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Enhancing the resolution of a sensor via negative correlation: a biologically inspired approach

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We demonstrate that a neuronal system, underpinned by "fire-then-reset" dynamics, can display an enhanced resolution $R \sim T_{ob}^{-1}$ where T_{ob} is the observation time of the measurement; this occurs when the interspike intervals are negatively correlated and $T_{ob} < \Delta/\varepsilon$ where ε is a parameter characterizing the level of correlation between inter-spike intervals and Δ is the average interspike interval. We also show that by introducing negative correlations into the *time domain* response of a *nonlinear* dynamical sensor it is possible to replicate this enhanced scaling of the resolution. Thus, we demonstrate the potential for designing a novel class of biomimetic sensors that afford improved signal resolution by functionally utilizing negative correlations.

It is well known that the resolution (defined as the smallest resolvable value of a measured quantity) of a sensor is constrained by the measurement noise. The commonest strategy for improving the resolution is to repeat the measurement N times and calculate the population mean (i.e. average) or, similarly, N measurements can be done simultaneously in parallel and averaged. When the measurements are statistically independent the uncertainty (i.e. standard deviation) of the population mean scales as $1/\sqrt{N}$, or, equivalently, as $T_{ob}^{-0.5}$ where $T_{ob} = N\delta t$ is the total observation time of the measurements and δt is the sampling period. However, it has been noted that if *negative* correlations between the individual measurements exist, then the uncertainty can be reduced as $1/N^{\alpha}$ where $\alpha > 0.5$ [1]. This suggests that, potentially, the resolution of a measurement can be improved by the introduction of negative correlation.

Negatively correlated interspike intervals (ISIs) are well documented in the neuroscience literature [2–4] and a number of models have been propounded to explain the origin of these correlations [5–7]. Additionally, studies have pointed to the functional benefit of these correlations; in particular, they have been shown to yield enhanced detectibility of stimuli [4] and, also, increased information transmission [5–7]. However a quantitative link between the resolution of a "neural" measurement and the degree of negative correlation has not previously been established. Additionally, the possibility of reverse engineering these newly identified neural coding mechanisms and exploiting them in a new class of "biomimetic sensors" has also not been addressed.

The aim of this Letter is threefold; first, we extend a recently introduced neural model of negatively correlated ISIs [7] to take into account *phase diffusion* - this extension is necessary to obtain physically meaningful results (relevant to real-world applications): phase diffusion occurs in all oscillating systems and, ultimately, governs the accuracy of any clock. This extension, also, enables us to describe systems with different degrees of negative correlation. Second, we obtain a relationship between the degree of negative correlation and the resolution of a measurement; the parameter regime for the enhanced resolution ($R \approx T_{ob}^{-1}$) is also established. Finally, we show how the configuration and operation of a nonlinear dynamic sensor (specifically, a single-core fluxgate magnetometer SCFG [9]) having a temporal *event*-based readout, can be adapted to mimic the dynamics of a neuron with negatively correlated ISIs. The sensor in this "biomimetic" (inspired by Nature) configuration yields the above-mentioned performance enhancement scaling quantified by a lower value of the resolution; further, it will become clear that the biomimetic configuration is applicable to other nonlinear dynamic sensors that subscribe to a time-domain based readout.

To assess the role that negatively correlated ISIs have on the resolution of a neural measurement we start with a Perfect Integrate Fire (PIF) model with noisy threshold [7],

$$\dot{v} = \beta + s, \tag{1}$$

with s the (constant) signal to be estimated, β a constant base current, and v the voltage across the nerve membrane. The threshold θ is a uniformly distributed random variable, $\theta \in [\theta_a - D_u, \theta_a + D_u]$, that is independently defined for every inter-spike interval. D_u is the noise intensity, and θ_a the mean threshold, $\theta_a = \langle\theta\rangle$. The mode of operation is as follows; when the voltage v reaches the threshold θ, a spike is fired, a new threshold is chosen, and the voltage is reset to a new level $\eta = \theta - \Lambda$. If $\Lambda = const$, as was proposed in [7], then the ISIs generated by the model (1) are strongly negatively correlated.

