Dynamics of Elastic Rods in Perfect Friction Contact

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Dynamics of elastic rods in perfect friction contact

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One of the most challenging and basic problems in elastic rod dynamics is a description of rods in contact that prevents any unphysical self-intersections. Most previous works addressed this issue through the introduction of short-range potentials. We study the dynamics of elastic rods with perfect rolling contact which is physically relevant for rods with rough surface. Such dynamics cannot be described by the introduction of any kind of potential. We show that, surprisingly, the presence of rolling contact in rod dynamics leads to highly complex behavior even for evolution of small disturbances.

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Introduction Take two rubber strings, stretch them a bit and cross them so they remain in contact, as shown on Figure 1. As long as the deformations are not too large, the strings will roll at the contact without sliding. Of course, that simple experiment is dominated by the energy loss from the internal deformation of strings at the contact point; however, improving the quality of strings will allow them to oscillate for a reasonable time before the energy loss takes over. Interestingly, this simple and familiar experiment has deep mathematical and physical implications that go beyond a toy problem.

There are many objects that can be represented as long elastic rods, from a rubber hose to DNA molecules. Typically, if these objects are put in a confined space, or undergo other non-trivial dynamics, self-contact of these rods typically appears. The true dynamics will not allow the rod to pass through itself, and it must preserve the side of contact under dynamics. While it is generally accepted that something like DNA at contact will slide freely, the dynamics of other molecules like dendronized polymers (DP) may be different. These compound molecular structures are formed by assembling multiple *dendrons* that are each connected by its base to a long polymeric backbone [1]. A simplified, coarse-grained rod model of such polymers must take into account the rough surface formed by tree-like structures that is likely to generate perfect rolling contact. Another case when the rods will be rolling rather than sliding at contact can be realized for highly adhesive surfaces. A simple physical estimate demonstrates how essential that phenomenon is. Suppose two (macroscopic) elastic rods in contact have coefficient of dry friction of k. For microscopic objects like DPs, the roughness of the rod's surface will introduce an effective value of k, although the exact value is not yet available in the literature. If the force due to contact $\boldsymbol{\lambda}$ (see below on how to compute that force) has a normal component to the surface $\boldsymbol{\lambda}_n$ and the tangential component $\boldsymbol{\lambda}_\tau$, then rolling contact occurs when $|\boldsymbol{\lambda}_\tau| \leq k|\boldsymbol{\lambda}_n|$. Thus, whenever the contact force lies in the cone with the opening angle $\arctan k$ with respect to the line connecting the centers, the contact will be rolling rather than sliding. For example, for a highly compacted configurations of a rod with many self-intersections, there will be a large fraction of contacts where the reaction force enforces rolling at contact; assuming that each contact generates a force that is uniformly randomly distributed within a hemisphere of possible forces enforcing contact, we conclude that the fraction of contacts experiencing rolling is $\sim 0.5k/\sqrt{k^2+1}$. Even for the rather small value of $k \sim 0.2$, approximately $\sim$10% of contacts will be rolling; larger values of $k \sim 1$ gives that the rolling contacts will be encountered in $\sim$30% of cases.

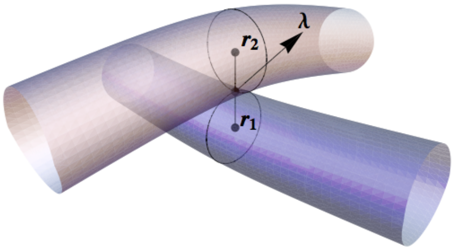

FIG. 1: A sketch of two strings in contact. The line going through the centers of contact disks $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ passes through the contact point. Contact reaction force is denoted as $\boldsymbol{\lambda}$.

Efficient numerical methods have been developed recently to deal with the self-contact forces of rods using short-range repulsive potential for statics [2] and dynamics [3–5]. Another avenue of studies of *stationary states* of elastic rods with self-contact [6–10] explicitly computes the contact forces from the existence of constraints. In our case, the rolling contact comes from friction, which does not admit any potential description. Thus, earlier works on rod dynamics based on nonlocal potential forces [11, 12] cannot be extended to the case of rolling contact. The extension of the latter approach to include the dynamics, and especially the rolling constraint is difficult. We also note that the true motion may be a combination of rolling and sliding friction, but in the absence of a consistent theory for rolling slippage we shall concentrate on the perfect rolling only. Our solution of the problem of rolling contact is similar in spirit of the second approach, as the contact force comes naturally from the constraint and not from short-range forces.

