Woltjer-Taylor State without Taylor's Conjecture: Plasma Relaxation at all Wavelengths

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Phys. Rev. Lett. **109**, 235001 — Published 3 December 2012

DOI: 10.1103/PhysRevLett.109.235001

Woltjer-Taylor state without Taylor's conjecture – plasma relaxation at all wavelengths

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Abstract

In the process of dissipative relaxation, there is strong astrophysical and laboratory evidence that plasmas tend to evolve towards the well-known Woltjer-Taylor state, specified by $\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}$ for constant α. To explain how such a state is reached, Taylor developed his famous theory based on the conjecture that relaxation is dominated by short wavelength fluctuations. However, there is no conclusive experimental or numerical evidence in support of Taylor's conjecture. A new theory is developed, which predicts that the system will evolve towards the Woltjer-Taylor state for an arbitrary fluctuation spectrum.

PACS numbers: 52.55.Lf, 52.55.Tn

In both astrophysical and laboratory environments [1–10] there is strong observational evidence that plasmas tend to evolve towards the Woltjer-Taylor state,

$$\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B}, \quad \alpha \text{ is a constant.} \tag{1}$$

It is a perplexing theoretical question how, why, and under what conditions such a state is reached. In 1958 Woltjer [11] first showed that Eq. (1) can be obtained by minimizing the global magnetic energy

$$W \equiv \int_V \boldsymbol{B}^2 d^3\boldsymbol{x}, \tag{2}$$

while keeping the global magnetic helicity

$$H \equiv \int_V \boldsymbol{A} \cdot \boldsymbol{B} d^3\boldsymbol{x} \tag{3}$$

constant. Here the integration domain V is the entire 3D volume with a perfectly-conducting boundary. The justification for this minimization procedure is that in the ideal magnetohydrodynamic (MHD) model, the global magnetic helicity, H [Eq. (3)] is a constant of motion [12]. Equation (1) is obtained through a simple variational procedure, with α the Lagrange multiplier. To be faithful to history, we remark that Lust [13] and Chandrasekhar [14] had previously discussed force free states with constant α in the context of astrophysics, although no convincing justification was given as to why this should be the most interesting force free field. Woltjer's work was the first theoretical attempt to explain its importance; however, a problem is that it does not specify exactly how this relaxed state can be reached. Within ideal MHD, the global helicity defined in Eq. (3) is not the only invariant. In particular, for *any* given flux surface $\varphi = const.$, the magnetic helicity $H_\varphi \equiv \int_\varphi \boldsymbol{A} \cdot \boldsymbol{B} d^3\boldsymbol{x}$ in the volume enclosed by the flux surface is a conserved quantity. This implies that the topological structure of the magnetic field is invariant. As a consequence [15–17], for an arbitrary initial condition the final state specified by Eq. (1) will not be dynamically accessible.

To explain this puzzle, Taylor suggested that if the plasma is slightly resistive then the topological structure of the magnetic field will be destroyed. Nonetheless, since the deviation of the magnetic field from the ideal case will be relatively small, the global helicity H [Eq. (3)] will still be approximately conserved. With this idea, Taylor argued that for any initial condition the system will relax towards a final state which minimizes the global magnetic energy while keeping the global helicity constant. Such a state is of course specified by

Eq. (1). For this reason we will refer this relaxed state as the Woltjer-Taylor state throughout this article.

Taylor's theory has been successful at predicting the reversal of the toroidal field in a series of RFP (Reverse Field Pinch) experiments [1–10]. Even though it has been extended in various directions since the 1980s [17–26], the theory maintains its popularity because of its simplistic beauty, and is widely accepted as a fundamental theory with great importance in plasma physics. Nevertheless, there is one element of Taylor's theory that remains unsatisfactory. The relaxation of the plasma is caused by resistivity, and rigorously speaking neither helicity, H, nor magnetic energy, W, are conserved quantities in resistive MHD. In order to justify keeping H constant in the variational procedure, Taylor (along with other researchers [10, 15, 16, 27]) observed that decay rates for H and W scale as

$$\frac{dH}{dt} \simeq -\frac{2Vc^2\eta}{4\pi}\sum_{\boldsymbol{k}} |\boldsymbol{k}|\boldsymbol{B}_{\boldsymbol{k}}^2 \tag{4}$$