However, in practice, this reset mechanism cannot be infinitely precise. Hence, we introduce a second noise source into the model to take this additional uncertainty into account. We assume that in the model (1) the reset happens in the presence of noise so that $\Lambda = \theta_a + \xi$, where ξ is a uniformly distributed random variable, $\xi \in [-D_d : D_d]$. We refer to this modified PIF model as the 'Perfect

Integrate and Fire model with Noisy Threshold and Reset' or PIF-NTR for short. The reset level is, now, given by $\eta = \theta - \theta_a - \xi$. As will become apparent, the effect of this second noise source is to introduce phase diffusion into the response of the neuron. The additive reset noise allows us to bridge smoothly between the non-renewal and renewal regimes - these regimes were described using two separate models in [7, 8]. Denoting the times of the level crossings as t_0, t_1, ..., t_k, the values of the threshold at crossing as θ_0, θ_1, ..., θ_k, and the reset levels as $\eta_0 = \theta_0 - \Lambda_0$, ..., $\eta_k = \theta_k - \Lambda_k$ it is straightforward to show from (1) that the k^{th} time interval is given by,

$$T_k = (t_k - t_{k-1}) = \frac{\theta_k - \theta_{k-1} + \theta_a + \xi_{k-1}}{\beta + s}. \quad (2)$$

We decompose the inter-spike interval T_k into three independent random variables so that $T_k = \delta_k + \Delta_k - \delta_{k-1}$, where we introduce $\delta_k = \theta_k/(\beta+s)$, $\delta_{k-1} = \theta_{k-1}/(\beta+s)$, and $\Delta_k = (\theta_a + \xi_{k-1})/(\beta+s)$. If the reset and threshold are noiseless, and the signal $s = const$, then $\Delta_k = \Delta = const$. In this case the process is exactly periodic because the spikes precisely occur at times $t_k = k\Delta + t_0$. It can now be observed that the second noise source ξ leads to a variation in the intervals Δ_k given by $\xi_{k-1}/(\beta+s)$ and hence this term introduces a form of phase diffusion into the dynamics. More generally we denote the average interspike interval $\langle T_k \rangle = \langle \Delta_k \rangle = \theta_a/(\beta+s) = \Delta$

The serial correlation coefficient is calculated as [8],

$$\rho(m) = \frac{\langle (T_k - \langle T_k \rangle)(T_{k+m} - \langle T_{k+m} \rangle) \rangle}{\sigma_{T_k} \sigma_{T_{k+m}}} = \begin{cases} 1 : & m = 0, \\ -\frac{1}{2} + \frac{\varepsilon}{2} : & m = 1, \\ 0 : & m > 1, \end{cases} \quad (3)$$

where we introduce the parameter $\varepsilon = \sigma_\Delta^2/(2\sigma_\delta^2 + \sigma_\Delta^2) = D_d^2/(2D_u^2 + D_d^2)$, m is the lag and the average is performed over index k. For systems that display strongly negatively correlated ISIs one typically has $\varepsilon << 1$ and hence $\varepsilon \sim \sigma_\Delta^2/(2\sigma_\delta^2) = \sigma_\xi^2/(2\sigma_\theta^2)$. This result reduces to that obtained in [7] when $\varepsilon = 0$. Clearly, $\varepsilon \in [0 : 1]$ governs the degree of negative correlation; ISIs are maximally (negatively) correlated (i.e. $\rho(1) = -0.5$) when $\varepsilon = 0$ and when $\varepsilon = 1$, $\rho(1) = 0$. Consequently, the degree of negative correlation at $\rho(1)$ is governed by the balance between the phase diffusion resulting from the noisy reset (that randomly varies the period) and the threshold noise (that 'jitters' the spikes about the average period). We note that that for non-zero reset noise, the point process describing the spike times is transformed into a diffusion process with the diffusion coefficient $D_{diff} = \sigma_\Delta^2/\Delta = \sigma_\xi^2/((\beta+s)^2\Delta)$.

We now introduce the sum of N inter-spike intervals $\sum_{k=1}^{N} T_k$; this is the observation time $\tau_{ob,N}$:

$$\tau_{ob,N} = \delta_0 - \delta_N + N\Delta + \frac{1}{\beta+s}\sum_{k=0}^{N-1} \xi_k. \quad (4)$$