Setup of the problem As is well known [13], the problem of rolling motion is essentially non-holonomic and, in general, cannot be represented from the potential point of view. We shall note that it is possible to define potential approach to some non-holonomic systems [14]. However, these cases seem to be more of an exception rather than a rule. For the problem in hand, we proceed with the Lagrange-d'Alembert (LdA) principle which is the fundamental method for the treatment of non-holonomic systems [13, 15]. Similar approach has been recently used to describe the motion of an elastic rod rolling on a plane [16]. In order to utilize LdA, we recast the motion of the rods in variational setting through the Simo-Marsden-Krishnaprasad (SMK) rod theory [17], which allows a variational formulation of string dynamics [11]. The rod is parameterized by a coordinate s that does not have to be arc length. We fix a reference frame and measure the local position $\boldsymbol{r}(s,t)$ and orientation $\Lambda(s,t)$ at time t with respect to the fixed frame. SMK theory formulates the dynamics in terms of the variables that do not depend on the choice of basic frame:

$$\boldsymbol{\Gamma}=\Lambda^{-1}\boldsymbol{r}'\,,\;\boldsymbol{\Omega}=\Lambda^{-1}\Lambda'\,,\;\boldsymbol{\gamma}=\Lambda^{-1}\dot{\boldsymbol{r}}\,,\;\boldsymbol{\omega}=\Lambda^{-1}\dot{\Lambda}\,. \quad (1)$$

Here, the prime denotes the partial derivative with respect to s and dot the derivative with respect to t. The physical meaning of $\boldsymbol{\gamma}$ and $\boldsymbol{\omega}$ is the linear and angular velocity in the body frame of reference, and $\boldsymbol{\Gamma}$ and $\boldsymbol{\Omega}$ the corresponding deformation rate. If $\ell(\boldsymbol{\Gamma},\boldsymbol{\Omega},\boldsymbol{\gamma},\boldsymbol{\omega})$ is the Lagrangian, then the equations of motion for a free elastic rod are given by the variational principle [11]

$$\delta\int \ell(\boldsymbol{\Gamma},\boldsymbol{\Omega},\boldsymbol{\gamma},\boldsymbol{\omega})\mathrm{d}s\mathrm{d}t=0\,, \quad (2)$$

appropriately computing the variations of variables $(\boldsymbol{\Gamma},\boldsymbol{\Omega},\boldsymbol{\gamma},\boldsymbol{\omega})$. In the case of constraints, this variational principle (2) has to be modified according to the LdA approach as follows. Assume, for simplicity, that the undeformed rod has a circular cross section and the acting forces are small enough so that the cross-section remains circular even at contact. Thus, if the strands at contact can be approximated by touching circular cylinders, then the contact point is always located at $c(t)=(\boldsymbol{r}_1+\boldsymbol{r}_2)/2$, and the vector from the center of the cylinder $\boldsymbol{r}_i(t)$ to the contact point is $\pm(\boldsymbol{r}_2-\boldsymbol{r}_1)/2$. Here, the index $i=1,2$ means evaluation at $s=s_i$ marking the disks at contact. Since the angular velocity in the fixed frame is $\dot{\Lambda}_i\Lambda_i^{-1}$, $i=1,2$, then the velocity of the material point associated with contact, also in the fixed frame, is

$$\dot{\boldsymbol{r}}_1+\frac{1}{2}\dot{\Lambda}_1\Lambda_1^{-1}(\boldsymbol{r}_2-\boldsymbol{r}_1)=\dot{\boldsymbol{r}}_2-\frac{1}{2}\dot{\Lambda}_2\Lambda_2^{-1}(\boldsymbol{r}_2-\boldsymbol{r}_1) \quad (3)$$