$$\frac{dW}{dt} \simeq -\frac{2Vc^2\eta}{4\pi}\sum_{\boldsymbol{k}} \boldsymbol{k}^2\boldsymbol{B}_{\boldsymbol{k}}^2 \tag{5}$$

where η is the resistivity, $\boldsymbol{k}$ is the wavenumber of the fluctuation and $\boldsymbol{B}_{\boldsymbol{k}}$ is the Fourier component of the magnetic field at $\boldsymbol{k}$. (A detailed derivation these expressions is given by Eqs. (13), (14) and (27), with their validity discussed near the end of the paper.) According to Eqs. (4) and (5) [10, 15, 16, 27], dissipation of both W and H is due to the finite resistivity, but the decay rate of W scales with $\boldsymbol{k}^2$, while that of H scales with $\boldsymbol{k}$. If the relaxation process is dominated by structures with wavelengths shorter than $\eta^{1/2}$, dissipation of W will be much larger than that of H. Taylor conjectured that this is indeed the case, providing natural justification for minimizing W at fixed H. "Unfortunately", as pointed out by Ortolani and Schnack [10], "in the RFP there is no experimental evidence that relaxation is produced by small scale turbulence. The dominant magnetic fluctuations associated with the relaxation process appear to have global, long wavelength structure. This view is supported by extensive numerical simulations, which show that relaxation is produced by the nonlinear interaction of long wavelength instabilities. (Many of these results will be described in detail in Chapter 5)." It is at least fair to surmise that experimental [1–10] and theoretical [28–35] studies in the last 40 years do not provide conclusive evidence to support the conjecture that plasma relaxation should be dominated by short wavelength structures. Realizing this shortcoming in Taylor's theory, Bhattacharjee *et al.* [17, 18, 20–22] discovered that for a single helicity long wavelength resistive tearing mode, the global helicity is nearly constant, along with an infinite set of other approximate invariants. A theory of relaxation has been developed using these invariants [17, 18, 20–22], and the relaxed state is in general different from the Woltjer-Taylor state.

In this paper, we present a new theory on how the Woltjer-Taylor state can be reached during resistive plasma relaxation, without invoking Taylor's conjecture. We do not need to assume that the fluctuation spectrum is dominated by short wavelength structures or that W decays faster than H. In our theory, the Woltjer-Taylor state is not reached by minimizing W at fixed H. We show that the Woltjer-Taylor state arises naturally as the final state of a resistive MHD relaxation process at low plasma β, for any fluctuation spectrum. We prove this fact as follows.

For any vector potential $\boldsymbol{A}$ and magnetic field $\boldsymbol{B}$, the well-known Cauchy-Schwartz inequality is

$$QW - H^2 \geq 0\,, \tag{6}$$

$$Q \equiv \int_V \boldsymbol{A}^2 d^3\boldsymbol{x}\,. \tag{7}$$

The equality is reached if and only if $\boldsymbol{B} = \alpha\boldsymbol{A}$ everywhere, with constant α. Amazingly, this condition is exactly Eq. (1); *i.e.*, the condition for the plasma to be in a Woltjer-Taylor state. We will further prove that in the resistive MHD model the difference between QW and H^2 decreases with time, *i.e.*,

$$\frac{d}{dt}\left(QW - H^2\right) \leq 0, \tag{8}$$

and that equality in inequality (8) holds if and only if Eq. (1) is satisfied. These inequalities [Eqs. (8) and (6)] thus imply that the non-negative quantity $QW - H^2$ ceases to decrease in time only when it is zero, demonstrating that the system must dynamically evolve towards a Woltjer-Taylor state. When the system is far from this state, the rate of change, $d(QW - H^2)/dt$, can be significantly negative, meaning the system is evolving towards a Woltjer-Taylor state at a fast pace. An illustration of the dynamical behavior of H^2 and QW is given in Fig. 1.