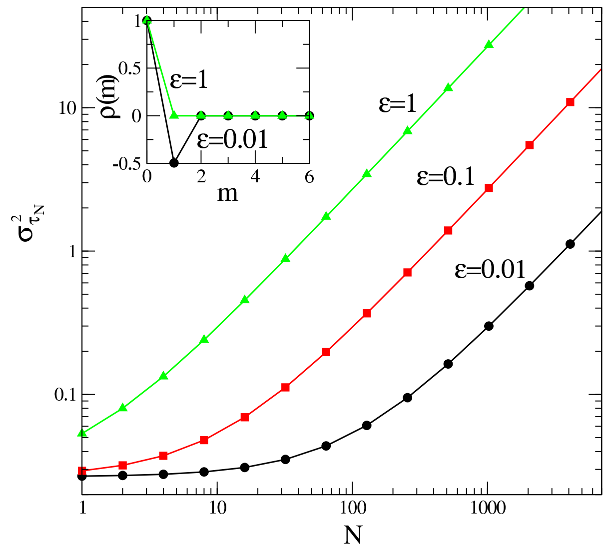

FIG. 1. The variance $\sigma^2_{\tau_{ob,N}}$ (Eq. 6) and (inset) the correlation coefficient (Eq. 3)) of the model (1). $\beta = 1$, $s = 0$, $\theta_a = 1$, $D_u = 0.2$.

$\tau_{ob,N}$, is the observable from which the (target) signal magnitude must be inferred. The reason for choosing this quantity as the observable can be understood by inspection of Equation (4). The threshold noise only enters via the jitter times δ_0 and δ_N; these are the jitters associated with the first and last spikes of a sequence of spikes. In other words, the spike timing jitter due to the threshold noise observed during time intervals $k = 1..N-1$ cancels out due to the negative correlation. This noise cancelation property is expected to lead to an improvement in the resolution. In contrast, the last term in Equation (4) demonstrates that the error due to the noisy reset is accumulated.

To understand the affect of these noise sources we obtain the average observation time from Equation (4)

$$T_{ob} = \langle \tau_{ob,N} \rangle = N\Delta = N\frac{\theta_a}{\beta+s} \quad (5)$$

and the variance of the observation time as,

$$\sigma^2_{\tau_{ob,N}} = 2\sigma_\delta^2(1 + \varepsilon N). \quad (6)$$

The quantity $\sigma_{\tau_{ob,N}}$ governs the error with which T_{ob} can be measured. Figure 1 demonstrates how this error scales with the number of time intervals included in the observation. For $\varepsilon N << 1$ (strong negative correlation) $\sigma_{\tau_{ob,N}} \approx \sqrt{2}\sigma_\delta$ and hence the error is independent of N; consequently increasing N will rapidly reduce the fractional error, $\sigma_{\tau_{ob,N}}/T_{ob}$, according to the scaling $1/N$. However, in the opposite limit, $\varepsilon N >> 1$, $\sigma_{\tau_{ob,N}}$ scales as $\sqrt{N}$ and hence the fractional error scales as $1/\sqrt{N}$ as normally expected.

The above analysis has not, however, taken into account how these quantities depend on the signal, s. As can be observed from (4), the average period of the ISI is dependent on the value of the signal - hence the signal can be estimated by measuring the period or, more accurately, by measuring the time to observe N periods. Thus, we now calculate the resolution, R, based on the measurement of T_{ob} using the definition [10, 11],

$$R = \left|\frac{\partial T_{ob,N}}{\partial s}\right|^{-1} \sigma_{\tau_{ob,N}}. \qquad (7)$$

Physically, the resolution represents the signal value that results in $\sigma_{\tau_{ob,N}}$ being equal to the difference in T_{ob} with and without the signal. Substituting Eqs. (5, 6) into Eq (7) and noting that the best resolution is obtained in the $\lim s \to 0$ yields,

$$\lim_{s\to 0} R = \frac{\sqrt{2}\sigma_\delta \beta^2}{\theta_a}\frac{\sqrt{1+\varepsilon N}}{N}. \qquad (8)$$

For short observation time (i.e. $N \ll 1/\varepsilon$):

$$\lim_{s\to 0} R = \frac{\sqrt{2}\sigma_\delta \beta^2}{\theta_a}\frac{1}{N} \sim \frac{1}{T_{ob}}. \qquad (9)$$

For long observation time ($N \gg 1/\varepsilon$) we have:

$$\lim_{s\to 0} R = \frac{\sqrt{2\varepsilon}\sigma_\delta \beta^2}{\theta_a}\frac{1}{\sqrt{N}} \sim \frac{1}{\sqrt{T}_{ob}}. \qquad (10)$$

The crossover between these two scalings relationships occurs when $\varepsilon \sim 1/N$, or, equivalently when $T_{ob} \sim \Delta/\varepsilon$. Consequently, these results clearly demonstrate that an enhanced scaling of the resolution can be observed for sufficiently short observation windows. The size of the window over which this enhanced scaling is observed is governed (approximately) by the ratio of the reset noise to the threshold noise; in general the larger the reset noise the shorter the duration of the enhanced scaling regime.