This condition can be reformulated in terms of invariant variables by multiplying (3) by Λ_1^{-1}:

$$\boldsymbol{\gamma}_1+\frac{1}{2}\boldsymbol{\omega}_1\times\boldsymbol{\kappa}_{12}=\xi_{12}\boldsymbol{\gamma}_2-\frac{1}{2}(\xi_{12}\boldsymbol{\omega}_2)\times\boldsymbol{\kappa}_{12}\,, \quad (4)$$

where $\xi_{12}:=\Lambda_1^{-1}\Lambda_2$ is the relative orientation, $\boldsymbol{\kappa}_{12}=\Lambda_1^{-1}(\boldsymbol{r}_2-\boldsymbol{r}_1)$, and other invariant variables are defined as in (4), with the index i meaning evaluation at $s=s_i$. Note that due to the uniformity of the strand, and the assumption of circular cross-section, the Lagrangian does not depend explicitly on s_i and $\dot{s}_i$. Then, LdA principle states that one has to replace *time derivatives only* in (4) by δ-variations and use that expression as an additional constraint on $(\delta\boldsymbol{\Gamma},\delta\boldsymbol{\Omega},\delta\boldsymbol{\gamma},\delta\boldsymbol{\omega})$ in (2). For example, $\boldsymbol{\gamma}_1=\Lambda_1^{-1}\dot{\boldsymbol{r}}_1\to\Lambda_1^{-1}\delta\boldsymbol{r}(s)\delta_{s_1}$ and similarly for other variables. Here we have denoted, for shortness, $\delta_{s_i}:=\delta(s-s_i)$. There is an unfortunate collision of notation in δ between the variational derivatives and Dirac's δ-functions; in this paper, the δ-function always has a subscript. Denote by $\boldsymbol{\lambda}(t)$ a vector that enforces the LdA constraints, and by $\mathrm{D}/\mathrm{D}t:=\partial_t+\boldsymbol{\omega}\times$, $\mathrm{D}/\mathrm{D}s:=\partial_s+\boldsymbol{\Omega}\times$ the full t- and s- derivatives. We get the following equations for the motion of strings with rolling contact:

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\ell}{\partial\boldsymbol{\gamma}}+\frac{\mathrm{D}}{\mathrm{D}s}\frac{\partial\ell}{\partial\boldsymbol{\Gamma}}=\boldsymbol{\lambda}\delta_{s_1}-\xi_{12}^{-1}\boldsymbol{\lambda}\delta_{s_2} \quad (5)$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\ell}{\partial\boldsymbol{\omega}}+\frac{\mathrm{D}}{\mathrm{D}s}\frac{\delta\ell}{\delta\boldsymbol{\Omega}}=\frac{\partial\ell}{\partial\boldsymbol{\gamma}}\times\boldsymbol{\gamma}+\frac{\partial\ell}{\partial\boldsymbol{\Gamma}}\times\boldsymbol{\Gamma}+\frac{1}{2}\boldsymbol{\kappa}_{12}\times\boldsymbol{\lambda}\delta_{s_1}+\frac{1}{2}\xi_{12}^{-1}\Big(\boldsymbol{\kappa}_{12}\times\boldsymbol{\lambda}\Big)\delta_{s_2}. \quad (6)$$

The physical meaning of $\boldsymbol{\lambda}$ as the force due to the constraint is now evident from the linear momentum (5). Correspondingly, the $\boldsymbol{\lambda}$ terms in equation (6) are identified as torques acting on the contact point due to the presence of the constraint. These equations have to be augmented by the compatibility conditions

$$\dot{\boldsymbol{\Gamma}}=\boldsymbol{\gamma}'-\boldsymbol{\omega}\times\boldsymbol{\Gamma}+\boldsymbol{\Omega}\times\boldsymbol{\gamma}$$
$$\dot{\boldsymbol{\Omega}}=\boldsymbol{\omega}'+\boldsymbol{\omega}\times\boldsymbol{\Omega}\,. \quad (7)$$