We now give proofs of inequality (8) and the fact that equality is reached if and only if the plasma is in a Woltjer-Taylor state. As in previous studies, we will use the resistive MHD equations [27] and assume that the thermal and kinetic energies are much smaller than the magnetic energy. For clarity we give two separate proofs: in the first we utilize a

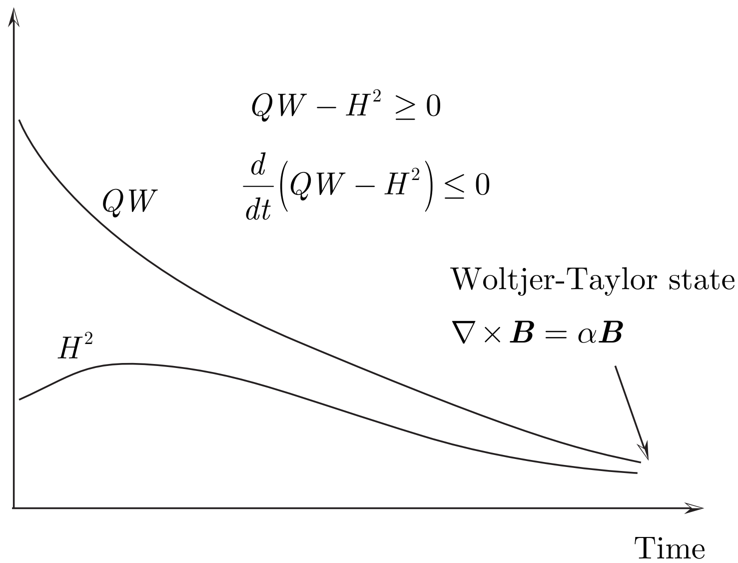

Figure 1: Away from a Woltjer-Taylor state, $QW - H^2$ is positive definite and decreasing in time at a non-vanishing rate. Thus, the system is relaxing towards the Woltjer-Taylor state. The state is reached when $QW - H^2 = 0$.

Fourier decomposition, while the second is a direct proof in real space, which is more succinct and transparent [36]. For notational simplicity, we use $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ to denote the following scalar product between two vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle \equiv \int_V \boldsymbol{a} \cdot \boldsymbol{b} d^3\boldsymbol{x} \,.$$

The first method proceeds as follows. Let

$$(\boldsymbol{B}, \boldsymbol{J}, \boldsymbol{A}) = \sum (\boldsymbol{B_k}, \boldsymbol{J_k}, \boldsymbol{A_k}) \exp(i\boldsymbol{k} \cdot \boldsymbol{x}) \,, \tag{9}$$

then Ampï¿œre's law is expressed as

$$\boldsymbol{j_k} = \frac{ic\boldsymbol{k} \times \boldsymbol{B_k}}{4\pi} \,, \tag{10}$$

and $\nabla \times \boldsymbol{A} = \boldsymbol{B}$ as

$$i\boldsymbol{k} \times \boldsymbol{A_k} = \boldsymbol{B_k} \,. \tag{11}$$

At $\boldsymbol{k} = 0$, we will set $\boldsymbol{A}_0 = 0$ as there is no magnetic field associated with $\boldsymbol{A}_0$ and the electromagnetic field vanishes on the boundary. To solve for $\boldsymbol{A_k}(\boldsymbol{k} \neq 0)$ in terms of $\boldsymbol{B_k}$, we choose to work with the Coulomb gauge $\boldsymbol{k} \cdot \boldsymbol{A_k} = 0$, which leads to

$$\boldsymbol{A_k} = \frac{i\boldsymbol{k} \times \boldsymbol{B_k}}{\boldsymbol{k}^2} \quad (\boldsymbol{k} \neq 0). \tag{12}$$

Note that $(\boldsymbol{B}_{\boldsymbol{k}}^*, \boldsymbol{J}_{\boldsymbol{k}}^*, \boldsymbol{A}_{\boldsymbol{k}}^*) = (\boldsymbol{B}_{-\boldsymbol{k}}, \boldsymbol{J}_{-\boldsymbol{k}}, \boldsymbol{A}_{-\boldsymbol{k}})$, since $(\boldsymbol{B}, \boldsymbol{J}, \boldsymbol{A})$ are real. Here $\boldsymbol{u}^*$ denotes the complex conjugate of $\boldsymbol{u}$, and we adopt the notation $\boldsymbol{u}^2 \equiv \boldsymbol{u} \cdot \boldsymbol{u}^* = |\boldsymbol{u}|^2$ for the magnitude of a vector $\boldsymbol{u}$. In resistive MHD, the rate of change of H is given by

$$\frac{dH}{dt} = -2c\eta \int_V \boldsymbol{j} \cdot \boldsymbol{B} d^3\boldsymbol{x} = -2c\eta \langle \boldsymbol{j}, \boldsymbol{B} \rangle \ , \tag{13}$$