We now assess whether the enhanced resolution can be realised in an engineered sensor, using the SCFG [9, 10] as a test example. To operate the magnetic sensor in a similar mode to model (1) we need to map the sensor dynamics onto the PIF-NTR model. We utilise the fact that the magnetisation of the ferromagnetic core can be made to switch states by application of a magnetic field $B_+ + B_0$ followed by the field $B_- + B_0$ where B_0 is the unknown external magnetic field and B_+ and B_- are constant 'control' fields used to cause the switching. We note that the fields B_+ and B_- have opposite polarities and, in practice, can be realised [9] by applying currents to the primary coil; also, in practice, one usually has $|B_0| << |B_\pm|$. The dynamics can now be forced to approximate those of the PIF-NTR model (1) by employing the following mode of operation. Application of the field $B_+ + B_0$ causes the total magnetic field, B, (where $B = B_0 + B_+ + \mu_0 M$ and μ_0 is the magnetic constant and M is the magnetisation of the ferromagnetic core), to increase at a rate governed by the magnetisation relaxation time τ_r. A threshold, θ, is now set such that when $B = \theta$ the reset field $B_- + B_0$ is applied *for a duration* τ before being switched back to $B_+ + B_0$ and the process repeats. The fact that the reset field is applied for a fixed time duration τ is essential because the value of the magnetisation reached during the reset period will depend on the value of the magnetisation when the threshold is crossed (the crossing point varies due to noise) and, hence, memory of this magnetisation is carried forward to the next cycle; this memory effect gives rise to negative correlations. Finally, we note that due to the nonlinearity of the magnetic hysteresis the magnetisation relaxation time τ_r will depend (nonlinearly) on the value of B_0. Consequently, B_0 can be estimated by measuring the period of oscillation $T = \tau + \tau_r$ or, by analogy with the neural system, the time τ_{ob} to complete N periods. It should be noted that we assume all noise sources are intrinsic to the SCFG, these are modelled as an effective threshold noise. An effective reset noise also occurs because of nonlinear relaxation; unlike Eq. 1, the reset value depends nonlinearly on the value of the magnetisation when the reset pulse is applied, this leads to an effective error (noise) in the reset level [14]. We also note that our measurement scheme requires a comparison of the field B with the threshold θ. We envisage this being undertaken by a device with a sigmoidal (almost binary) dependence on B (e.g. like a phase transition device) - such a device would then trigger a circuit (e.g. monostable multivibrator) to generate a pulse of duration τ. Due to the binary response this unit cannot be used for direct measurement of magnetic fields.

To verify these concepts we model the dynamics of the magnetization of a ferromagnetic material with the simplified equation [12],

$$\tau_a \frac{dM}{dt} = -M + M_s \tanh\left(\frac{cB}{\mu_0}\right), \qquad (11)$$

where M_s is the saturation level of the magnetization, τ_a is a characteristic time of the magnetization relaxation. In Eq. (11), c is a non-linearity parameter that is proportional to the Curie temperature-to-temperature ratio. Analogous with [7] we introduce noise in the threshold θ as an uniformly distributed variable in the interval $[\theta_a - D_u : \theta_a + D_u]$.

Numerical simulations of Eq. (11) show that the level that the magnetization is reset to is strongly dependent on τ. For large τ (see inset of Fig. 2(a) for $\tau = 3$) the magnetisation approaches the saturation value and this reduces the negative correlation at $\rho(1)$ (Fig. 2(a) for $\tau = 3$). $\rho(1)$ reduces because partial saturation of the

