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# Long-range correlations and coherent structures in magnetohydrodynamic equilibria

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The equilibrium theory of the 2D magnetohydrodynamic equations is derived, accounting for the full infinite hierarchies of conserved integrals. An exact description in terms of two coupled elastic membranes emerges, producing long-ranged correlations between the magnetic and velocity fields. This is quite different from the results of previous variational treatments, which relied on a local product ansatz for the thermodynamic Gibbs distribution. The equilibria display the same type of coherent structures, such as compact eddies and zonal jets, previously found in pure fluid equilibria. Possible consequences of this for recent simulations of the solar tachocline are discussed.

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The ideal magnetohydrodynamic (MHD) equations

$$\begin{aligned}\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + 2\boldsymbol{\Omega} \times \mathbf{v} &= -\nabla P + \mathbf{J} \times \mathbf{B} \\ \partial_t \mathbf{B} &= \nabla \times (\mathbf{v} \times \mathbf{B})\end{aligned}\quad (1)$$

describe the evolution (in a frame of reference with angular velocity  $\boldsymbol{\Omega}$ ) of the velocity field  $\mathbf{v}$  of a perfectly conducting fluid under the influence of external and internal forces ( $P$  includes pressure, centrifugal force, gravity, etc.), and advection of the magnetic field  $\mathbf{B}$  by the velocity field [1]. The equations are closed through Ampere's law  $\mathbf{J} = \nabla \times \mathbf{B}$  and the incompressibility constraints  $\nabla \cdot \mathbf{v} = 0$ ,  $\nabla \cdot \mathbf{B} = 0$ .

A number of systems such as the solar tachocline [2], and possibly the Earth's core-mantle boundary [3], are approximately governed by a 2D approximation in which  $\mathbf{v}, \mathbf{B}$  are horizontal, depending only on the horizontal coordinates  $\mathbf{r} = (x, y)$  in a domain  $D$  of the  $xy$ -plane (with 1D boundary  $\partial D$ ) [4]. The current  $\mathbf{J} = J\hat{\mathbf{z}}$  and vorticity  $\nabla \times \mathbf{v} = \omega\hat{\mathbf{z}}$  may then be treated as scalars. One may also express  $\mathbf{B}, \mathbf{v}$  in terms of a potential  $A$  and stream function  $\psi$ :  $\mathbf{B} = \nabla \times (A\hat{\mathbf{z}})$ ,  $\mathbf{v} = \nabla \times (\psi\hat{\mathbf{z}})$ , with the 2D Laplacian relationships  $J = -\nabla^2 A$ ,  $\omega = -\nabla^2 \psi$ . The equations then reduce to two scalar equations

$$\begin{aligned}\partial_t \omega + \mathbf{v} \cdot \nabla (\omega + f) &= \mathbf{B} \cdot \nabla J \\ \partial_t A + \mathbf{v} \cdot \nabla A &= 0\end{aligned}\quad (2)$$

with Coriolis parameter  $f(\mathbf{r}) = 2\Omega \sin(\varphi)$ , with latitude  $\varphi(y)$ . In the beta-plane approximation one linearizes  $f(y) = f_0 + \beta y$  about a reference  $\varphi_0$ . The total energy is

$$E = \frac{1}{2} \int d^2r (|\mathbf{v}|^2 + |\mathbf{B}|^2) = \frac{1}{2} \int d^2r (|\nabla \psi|^2 + |\nabla A|^2). \quad (3)$$

Its conservation requires lossless boundary conditions, e.g., periodic or "free-slip":  $\mathbf{B} \cdot \hat{\mathbf{n}} = 0$ ,  $\mathbf{v} \cdot \hat{\mathbf{n}} = 0$ , with  $\hat{\mathbf{n}}$  the local normal to  $\partial D$ . The latter imply constant values of  $A, \psi$  on each connected component of  $\partial D$ .

The pure 2D fluid equation ( $\mathbf{B} \equiv 0$ ) exhibits a turbulent inverse cascade: small scale vortices self-organize into large scale flows, limited only by the domain size.

In a forced system, this leads to growing flows, limited only by dissipation. These take the form of strong jet-like structures, or large coherent vortices, which have strong implications for geophysical flow stability and global transport. The weather bands and Great Red Spot of Jupiter are famous examples. For weak driving and dissipation, these flows may be modeled as near-equilibrium. Application of statistical mechanics to the fluid equations indeed produces such structures [5–12].