We still have to close the system by computing the equations of motion for the contact points s_i, which is done from the tangency conditions stating that the strand at a contact point, which is locally a cylinder, touches itself tangentially and there is no intersection of the material. These conditions state that

$$\boldsymbol{\kappa}_{12}(s_1,s_2,t)\cdot\mathbf{E}_1^3=0,\quad \boldsymbol{\kappa}_{12}(s_1,s_2,t)\cdot\xi_{12}\mathbf{E}_2^3=0, \quad (8)$$

where $\mathbf{E}_i^3$ is the axis of the i-th rod in that rod's coordinate frame, assumed to be constant. Note that these expressions do not contain any time derivatives of the variables and are thus *holonomic*. They also imply that the distance between the centers of disks in contact $|\boldsymbol{r}_1-\boldsymbol{r}_2|$ is preserved. Since the Lagrangian ℓ does not depend on s_i and $\dot{s}_i$, they can be imposed after the equations have been derived. In principle, such conditions already determine s_i through an implicit relation; however, they are difficult to use. Instead, time derivatives of (8) give

$$\mathbb{A}\cdot(\dot{s}_1\,,\,\dot{s}_2)^T+\mathbf{v}=0\,, \qquad (9)$$

where the 2×2 matrix $\mathbb{A}$ and 2-vector $\mathbf{v}$ depends on the dynamical properties at contact, with $\det(\mathbb{A})\neq 0$ when the rods are not locally parallel at the contact. Equations (5,6) and the conditions (8,9) give the complete system of equations of elastic rods in rolling contact. These equations are different from earlier works as they take into account the forces caused by rolling conditions.

Discrete strands in contact It is also interesting to consider the application of this theory to discrete, chain-like elastic structures in contact. In that case, we need to clarify the physical meaning of the δ-function at the contact position. Apart from its physical relevance, this consideration is also useful for consistent numerical discretization of (5,6). Here, care must be taken in deriving the equations of motion without breaking their variational structure [18]. Suppose that we have a string consisting of discrete set of points along the line, $s=s^k$, with k being integer. If the orientation and position of a material frame at $s=s^k$ are given by an orientation matrix Λ_k and a vector $\boldsymbol{r}_k$, the invariant variables are $p_k=\Lambda_k^{-1}\Lambda_{k+1}$ and $\boldsymbol{q}_k=\Lambda_k^{-1}(\boldsymbol{r}_{k+1}-\boldsymbol{r}_k)$. The purely elastic Lagrangian is $\ell=\ell(\boldsymbol{\omega}_k,\boldsymbol{\gamma}_k,p_k,\boldsymbol{q}_k)$. One also needs to define a "smeared-out" version of (4):

$$\alpha_k\big(\boldsymbol{\gamma}_k+\frac{1}{2}\beta_m\boldsymbol{\omega}_k\times\boldsymbol{\kappa}_{km}\big)$$
$$-\alpha_k\beta_m\big(\xi_{km}\boldsymbol{\gamma}_m-\frac{1}{2}(\xi_{km}\boldsymbol{\omega}_m)\times\boldsymbol{\kappa}_{km}\big)=0\,, \qquad (10)$$

(summation over k,m). Here, we have defined the averaging coefficients: $\alpha_k:=G\big(s_1-s^k\big)$, $\beta_k:=G\big(s_2-s^k\big)$ arising from a "bump" function $G(s)$ that rapidly decays away from $s=0$, and $\xi_{km}:=\Lambda_k^{-1}\Lambda_m$, $\boldsymbol{\kappa}_{km}:=\Lambda_k^{-1}\big(\boldsymbol{r}_m-\boldsymbol{r}_k\big)$. Note that the positions of the contact is defined by the continuous variables s_1 and s_2. Here, the width of $G(s)$ corresponds to the size of discrete elements on the string or, for numerical discretization, to the distance along s between the discrete points of the rod. The function $G(s)$ should satisfy three criteria: a) it is sufficiently smooth in order to avoid artificial accelerations of s_i; b) having a single maximum so there is no ambiguity about the position of contact; and c) decaying rapidly so as not to introduce any long-term interactions between the contact points on the rod. The physical meaning of (10) consists in spreading the point wise contact condition (4) to a few neighboring points surrounding the contact. Then, the LdA principle gives a discrete analogue of (5,6):