where the Ohm's law $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}/c = \eta \boldsymbol{j}$ has been used. Note that dH/dt can be both positive and negative. The integral in Eq. (13) can be evaluated using Fourier components,

$$\begin{aligned} \langle \boldsymbol{j}, \boldsymbol{B} \rangle = \int_V \boldsymbol{j} \cdot \boldsymbol{B} d^3\boldsymbol{x} &= \sum_{\boldsymbol{k},\boldsymbol{l}} \boldsymbol{j}_{\boldsymbol{k}} \cdot \boldsymbol{B}_{\boldsymbol{l}} \int_V \exp[i(\boldsymbol{k}+\boldsymbol{l}) \cdot \boldsymbol{x}] d^3\boldsymbol{x} = V \sum_{\boldsymbol{k}} \boldsymbol{j}_{\boldsymbol{k}} \cdot \boldsymbol{B}_{-\boldsymbol{k}} \\ &= V \sum_{\boldsymbol{k}} \boldsymbol{j}_{\boldsymbol{k}} \cdot \boldsymbol{B}_{\boldsymbol{k}}^* = V \sum_{\boldsymbol{k}} \frac{ic\boldsymbol{B}_{\boldsymbol{k}} \times \boldsymbol{B}_{\boldsymbol{k}}^*}{4\pi} \cdot \boldsymbol{k} = V \sum_{\boldsymbol{k} \neq 0} \frac{2c\boldsymbol{B}_{\boldsymbol{k}R} \times \boldsymbol{B}_{\boldsymbol{k}I}}{4\pi} \cdot \boldsymbol{k} \ , \end{aligned} \tag{14}$$

where V is the volume of the system, $\boldsymbol{B}_{\boldsymbol{k}R}$ and $\boldsymbol{B}_{\boldsymbol{k}I}$ are real and imaginary parts of $\boldsymbol{B}_{\boldsymbol{k}}$, and use is made of the following identities

$$\int_V \exp[i(\boldsymbol{k}+\boldsymbol{l}) \cdot \boldsymbol{x}] d^3\boldsymbol{x} = \begin{cases} 0, & \boldsymbol{k} \neq -\boldsymbol{l} \,, \\ V, & \boldsymbol{k} = -\boldsymbol{l} \,, \end{cases} \tag{15}$$

$$\boldsymbol{B}_{\boldsymbol{k}} \times \boldsymbol{B}_{\boldsymbol{k}}^* = -2i \boldsymbol{B}_{\boldsymbol{k}R} \times \boldsymbol{B}_{\boldsymbol{k}I} \,. \tag{16}$$

Similarly,

$$H = V \sum_{\boldsymbol{k} \neq 0} \frac{i\boldsymbol{B}_{\boldsymbol{k}} \times \boldsymbol{B}_{\boldsymbol{k}}^*}{\boldsymbol{k}^2} \cdot \boldsymbol{k} = V \sum_{\boldsymbol{k} \neq 0} \frac{2\boldsymbol{B}_{\boldsymbol{k}R} \times \boldsymbol{B}_{\boldsymbol{k}I}}{\boldsymbol{k}^2} \cdot \boldsymbol{k} \,. \tag{17}$$

From Eqs. (13), (14), and (17), we have

$$\frac{dH^2}{dt} = -\frac{V^2 c^2 \eta}{\pi} \sum_{\boldsymbol{k}} \boldsymbol{B}_{\boldsymbol{k}R} \times \boldsymbol{B}_{\boldsymbol{k}I} \cdot \boldsymbol{k} \sum_{\boldsymbol{k} \neq 0} \frac{\boldsymbol{B}_{\boldsymbol{k}R} \times \boldsymbol{B}_{\boldsymbol{k}I}}{\boldsymbol{k}^2} \cdot \boldsymbol{k} \,. \tag{18}$$