Similar flows in the solar tachocline, which sharply divides the rigidly rotating interior radiation zone from the differentially rotating outer convection zone, would have strong implications for angular momentum transport between the two zones [2]. Recent 2D MHD simulations [2, 4], however, have found that the presence of even weak  $\mathbf{B}$  tends to break up large scale flows. Although  $\mathbf{B}$  indeed destroys the infinite set of vorticity conservation laws that lead to the fluid inverse cascade, they are replaced by an entirely new infinite set of magnetic conservation laws [13]. This motivates the present investigation of the equilibria allowed by these new laws, and what conditions might limit their amplitude.

It will be shown that the new conservation laws indeed produce large scale flows, but of distinctly different character. In contrast to Euler flow, where all initial energy cascades to large scales, in 2D MHD a finite fraction flows to small scales, generating large microscale fluctuations. For small initial  $\mathbf{B}$ , these can be much larger than the mean flow [2, 4]. However, dissipation effects differentially tend to suppress these, and some form of driving might amplify the mean flow to more visible levels. The results here may then point to more careful studies of the types driving and initial conditions that could produce stronger large scale flows, and their physical realizability. In addition, unlike Euler flow, in which the small scale fluctuations are completely uncorrelated on the microscale [5, 7], the 2D MHD flows are shown to exhibit long-ranged power-law correlations, that could also be confirmed in simulations. Such correlations were invisible to earlier treatments which used a local product approximation for the Gibbs distribution [15, 16].

The MHD equations are nonlinear, thereby producing turbulent initial evolution from a spatially complex initial condition. However, at late time, the flow may equilibrate to a steady state. Unusually, such 2D fluid steady states are not necessarily quiescent, but can exhibit spatial structure such as large scale vortices or zonal jets [5–12]. The origin, and diversity, of these structures lies in the infinite number of conservation laws, beyond the usual energy (along with momentum or angular momentum if the relevant translation or rotation symmetry exists) constraining the flow [13]. If one sets  $\mathbf{B} \equiv 0$ , the first line of (2) is the Euler equation, and the resulting advective conservation of the potential vorticity  $\omega_p = \omega + f$  implies conservation of all spatial integrals (Casimirs)  $K_g = \int_D d^2r g(\omega_p)$ , with  $g$  an arbitrary 1D function. However, any nonzero  $\mathbf{B} \cdot \nabla J$  breaks this conservation [14], replacing it by advective conservation of  $A$  [second line of (2)]. The new Casimirs are [13]

$$J_g = \int d^2r g(A), \quad K_g = \int_D d^2r (\omega + f)g(A) \quad (4)$$

for arbitrary 1D functions  $g$ . Integrating by parts, one may also replace  $\omega g(A) \rightarrow (\mathbf{v} \cdot \mathbf{B})g'(A)$ . These may be parameterized as conservation of the functions  $j(\sigma), k(\sigma)$  for all  $\sigma$  obtained using  $g(s) = \delta(s - \sigma)$ . Finally, any conserved momentum may be expressed in the form

$$P = \int d^2r \alpha \omega = \int d^2r (\nabla \alpha) \times \mathbf{v} = - \int d^2r \nabla \alpha \cdot \nabla \psi, \quad (5)$$

for some fixed function  $\alpha(\mathbf{r})$  [17].

Under the usual ergodic assumptions (whose validity is far from obvious, and known to be violated for some initial conditions [18]), the equilibrium statistics are obtained from the grand canonical partition function

$$Z[\beta, \lambda, \mu, \nu] = \int D[A, \psi] e^{-\beta \mathcal{G}[A, \psi]}, \quad (6)$$

and associated free energy  $\mathcal{F} = -T \ln(Z)$ . The functional integral is over all fields  $A, \psi$  (obeying the appropriate boundary conditions),  $\beta = 1/T$  is the inverse temperature, and the Gibbs functional is  $\mathcal{G} = E - J_\mu - K_\nu - \lambda P$ :

$$\mathcal{G} = \int d^2r \left\{ \frac{1}{2} |\nabla A|^2 + \frac{1}{2} |\nabla \psi|^2 - \nu'(A) \nabla A \cdot \nabla \psi + \lambda \nabla \alpha \cdot \nabla \psi - [\mu(A) + f\nu(A)] \right\}, \quad (7)$$