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\ell}{\partial\boldsymbol{\gamma}_k}-p_{k-1}^{-1}\frac{\partial\ell}{\partial\boldsymbol{q}_{k-1}}+\frac{\partial\ell}{\partial\boldsymbol{q}_k}=\sum_m\big(\alpha_m\beta_k\xi_{mk}^{-1}-\alpha_k\big)\boldsymbol{\lambda}, \qquad (11)$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\ell}{\partial\boldsymbol{\omega}_k}+\frac{\partial\ell}{\partial p_k}p_k^{-1}-p_{k-1}^{-1}\frac{\partial\ell}{\partial p_{k-1}}=\frac{\partial\ell}{\partial\boldsymbol{\gamma}_k}\times\boldsymbol{\gamma}_k+\frac{\partial l}{\partial\boldsymbol{q}_k}\times\boldsymbol{q}_k$$
$$-\sum_m\frac{1}{2}\left(\alpha_k\beta_m\boldsymbol{\kappa}_{km}\times\boldsymbol{\lambda}+\beta_k\alpha_m\xi_{mk}^{-1}\big(\boldsymbol{\kappa}_{mk}\times\boldsymbol{\lambda}\big)\right). \qquad (12)$$

For a Lagrangian that is quadratic in all variables, corresponding to linear elasticity, an explicit computation of $\boldsymbol{\lambda}$ in (11,12) is possible, but cumbersome. Equations (11,12) are augmented by conditions for the variables s_1 and s_2 similar to (9) obtained by differentiating a discrete version of (8). Note that one could have, in principle, guessed the equations of motion for continuum rods in contact (5,6) using the standard Kirchhoff model and physical intuition, but we do not see any way to derive equations (11,12) without using the methods of this paper.

Linear strings in contact One may wonder if the equations of motion we have derived allow to deduce analytical expressions for the propagation of the disturbances along the rods at contact, such as the dispersion relation. The answer to this question is, unfortunately, no: the contact condition makes the disturbances essentially nonlinear. Let us consider two strings in contact, and denote for shortness $\mathbf{a}=(\boldsymbol{\gamma},\boldsymbol{\omega})^T$ and $\mathbf{A}=(\boldsymbol{\Gamma},\boldsymbol{\Omega})^T$. For linear elastic materials $\partial\ell/\partial\mathbf{a}=\mathbb{V}\mathbf{a}$ and $\partial\ell/\partial\mathbf{A}=-\mathbb{Q}\mathbf{A}$, where $\mathbb{V}$ and $\mathbb{Q}$ are 6x6 matrices. The linearized compatibility conditions (7) allow to introduce a vector potential $\boldsymbol{\phi}$ as $\mathbf{a}=\partial_t\boldsymbol{\phi}$, $\mathbf{A}=\partial_s\boldsymbol{\phi}$. Neglecting all nonlinear terms in the dynamic variables and assuming that the rod is naturally straight in its undeformed state, we can transform (5,6) into a vector wave equation [19]

$$\mathbb{V}\frac{\partial^2\boldsymbol{\phi}}{\partial t^2}-\mathbb{Q}\frac{\partial^2\boldsymbol{\phi}}{\partial s^2}=\begin{pmatrix}\mathrm{Id}\\ \frac{1}{2}\boldsymbol{\kappa}_{12}\times\end{pmatrix}\boldsymbol{\lambda}\delta_{s_1}+\xi_{12}^{-1}\begin{pmatrix}-\mathrm{Id}\\ \frac{1}{2}\boldsymbol{\kappa}_{12}\times\end{pmatrix}\boldsymbol{\lambda}\delta_{s_2}. \qquad (13)$$

The condition (4) can be expressed in this vector form as

$$\big(\mathrm{Id},-\frac{1}{2}\boldsymbol{\kappa}_{12}\times\big)^T\boldsymbol{\phi}_t(s_1,t)=\big(\xi_{12},\frac{1}{2}\boldsymbol{\kappa}_{12}\times\xi_{12}\big)^T\boldsymbol{\phi}_t(s_2,t) \qquad (14)$$

Thus, the evolution of small disturbances on the rod is governed by linear equations (13), linear rolling constraint (14) and *nonlinear* evolution equations for $s_{1,2}$ (9). We can illustrate the complexity of this problem on a pedagogical simplistic example of two straight rods in contact with only one mode being relevant in (13) for each rod. The one-dimensionality of disturbances is chosen just for the simplicity of computations. The chaotic behavior comes from the nonlinearity of coupling the forcing in equations (13) to the solution through the contact conditions (9), and is independent on the dimensionality of the system. Physically, such a mode can be realized for a rod with special elastic and inertia matrices $\mathbb{V}$ and $\mathbb{Q}$, *e.g.* for rods made out of composite material. No further simplifications of equations or analytical progress is possible. Thus, the answer to a deceptively simple question on how the disturbances propagate along the rod is surprisingly complex, and is due exclusively to the contact condition.

