, the elementary excitations of the Wigner crystal are *phonons*. These phonons are nothing but the waves of density, with a typical for phonons linear dispersion at small momenta, i.e., the phonons coincide with the bosons of the effective Luttinger liquid theory. The boson spectrum ω_q in the leading (zero) order in $\hbar$ can be found by expanding the potential energy in Eq. (4) to second order in the displacements of the particles from the corresponding lattice sites $u_l = x_l - l/\rho$, and solving classical equations of motion [11]. For small q, this yields Eq. (3) with model-dependent s and ξ [12].

Interaction between the bosons arises from the higher-order (anharmonic) terms in the expansion of the potential energy in Eq. (3) in the displacements u_l. A scattering process with two bosons both in the initial and in the final states, see Fig. 1(b), can occur either in the first order in the quartic anharmonicity, or in the second order in the cubic anharmonicity [11], and the corresponding contributions to the on-shell scattering amplitude $t_{q_1q_2;q_1'q_2'}$ are of the same order of magnitude. If all four wave numbers are small, the amplitude simplifies [12] to

$$t_{q_1q_2;q_1'q_2'} = \frac{\lambda}{N}\frac{\hbar^2\rho^2}{m}\left|q_1q_2q_1'q_2'\right|^{1/2}. \tag{5}$$

This expression is easy to understand if one notices that each boson with wave number q participating in scattering contributes a factor of $(\hbar/\omega_q)^{1/2}|q| \propto (\hbar|q|)^{1/2}$ to the amplitude. The dimensionless parameter λ in Eq. (5) depends on the functional form of $V(x)$ [12]. In particular, $\lambda = 0$ for $V(x) \propto 1/\sinh^2(c\rho x)$ and $V(x) \propto 1/x^2$ [12], as expected for integrable models [13] exhibiting no relaxation. For a generic interaction potential, however, $|\lambda|$ is of order unity. For the screened Coulomb interaction $\lambda = -3/4$, see [12].

In the absence of integrability, inelastic scattering leads to relaxation of the non-equilibrium boson distribution function N_q towards thermal equilibrium. Such thermalization of N_q is described by the Boltzmann equation, which for a homogeneous system in the absence of external fields has the form [14]

$$\frac{\partial N_q}{\partial t} = \mathcal{I}_{\rm out}\left[N_q\right] + \mathcal{I}_{\rm in}\left[N_q\right], \tag{6}$$

where the two terms in the right-hand side describe, respectively, the scattering out of single-boson state q, and the scattering into this state. In the leading order in $\hbar$, these terms are given by

$$\begin{aligned}\mathcal{I}_{\rm out}\left[N_q\right] &= -\sum_p \sum_{q_1>q_2} W_{q,p;q_1,q_2} N_q N_p (1+N_{q_1})(1+N_{q_2}),\\ \mathcal{I}_{\rm in}\left[N_q\right] &= \sum_p \sum_{q_1>q_2} W_{q,p;q_1,q_2} (1+N_q)(1+N_p) N_{q_1} N_{q_2}\end{aligned}$$

with

$$\begin{aligned}W_{q_1,q_2;q_1',q_2'} &= \frac{2\pi}{\hbar^2}\left|t_{q_1q_2;q_1'q_2'}\right|^2 \delta_{q_1+q_2,q_1'+q_2'}\\ &\quad\times \delta(\omega_{q_1}+\omega_{q_2}-\omega_{q_1'}-\omega_{q_2'}).\end{aligned} \tag{7}$$