where Lagrange multipliers have been introduced for each conserved integral: functions  $\mu(s), \nu(s)$  and parameters  $\beta, \lambda$  are adjusted to obtain the values of  $j(\sigma), k(\sigma), P, E$  defined by the initial flow. The physical model associated with  $\mathcal{G}$  is that of two membranes, with “heights”  $A, \psi$ , and unit surface tension, coupled through their gradients. The  $P$  term also acts to bias  $\nabla \psi$ ; in particular,  $\mathcal{G}$  favors  $\mathbf{B}$  parallel to  $\nu'(A)\mathbf{v} + \lambda \nabla \times \alpha$ . The term  $\mu(A) + f\nu(A)$

is an external potential, confining  $A$  near its minimum, and depends (smoothly) on position through  $f(\mathbf{r})$ .

The equilibria governed by  $\mathcal{G}$  have been previously investigated [15, 16] using a variational approach in which  $\mathbf{v}, \mathbf{B}$  were treated as spatially uncorrelated. The exact physics of  $\mathcal{G}$ , however, dictates something quite different. The elastic interactions generate very long-ranged correlations, with distortion in the surfaces interacting in a Coulomb-like fashion, leading to log-divergent fluctuations, and dipole-like correlations for  $\mathbf{v}, \mathbf{B}$ .

To understand the nature of the equilibrium states, it is extremely useful begin with a discrete spatial mesh, and then consider the continuum limit. To this end, using a square lattice with mesh size  $a$  one obtains

$$\begin{aligned} \beta \mathcal{G}_a = & \frac{1}{2} \beta \sum_{\langle i, j \rangle} \{ (A_i - A_j)^2 + (\psi_i - \psi_j)^2 \\ & - [\nu'(A_i) + \nu'(A_j)](A_i - A_j)(\psi_i - \psi_j) \\ & + 2\lambda(\alpha_i - \alpha_j)(\psi_i - \psi_j) \} \\ & - \beta a^2 \sum_i [\mu(A_i) + f_i \nu(A_i)], \end{aligned} \quad (8)$$

in which  $\langle i, j \rangle$  are nearest neighbors. The functional integral  $D[A, \psi] \rightarrow \prod_i \int dA_i d\psi_i$  now becomes (up to overall normalization) a product of single site integrals [19]. We define also  $\mathbf{B}_i = a^{-1}(A_{i+\hat{\mathbf{y}}} - A_i, A_i - A_{i+\hat{\mathbf{x}}})$ ,  $\mathbf{v}_i = a^{-1}(\psi_{i+\hat{\mathbf{y}}} - \psi_i, \psi_i - \psi_{i+\hat{\mathbf{x}}})$ . It is apparent here that if  $\beta$  remains finite as  $a \rightarrow 0$  then the first term will yield  $O(\sqrt{T})$  fluctuations between neighboring sites, and the second term becomes negligible [20]. Well defined hydrodynamic equilibria, with nontrivial competition between kinetic and potential energies requires that  $\beta$  scale with  $a$ . Specifically, taking  $\beta(a) = \bar{\beta}/a^2$  yields  $O(a\bar{T}^{1/2})$  site-to-site fluctuations, and continuous  $A(\mathbf{r}), \psi(\mathbf{r})$ . On the other hand  $\mathbf{B}_i, \mathbf{v}_i$  will have finite,  $O(\sqrt{T})$ , site-to-site fluctuations, yielding non-differentiable continuum  $A, \psi$ . With this scaling there are two surviving contributions to  $\mathcal{G}$  as  $a \rightarrow 0$ . Decomposing  $A = A_0 + \delta A$ ,  $\psi = \psi_0 + \delta \psi$  in which  $A_0 = \langle A \rangle$ ,  $\psi_0 = \langle \psi \rangle$  are the equilibrium averages (self-consistently determined below) one obtains

$$\begin{aligned} \mathcal{G}[A, \psi] &= \mathcal{G}[A_0, \psi_0] + \mathcal{G}_2[\delta A, \delta \psi; A_0] + O(a), \\ \mathcal{G}_2 &= \frac{1}{2} \int d^2r [|\nabla \delta A|^2 + |\nabla \delta \psi|^2 \\ &\quad - 2\nu'(A_0)(\nabla \delta A) \cdot (\nabla \delta \psi)] \\ &= \frac{1}{2} \int d^2r [ [1 + \nu'(A_0)] |\nabla \delta \phi^-|^2 \\ &\quad + [1 - \nu'(A_0)] |\nabla \delta \phi^+|^2 ], \end{aligned} \quad (9)$$