The rate of change of H^2 given by Eq. (18) will be compared with that of QW,

$$\frac{d(QW)}{dt} = Q\frac{dW}{dt} + W\frac{dQ}{dt} \,, \tag{19}$$

where W and Q can be expressed in terms of the Fourier components of $\boldsymbol{B}$,

$$\langle \boldsymbol{B}, \boldsymbol{B} \rangle \equiv W \equiv \int_V \boldsymbol{B}^2 d^3\boldsymbol{x} = V \sum_{\boldsymbol{k}} \boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{B}_{\boldsymbol{k}}^*, \tag{20}$$

$$\langle \boldsymbol{A}, \boldsymbol{A} \rangle \equiv Q \equiv \int_V \boldsymbol{A}^2 d^3\boldsymbol{x} = V \sum_{\boldsymbol{k} \neq 0} \frac{\boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{B}_{\boldsymbol{k}}^*}{\boldsymbol{k}^2} \,. \tag{21}$$

To calculate the rate of change of Q and W, we need to know $d\boldsymbol{B}_{\boldsymbol{k}}/dt \cdot \boldsymbol{B}^*_{\boldsymbol{k}}$, which can be calculated from Faraday's law $d\boldsymbol{B}_{\boldsymbol{k}}/dt = -ic\boldsymbol{k} \times \boldsymbol{E}_{\boldsymbol{k}}$ and Ohm's law as

$$\frac{d\boldsymbol{B}_{\boldsymbol{k}}}{dt} \cdot \boldsymbol{B}^*_{\boldsymbol{k}} = -ic\eta\boldsymbol{k} \times \boldsymbol{j}_{\boldsymbol{k}} \cdot \boldsymbol{B}^*_{\boldsymbol{k}} + i\boldsymbol{k} \times (\boldsymbol{v} \times \boldsymbol{B})_{\boldsymbol{k}} \cdot \boldsymbol{B}^*_{\boldsymbol{k}} \,. \tag{22}$$

The second term on the right-hand-side of Eq. (22) can be expressed in terms of the current

$$i\boldsymbol{k} \times (\boldsymbol{v} \times \boldsymbol{B})_{\boldsymbol{k}} \cdot \boldsymbol{B}^*_{\boldsymbol{k}} = \frac{4\pi}{c} \sum_{\boldsymbol{l}} \boldsymbol{B}_{\boldsymbol{l}} \times \boldsymbol{j}_{-\boldsymbol{k}} \cdot \boldsymbol{v}_{\boldsymbol{k}-\boldsymbol{l}} \,, \tag{23}$$

which is associated with the variation of kinetic energy due to the Lorentz force. To see this, we observe that the energy conservation law in the low-β limit takes the form

$$\frac{\partial}{\partial t}\left(\rho \frac{\boldsymbol{v}^2}{2}\right) + \nabla \cdot \left(\rho \boldsymbol{v} \frac{\boldsymbol{v}^2}{2}\right) - \frac{\boldsymbol{v}}{c} \cdot (\boldsymbol{j} \times \boldsymbol{B}) = 0 \,. \tag{24}$$

Integrating over the entire volume gives

$$\frac{\partial}{\partial t}\left(\int_V \rho \frac{\boldsymbol{v}^2}{2} d^3\boldsymbol{x}\right) = \int_V \frac{\boldsymbol{v}}{c} \cdot (\boldsymbol{j} \times \boldsymbol{B})\, d^3\boldsymbol{x} = V \sum_{\boldsymbol{k},\boldsymbol{l}} \frac{1}{c} \boldsymbol{v}_{\boldsymbol{k}-\boldsymbol{l}} \cdot \boldsymbol{j}_{-\boldsymbol{k}} \times \boldsymbol{B}_{\boldsymbol{l}} = -\frac{V}{4\pi} \sum_{\boldsymbol{k}} i\boldsymbol{k} \times (\boldsymbol{v} \times \boldsymbol{B})_{\boldsymbol{k}} \cdot \boldsymbol{B}^*_{\boldsymbol{k}} \,, \tag{25}$$

which indicates that the work done by the Lorentz force is converted into the kinetic energy of the plasma. Since we have assumed that the kinetic energy is much smaller than the magnetic energy, $\int_V \rho \boldsymbol{v}^2 d\boldsymbol{x}/2 \ll \int_V \boldsymbol{B}^2 d^3\boldsymbol{x}/8\pi$, it is clear that work due to the Lorentz force contributes little to the time rate of change of magnetic energy, *i.e.*,

$$i \sum_{\boldsymbol{k}} \boldsymbol{k} \times (\boldsymbol{v} \times \boldsymbol{B})_{\boldsymbol{k}} \cdot \boldsymbol{B}^*_{\boldsymbol{k}} \ll \sum_{\boldsymbol{k}} \frac{d\boldsymbol{B}_{\boldsymbol{k}}}{dt} \cdot \boldsymbol{B}^*_{\boldsymbol{k}}. \tag{26}$$