We begin the analysis of Eqs. (6)-(7) by considering the relaxation rate of a single high-energy boson. Specifically, we assume that the distribution function N_q differs from its equilibrium form, the Bose distribution $n_q = \left(e^{\hbar\omega_q/T} - 1\right)^{-1}$, in the population of a single state with q in the range $T/\hbar s \ll q \ll 1/\sqrt{\xi}$. In this limit $\mathcal{I}_{\rm in}\left[N_q\right]$ is exponentially suppressed, and Eq. (6) reduces to $\partial N_q/\partial t = -N_q/\tau_q$ with the relaxation rate

$$\tau_q^{-1} = \frac{\lambda^2K^2s}{48\pi^3} \times \begin{cases} (T/\hbar s)q^4, & q \ll (T/\hbar s\xi)^{1/3}\\[2mm] \dfrac{a(T/\hbar s)^3}{(\xi q)^2}, & q \gg (T/\hbar s\xi)^{1/3}\end{cases}, \tag{8}$$

where $a = 32\zeta(3)/3$. Here $\zeta(x)$ is the Riemann's zeta-function, $\zeta(3) \approx 1.2$. Although Eq. (8) is not directly applicable to *thermal* bosons with energy of the order of T, setting $q \sim T/\hbar s$ in Eq. (8) results in a correct order-of-magnitude estimate of the typical scattering rate, see Eq. (16) below.

Independently of the initial state, at $t \to \infty$ the distribution function N_q relaxes to n_q. In order to study the approach to equilibrium, we substitute

$$N_q = n_q + g_q f_q, \quad g_q = \sqrt{n_q(1+n_q)} \tag{9}$$

into Eqs. (6)–(7), neglect all but linear in f_q contributions, and obtain

$$\frac{\partial f_q}{\partial t} = -\frac{2\pi}{\hbar^2}\sum_p \sum_{q_1>q_2} \left|t_{qp;q_1q_2}\right|^2 \left(\frac{f_q}{g_q}+\frac{f_p}{g_p}-\frac{f_{q_1}}{g_{q_1}}-\frac{f_{q_2}}{g_{q_2}}\right) \times g_p g_{q_1} g_{q_2} \delta(\omega_q+\omega_p-\omega_{q_1}-\omega_{q_2})\, \delta_{q+p,q_1+q_2}. \tag{10}$$

The linearized Boltzmann equation (10) is applicable for both positive and negative q. Focusing from now on on $q>0$, we note that Eq. (10) simplifies considerably if

$$\xi(T/\hbar s)^3 \ll q \ll (T/\hbar s\xi)^{1/3}. \tag{11}$$

The first inequality in Eq. (11) ensures that contributions from the processes with all bosons but q on the left-moving branch of the spectrum are exponentially suppressed. The second inequality in Eq. (11) guarantees that the wave number of the only left-moving boson participating in the remaining scattering processes is much smaller than $T/\hbar s$. Under these conditions, the spectrum in the right-hand side of Eq. (10) can be linearized, which amounts to neglecting corrections of order $\xi(T/\hbar s)^2 \ll 1$ [this inequality is implicit in Eq. (11)]. This approximation corresponds to the substitution into Eq. (10)

$$\delta\big(\omega_q+\omega_p-\omega_{q_1}-\omega_{q_2}\big)\, \delta_{q+p,q_1+q_2} \approx \frac{1}{2s}\big[\delta(p+0)\,\delta_{q,q_1+q_2} + \delta(q_2+0)\,\delta_{q+p,q_1}\big], \tag{12}$$

where $\delta(k+0)$ indicates that k is an infinitesimal wave number on the left-moving branch. This yields

$$\frac{\partial f_q}{\partial t} = -\frac{1}{4\pi^3}\lambda^2 K^2 s\,(T/\hbar s)\, q \int_0^\infty dq_1 \left\{ \frac{1}{2}\int_0^\infty dq_2\, \delta(q-q_1-q_2)\, q_1 g_{q_1} q_2 g_{q_2} \left(\frac{f_q}{g_q}-\frac{f_{q_1}}{g_{q_1}}-\frac{f_{q_2}}{g_{q_2}}\right) + \int_0^\infty dp\, \delta(q+p-q_1)\, p g_p q_1 g_{q_1} \left(\frac{f_q}{g_q}+\frac{f_p}{g_p}-\frac{f_{q_1}}{g_{q_1}}\right)\right\}, \tag{13}$$

where g_q, g_p, g_{q_1}, and g_{q_2} are given by Eq. (9) with a linearized spectrum, e.g., $g_q = \big[2\sinh(\hbar s q/2T)\big]^{-1}$. The factor $T/\hbar s$ in the right-hand side of Eq. (13) is a remnant of the left-moving boson. Indeed, its wave number k [k is either p or q_2, see Eq. (12)] appears in Eq. (10) in combination $|k|g_k$, where the factor $|k|$ comes from the square of the amplitude (5). For $|k| \ll T/\hbar s$, we have $g_k = (T/\hbar s)|k|^{-1}$, which gives $|k|g_k = T/\hbar s$.