in which  $\delta \phi^\pm = (\delta A \pm \delta \psi)/\sqrt{2}$  are independent Gaussian fields [21]. We have reverted, for compactness, to continuum notation. For smooth  $A_0, \psi_0$ , all other terms, including those linear in  $\delta A, \delta \psi$ , vanish with  $a \rightarrow 0$ . From  $\mathcal{G}_2$  one therefore obtains the exact free energy

$$\mathcal{F}[\bar{\beta}, \lambda, \mu, \nu] = \mathcal{G}[A_0, \psi_0] + \mathcal{F}_2[A_0] \quad (10)$$

in which the Gaussian fluctuation free energy  $\mathcal{F}_2$  has a well defined  $a \rightarrow 0$  limit. It is not computable in closed form for general  $A_0$ , but for constant  $\nu'(A_0) \equiv \nu'_0$  and periodic boundary conditions one obtains

$$\mathcal{F}_2 = \frac{|D|}{\beta} \int_{\text{BZ}} \frac{d^2 k}{(2\pi)^2} \ln \left[ \frac{\bar{\beta}}{2\pi} \sqrt{1 - \nu_0'^2 E(\mathbf{k})} \right]$$

$$E(\mathbf{k}) \equiv 4 \left[ \sin^2(k_x/2) + \sin^2(k_y/2) \right], \quad (11)$$

in which  $|D|$  is the area of  $D$ , and the Brillouin zone is defined by  $\pi < k_x, k_y \leq \pi$ . More generally,  $\mathcal{F}_2$  is obtained from the log-sum of the eigenvalues of the (generalized Laplacian) operators  $\mathcal{O}^\pm = -\nabla \cdot [1 \mp \nu'(A_0)] \nabla$ .

The divergence of  $\beta = \bar{\beta}/a^2$  in (6) means that  $A_0, \psi_0$  are determined self-consistently by minimizing  $\mathcal{F}$ . The conditions  $\delta\mathcal{G}_0/\delta\psi_0(\mathbf{r}) = 0$ ,  $\delta(\mathcal{G}_0 + \mathcal{F}_2)/\delta A_0(\mathbf{r}) = 0$  produce, respectively, the equilibrium equations

$$\mathbf{v}_0 = \nu'(A_0) \mathbf{B}_0 - \lambda \nabla \times \alpha$$

$$\omega_0 + \nu'(A_0) J_0 = \mu'(A_0) + f \nu'(A_0) + \nu''(A_0) \gamma(\mathbf{r}, \mathbf{r}; A_0), \quad (12)$$

where  $\mathbf{B}_0 = \nabla \times (A_0 \hat{\mathbf{z}})$ , etc., are the other derived equilibrium fields. The first equation provides a direct relation between the equilibrium velocity and magnetic field:  $\mathbf{B}_0$  is colinear with  $\mathbf{v}_0$  up to a mean flow subtraction. Substituting the curl of the first equation into the second, one obtains a closed equation for  $A_0$ :

$$-\nabla \times \{ [1 - \nu'(A_0)^2] \mathbf{B}_0 \} = \mu'(A_0) + f \nu'(A_0) + \lambda \nabla^2 \alpha + \frac{1}{2} [\nu'(A_0)^2]' |\mathbf{B}_0|^2 + \nu''(A_0) \gamma(\mathbf{r}, \mathbf{r}). \quad (13)$$

The fluctuation-derived nonlocal term at the end is obtained from the cross-correlation function

$$\gamma(\mathbf{r}, \mathbf{r}') = \langle \nabla \delta A(\mathbf{r}) \cdot \nabla \delta \psi(\mathbf{r}') \rangle_2 = \langle \delta \mathbf{B}(\mathbf{r}) \cdot \delta \mathbf{v}(\mathbf{r}') \rangle_2. \quad (14)$$