Let us denote by $u(x,t)$ and $v(y,t)$, the one-dimensional deflection for the first and second rod, respectively. In this case, the rolling constraint and the motion of the contact points are written simply as

$$u_t(s_1) = F v_t(s_2)\,, \quad \dot{s}_i = G_i u_t(s_i)\,,\, i = 1, 2\,, \qquad (15)$$

where F and G_i are constants depending on the material parameters and the base state of the rods. The equations of motion for (u, v) in this reduced setting are:

$$u_{tt} - c^2 u_{xx} + \lambda \delta_{s_1} = 0\,,\, v_{tt} - c^2 v_{yy} - \lambda F \delta_{s_2} = 0, \quad (16)$$

where λ enforces the first constraint of (15).

A discrete version of these equations can be derived, similarly to the full equation described above in (12). We shall emphasize that the complexity caused by the nonlinear contact conditions is the same in the full equation (13) and its one-dimensional counterpart (16). While the complex dynamics caused by the nonlinear rod equations have been well studied, as far as we are aware, there has been no work on the complex dynamics caused by the rolling contact condition. In the absence of constraints, the wave equations provide a simple harmonic oscillation of the string. However, when the constraint is present, the motion of the string is challenging and complex.

Contact chaos In the case when the boundary conditions for the strings are periodic, exact solution of equations can be found, which we omit here. In the more realistic fixed boundary conditions for u and v, the behavior is highly complex. The motion, as one can show from equations (16), conserves energy; however, in reality, friction with air and, more importantly, rolling friction will lead to energy dissipation. It is nevertheless interesting to see the structure of that dynamics, with the nonlinearity obtained only from the contact condition. As we see from Figure 2, the system produces a complex spatio-temporal dynamics of both strings. It is also relevant to present another measure of complexity, computed from the dynamics of the base harmonic of $u(x, t)$, call it $\hat{u}_1(t)$, as a function of t. If the string were vibrating in the air, the sound sufficiently far away from the string will primarily contain the contribution from the first harmonic. On Figure 2, right, we plot

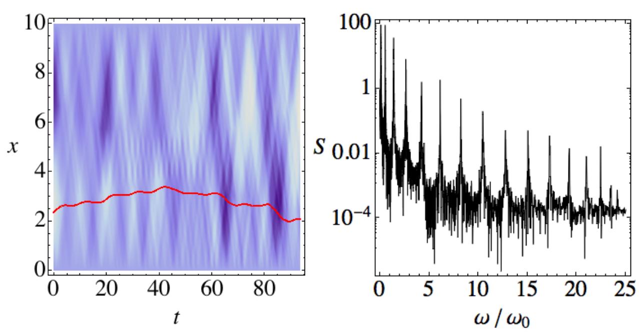

FIG. 2: Left: the initial spatio-temporal evolution of one of the strings in contact, with the red line marking the point of contact. The color, dark blue to white, denotes the deviation of the string from its equilibrium position, from high positive to high negative. Right: spectrum $S(\omega)$ vs ω/ω_0 of time signal produced by the lead harmonic (in x) of $u(x, t)$, with $\omega_0 = 2\pi c/l$ being the basic frequency of a string without contact.

the time spectrum $S(\omega)$ as a function of temporal frequency ω, obtained from the time signal of the first harmonic $\hat{u}_1(t)$. Starting with $u(x, 0) = \sin x$, a linear rod with $0 < x < 2\pi$ without rolling contact will generate a purely monochromatic signal; however, when the contact is present, there is a persistence of high overtones to the signal. A sound file in the supplementary materials gives the reader an impression about the quality of that signal. We have found out that the chaotic behavior caused by the contact condition persists for all initial conditions we have tried. The appearance contact-caused chaos is highly interesting and important for many physical applications, and yet it has not been discussed previously.

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