Note that the parameter ξ [see Eq. (3)] does not appear explicitly in Eq. (13). This is consistent with the above result for the relaxation rate of high-energy bosons: τ_q^{-1} is independent of ξ at $q \ll (T/\hbar s\xi)^3$, see Eq. (8). Note also that all wave numbers in Eq. (13) are strictly positive: coupling between bosons moving in opposite directions appears only in higher orders in $\xi(T/\hbar s)^2$. Accordingly, the right-hand side of Eq. (13) involves only three bosons moving in the same direction. This kind of scattering processes has a divergent rate when the spectrum is taken to be strictly linear from the outset, see Eq. (1) and Fig. 1(a). While Eq. (13) also corresponds to the limit of vanishing spectrum nonlinearity, it is crucial that the spectrum is linearized *after* the scattering amplitudes are evaluated and the divergences are regularized.

After integration over q_2 and p, Eq. (13) assumes the form

$$\frac{\partial}{\partial t} f(x,t) = -\tau_0^{-1}\int_0^\infty dy\, \mathcal{G}(x,y) f(y,t), \tag{14}$$

where $f(x,t) = f_q(t)\big|_{q=2\pi(T/\hbar s)x}$. The kernel $\mathcal{G}(x,y)$ is given by

$$\mathcal{G}(x,y) = \frac{xy(x+y)}{\sinh\big[\pi(x+y)\big]} - \frac{xy(x-y)}{\sinh\big[\pi(x-y)\big]} + \frac{1}{6}x^2(x^2+1)\delta(x-y)\,, \tag{15}$$

and the typical scattering rate is

$$\tau_0^{-1} = 2\pi\lambda^2 K^2 s\,(T/\hbar s)^5. \tag{16}$$

The integro-differential equation (14)-(15) can be solved exactly. The solution reads [12]

$$f(x,t) = \alpha_0\varphi_0(x) + \int_0^\infty d\nu\, \alpha_\nu \varphi_\nu(x)\, e^{-\eta_\nu t/\tau_0}, \tag{17}$$

where $\eta_\nu = \nu^2(\nu^2+1)/6$ and

$$\varphi_0(x) = \sqrt{6\pi}\,\frac{x}{\sinh(\pi x)}, \quad (18)$$

$$\varphi_\nu(x) = \frac{1}{\sqrt{(\nu^2+1)(4\nu^2+1)}}\Big\{(2\nu^2-1)\delta(x-\nu) + \frac{3x}{\sinh[\pi(x+\nu)]} + \frac{3x}{\sinh[\pi(x-\nu)]}\Big\}. \quad (19)$$

(The singularity in the right-hand side of Eq. (19) is to be understood as the principal value.) The coefficients α_0 and α_ν in Eq. (17) are determined by the initial conditions, $\alpha_\mu = \int_0^\infty dx\, \varphi_\mu(x) f(x,0)$ for $\mu = 0, \nu$.

The first term in the right-hand side of Eq. (17) represents a stationary (independent of t) contribution to $f(x,t)$. At $t\to\infty$ Eqs. (9) and (17) yield

$$\delta N_q = N_q\big|_{t\to\infty} - n_q = \alpha_0 g_q \varphi_0(x)\big|_{x=\hbar s q/2\pi T}. \quad (20)$$

This result has a clear physical meaning. In general, a stationary (equilibrium) solution of the Boltzmann equation $N_q\big|_{t\to\infty}$ is not unique. All such solutions, however, have the form of the Bose function n_q, parametrized by temperature T. A change of T by δT generates a correction to $N_q\big|_{t\to\infty}$, which, to linear order in δT, indeed has the form (20) with $\alpha_0 = \sqrt{\pi/6}\,(\delta T/T)$. On the other hand, the energy of the system at $t\to\infty$ coincides with that in the initial non-equilibrium state. Thus, the temperature T characterizing the equilibrium distribution at $t\to\infty$ is uniquely determined by the initial conditions. Choosing n_q as the Bose distribution with this temperature, one ensures that $\alpha_0 = 0$ in Eq. (17).