If one defines the Green functions of the operators  $\mathcal{O}^\pm$ ,

$$-\nabla \cdot [1 \mp \nu'(A_0)] \nabla G^\pm(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'), \quad (15)$$

which characterize the membrane fluctuations,

$$G^\pm(\mathbf{r}, \mathbf{r}') = \frac{\bar{\beta}}{a^2} \langle [\delta \phi^\pm(\mathbf{r}) - \delta \phi^\pm(\mathbf{r}')]^2 \rangle_2, \quad (16)$$

then one obtains the relation

$$\gamma(\mathbf{r}, \mathbf{r}') = \frac{1}{2} \nabla \cdot \nabla' [G^+(\mathbf{r}, \mathbf{r}') - G^-(\mathbf{r}, \mathbf{r}')]. \quad (17)$$

Solutions to (15) yield the electrostatic potential generated by a unit charge at  $\mathbf{r}'$  with spatially varying dielectric function  $\epsilon^\pm(\mathbf{r}) = 1 \mp \nu'(A_0)$ . By Gauss's law, the displacement field  $\epsilon^\pm \nabla G^\pm$  has unit flux through any bounding contour, and it follows that  $G^\pm(\mathbf{r} - \mathbf{r}') \sim (1/2\pi\epsilon^\pm) \ln(|\mathbf{r} - \mathbf{r}'|/a)$  grows logarithmically with distance. This unbounded wandering reflects the usual thermal roughening result for 2D membranes [22].

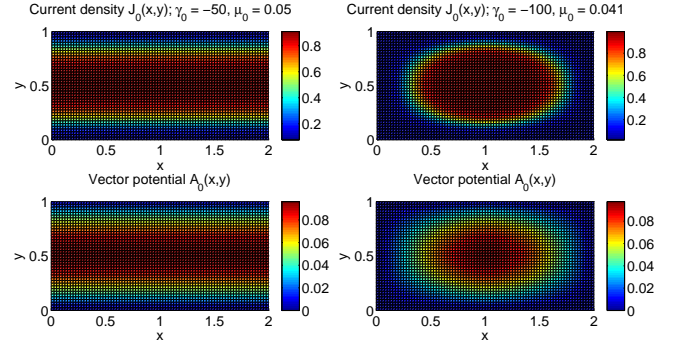


FIG. 1: **Left:** Equilibrium jet solution obtained using periodic boundary conditions in  $x$ . **Right:** Equilibrium vortex solution obtained using free slip boundary conditions in both  $x$  and  $y$ . Vortex solutions exist as well for periodic boundary conditions, but the numerics are more involved. Stream lines and magnetic field lines are level curves of constant potential  $A_0$  (bottom panels) or current density  $J_0$  (top panels). For simplicity we set  $\bar{T} = 0$  (no microscale fluctuations),  $f = 0$  (no rotation),  $\lambda = 0$  (no net mean flow), take constant  $\nu'(A) = \nu'_0$  (hence  $\mathbf{B}_0 = \mathbf{v}_0/\nu'_0$ ), and use the form  $\mu(A) = -\gamma^{-1} \ln[1 + e^{-\gamma(A-\mu_1)}]$ . For large  $\gamma < 0$ , the latter tends to produce a two level system, with  $J_0 = \omega_0/\nu'_0$  taking values 0 or  $q = (1 - \nu_0'^2)^{-1}$ , and the parameter  $\mu_1$  controls the proportion of each level. Plotted are actually the scaled quantities  $A_0/q$ ,  $J_0/q$ , with parameters  $\gamma_0 = \gamma q$ ,  $\mu_0 = \mu_1 q$  specified in the titles. The results are then independent of  $q$  (and hence of the choice of  $\nu'_0$ ).

For a translation invariant system (i.e., periodic, uniform), one may replace  $\nabla' \rightarrow -\nabla$  and (15) implies that  $\gamma \equiv 0$  for  $\mathbf{r} \neq \mathbf{r}'$ , and  $\gamma(\mathbf{r}, \mathbf{r}) = \nu'_0/2\bar{\beta}[1 - \nu_0'^2]$ . For this case, the locality assumption [15] is valid [23]. Otherwise one obtains nonlocal contributions to  $\gamma(\mathbf{r}, \mathbf{r})$ , even for a uniform system with free slip boundary conditions: nonlocal contributions then arise from image charges outside  $D$  that enforce the boundary conditions on  $G^\pm$ .