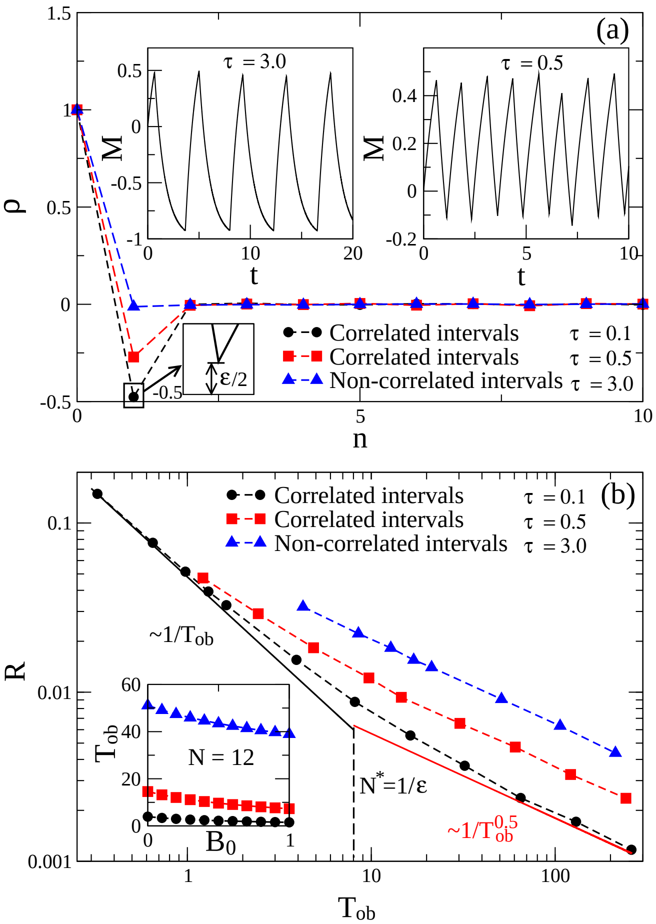

FIG. 2. (a) The correlation coefficient $\rho(n)$. Insets: the magnetization M as a function of time t for the model described via (11). The time series corresponds to non-correlated intervals on the left ($\tau = 3.0$) and negative correlated intervals on the right ($\tau = 0.5$). (b) The resolution, R, vs. the length of the time series, T_{ob}. Inset: the observation time T_{ob} as a function of the external magnetic field B_0. $\theta_a = 2.5$, $D_u = 0.05$, $\mu_0 = 1$, $B_- = -B_+ = 2$, $c = 3$, and $B_0 = 0$ except inset in (b).

magnetisation results in a loss of memory of the initial magnetisation when the threshold is crossed (if the reset fully saturates the core then all memory effects are removed). For small τ, the level that the magnetization is reset to strongly depends on the value of the magnetisation when the threshold was crossed (see inset of Fig. 2(a) for $\tau = 0.5$) and hence strong negative correlation is observed (see Fig. 2(a) for $\tau = 0.5$).

The target magnetic field B_0 can be estimated from N time intervals, T_i, $i = 1, 2, ..., N$ as the total observation time $\tau_{ob} = \sum_{i=1}^{N} T_i$. The resolution, R, of the magnetic sensor is defined via (7) with the replacement $\sigma_{\tau_{ob,N}} = \sigma_{\tau_{ob}} = \sqrt{\langle[\tau_{ob} - T_{ob}]^2\rangle}$, with the mean observation time identified as $T_{ob} = \langle\tau_{ob}\rangle$, and $s = B_0$ the target signal.

The dependence of the observation time T_{ob} on B_0 is shown in the inset of Fig. 2(b). This monotonic dependence can be used to estimate the target field. Fig. 2(b) shows that R scales as $T_{ob}^{-0.5}$ for the non-correlated intervals (e.g. for parameter value $\tau = 3$). However, when strong negative correlation exists (e.g. $\tau = 0.1$) the scaling is more complex. The scalings T_{ob}^{-1} and $T_{ob}^{-0.5}$ are shown as the black and red straight lines and these are seen to asymptote to the $\tau = 0.1$ data at small and large observation times respectively. This provides clear evidence that at short observation times the enhanced scaling T_{ob}^{-1} is observed; this scaling crosses over to $T_{ob}^{-0.5}$ at large observation time. Theoretical we have shown [13] that this dual scaling appears to be a universal property; it occurs for linear and nonlinear reset mechanisms and in models of sensors and neural models. Moreover, our theory [13] predicts the number of periods N^* at which the scaling crosses over from T_{ob}^{-1} to $T_{ob}^{-0.5}$; the result is $N^* \simeq 1/\varepsilon$ (see Fig 2(b)) (note ε can be estimated directly from the numerical results presented in Fig. 2(a).

We conclude that operating a nonlinear sensor in a 'biomimetic mode' can in principle improve its performance by exploiting negative correlation. In particular, absent the luxury of a long observation time, this operational mode might be helpful. Finally other methods of introducing negative correlation can be envisaged. Further research is required to establish how to exploit this effect for maximal advantage in sensor design.

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