The remaining (time-dependent) term in the right-hand side of Eq. (17) describes approach to equilibrium. At short times, $t \ll \tau_0$, only the relaxation modes with $\nu \gtrsim (\tau_0/t)^{1/4} \gg 1$ are affected. Since $\varphi_\nu(x) \approx \delta(x-\nu)$ at $\nu \gg 1$, Eq. (17) gives $f(x,t) \propto e^{-\eta_x t/\tau_0}$, which describes exponential relaxation with the rate given by the appropriate limit of Eq. (8) [$q \ll (T/\hbar s\xi)^{1/3}$, see Eq. (11)].

At $t \gg \tau_0$ the high-energy bosons have already relaxed, and thermal bosons (with $x \sim 1$ or $q \sim T/\hbar s$) have equilibrated among themselves, although at temperature that has not yet reached its equilibrium value. Indeed, at large t the main contribution to the integral in Eq. (17) comes from small ν. Approximating $\varphi_\nu(x) \approx -\delta(x-\nu) + \sqrt{6/\pi}\,\varphi_0(x)$ and $\eta_\nu \approx \nu^2/6$, we find $\alpha_\nu = -f(\nu,0)$, and Eq. (17) yields

$$f(x,t) = F(x,t) - \sqrt{6/\pi}\,\varphi_0(x)\int_0^\infty d\nu\, F(\nu,t). \quad (21)$$

Here $F(x,t) = f(x,0)\, e^{-x^2 t/6\tau_0}$ corresponds to exponential relaxation of the boson distribution at $q \ll T/\hbar s$ with the rate

$$\tau_q^{-1} = \frac{1}{12\pi}\lambda^2 K^2 s\,(T/\hbar s)^3 q^2. \quad (22)$$

The role of the second term in Eq. (21) is to ensure the energy conservation. The corresponding correction to the distribution function [see Eq. (9)] can be cast in the form

$$\delta N_q = \frac{\partial n_q}{\partial T}\,\delta T(t), \quad \delta T(t) = -\frac{3\hbar s}{\pi^2}\int_0^\infty dq\, f_q(0)\, e^{-t/\tau_q}$$

with $1/\tau_q$ given by Eq. (22). For generic $f_q(0)$, the correction to temperature $\delta T(t)$ exhibits non-exponential dependence on time.

To summarize, elementary excitations of one-dimensional interacting systems are often described in the framework of the effective Luttinger liquid theory. Both the conventional Luttinger liquid theory [1, 3, 4] and its recent extensions [6, 7, 15] provide a set of efficient tools for evaluation of various correlation functions. However, none of these approaches is capable of describing the thermalization of bosonic quasiparticles in non-integrable systems because interaction between bosons with linear spectrum results in a divergent inelastic scattering rate.

In this Letter we demonstrated that the divergences are regularized when the nonlinearity of boson spectrum is taken into account. We derived and solved the Boltzmann equation describing the fastest equilibration process in the system, namely, thermalization of bosons moving in the same direction. The equation describes bosons with a linearized (as opposed to strictly linear) spectrum and results in a finite relaxation rate, see Eqs. (16) and (22).

At $t \ll \tau_0$ the relaxation is controlled by the inverse lifetime of a boson, Eq. (8). Exponential decay of the distribution function in this regime manifests itself in Lorentzian broadening of the peak in the dynamic structure factor at $q \gg \max\{T/\hbar s, \sqrt{K}/\xi\}$. In principle, such broadening can be observed using the two-photon Bragg spectroscopy technique [16] in fermionic [17] and bosonic [18] strongly interacting ultracold dipolar quantum gases [19] confined in one-dimensional traps.

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