Once the equilibrium solution is obtained, the values of the conserved variables may be obtained from derivatives of  $\mathcal{F}$  at fixed  $A_0, \Psi_0$ :

$$E = \frac{1}{2} \int_D d^2 r [|\mathbf{v}_0|^2 + |\mathbf{B}_0|^2 + \varepsilon(\mathbf{r}, \mathbf{r})]$$

$$P = -\frac{\partial \mathcal{F}}{\partial \lambda} = \int_D d^2 r \alpha \omega_0$$

$$j(\sigma) = -\frac{\delta \mathcal{F}}{\delta \mu(\sigma)} = \int_D d^2 r \delta[\sigma - A_0(\mathbf{r})]$$

$$k(\sigma) = -\frac{\delta \mathcal{F}}{\delta \nu(\sigma)} = \int_D d^2 r \{ (\omega_0 + f) \delta[\sigma - A_0(\mathbf{r})] - \gamma(\mathbf{r}, \mathbf{r}) \delta'[\sigma - A_0(\mathbf{r})] \}. \quad (18)$$

where  $\varepsilon(\mathbf{r}, \mathbf{r}') = -\nabla \cdot \nabla' [G^+(\mathbf{r}, \mathbf{r}') + G^-(\mathbf{r}, \mathbf{r}')] determines the microscale fluctuation part of the energy. The extremum condition ensures that all derivatives with respect to the implicit dependence of  $A_0, \psi_0$  on the Lagrange multipliers cancels out.$

Comparing (4), it is seen, due to continuity of  $A$ , that  $j(\sigma)$  is a large-scale quantity: it is its own equilibrium average. Hence the level sets of the initial  $A(\mathbf{r}, t = 0)$  are exactly preserved in the equilibrium  $A_0(\mathbf{r})$ . One implication is that a low-amplitude initial  $A$  will produce an identically low-amplitude  $A_0$ . However, e.g., for a spatially irregular initial condition with comparatively large energy, one will have  $\varepsilon \gg |\mathbf{v}_0|^2, |\mathbf{B}_0|^2$  and the physical equilibrium fields will be masked by fluctuations. This may help explain what is observed in simulations [2, 4]. For increasing initial  $A$ , especially in the presence of forcing and weak dissipation, one may expect clear equilibria to emerge more strongly, but the exact conditions required for this remain to be determined.

Equations (12), (17) and (18) are the basic results of this paper. These equations are nonlocal and highly nonlinear, and generally require a numerical solution. However, some general properties may be inferred from the basic form of the effective Hamiltonian (9). The  $\mathcal{G}_0$  term reflects a straightforward classical surface tension minimization problem. The fluctuations locally stretch the membranes and hence act to renormalize the surface tension, but in a way that depends self-consistently on the average membrane position  $A_0, \psi_0$ , which in turn respond to the external forces provided by the  $\mu + f\nu$  and  $\lambda\alpha$  terms in (7). Thus, a stretched region of a membrane may be expected to have reduced amplitude fluctuations. The minimization of  $\mathcal{F}$  accounts fully for both effects. Figure 1 shows examples of very simple equilibrium solutions in a rectangular domain (with model parameters specified in the caption), demonstrating the existence of both vortex and jet structures.

Unlike the Euler case, where the conserved integrals ensure bounded  $\omega = \nabla \times \mathbf{v}$ , thereby generating continuous  $\mathbf{v}$  and differentiable  $\psi$ , for 2D MHD only continuity of  $A, \psi$  are provided, and this occurs now in response to the squared-gradients in the energy, not in response to the conserved integrals. The latter also ensures bounded  $\delta\mathbf{v}, \delta\mathbf{B}$ , but this in turn allows for finite microscale energy density  $\varepsilon$ . The second derivatives  $\omega, J$  then have unbounded fluctuations. Since simulations often propagate  $A, \omega$  using (2), extra care may then be required to ensure reasonable equilibration. In particular, if (e.g., hyperviscous) dissipation acts too strongly to quell the micro-fluctuations, it may also bleed energy out of the large scale flow. There could also be physical analogues of this effect, depending on the precise nature, e.g., of the true solar tachocline dissipation mechanisms. These, and presumably many other, considerations must enter the implications of the theory developed here.

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  - [22] This log-divergence reflects only the physics of  $\mathcal{G}_2$ . The full physics of  $\mathcal{G}_0 + \mathcal{G}_2$  confines the membranes close to  $A_0, \psi_0$  on length scales  $\gg a$ , but still microscopic.
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