It follows that variation of the magnetic energy is mainly due to the finite resistivity,

$$\frac{dW}{dt} = \int_V 2\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} d^3\boldsymbol{x} = 2V \sum_{\boldsymbol{k}} \boldsymbol{B}_{\boldsymbol{k}} \cdot d\boldsymbol{B}^*_{\boldsymbol{k}}/dt = -\frac{2Vc^2\eta}{4\pi} \sum_{\boldsymbol{k}} \boldsymbol{k}^2 \boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{B}^*_{\boldsymbol{k}} \,. \tag{27}$$

We further assume that the condition that kinetic energy be much smaller than magnetic energy holds at large scales. Thus, when each term in Eq. (26) is weighted by $1/\boldsymbol{k}^2$, the total contribution from the Lorentz force term is still small, *i.e.*,

$$i \sum_{\boldsymbol{k}} \frac{1}{\boldsymbol{k}^2} \boldsymbol{k} \times (\boldsymbol{v} \times \boldsymbol{B})_{\boldsymbol{k}} \cdot \boldsymbol{B}^*_{\boldsymbol{k}} \ll \sum_{\boldsymbol{k}} \frac{1}{\boldsymbol{k}^2} \frac{d\boldsymbol{B}_{\boldsymbol{k}}}{dt} \cdot \boldsymbol{B}^*_{\boldsymbol{k}}. \tag{28}$$

Under these conditions, the rate of change of Q can be expressed in terms of $\boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{B}^*_{\boldsymbol{k}}$ as

$$\frac{dQ}{dt} = \int_V 2\boldsymbol{A} \cdot \frac{\partial \boldsymbol{A}}{\partial t} d^3\boldsymbol{x} = 2V \sum_{\boldsymbol{k} \neq 0} \frac{\boldsymbol{B}_{\boldsymbol{k}} \cdot d\boldsymbol{B}^*_{\boldsymbol{k}}/dt}{\boldsymbol{k}^2} = -\frac{2Vc^2\eta}{4\pi} \sum_{\boldsymbol{k} \neq 0} \boldsymbol{B}_{\boldsymbol{k}} \cdot \boldsymbol{B}^*_{\boldsymbol{k}} \,. \tag{29}$$

We will give a more detailed discussion of the validity of Eqs. (5), (27) and (29) near the end of the article. Equations (21), (20), (27), and (29) can be assembled together to give

$$
\begin{aligned}
\frac{d(QW)}{dt} &= -\frac{2c^2V^2\eta}{4\pi}\left[\boldsymbol{B}_0^2\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2+\left(\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2\right)^2+\sum_{\boldsymbol{k}\neq 0}\frac{\boldsymbol{B}_{\boldsymbol{k}}^2}{\boldsymbol{k}^2}\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2\boldsymbol{k}^2\right] \\
&= -\frac{2c^2V^2\eta}{4\pi}\left[\boldsymbol{B}_0^2\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2+2\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^4+\frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{l}\neq 0}^{\boldsymbol{k}\neq\boldsymbol{l}}\left(2+\frac{\boldsymbol{l}^2}{\boldsymbol{k}^2}+\frac{\boldsymbol{k}^2}{\boldsymbol{l}^2}\right)\boldsymbol{B}_{\boldsymbol{k}}^2\boldsymbol{B}_{\boldsymbol{l}}^2\right] \\
&\leq -\frac{2c^2V^2\eta}{4\pi}\left[\boldsymbol{B}_0^2\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2+2\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^4+\sum_{\boldsymbol{k},\boldsymbol{l}\neq 0}^{\boldsymbol{k}\neq\boldsymbol{l}}\left(\frac{|\,\boldsymbol{l}\,|}{|\,\boldsymbol{k}\,|}+\frac{|\,\boldsymbol{k}\,|}{|\,\boldsymbol{l}\,|}\right)\boldsymbol{B}_{\boldsymbol{k}}^2\boldsymbol{B}_{\boldsymbol{l}}^2\right], \qquad (30)
\end{aligned}
$$

where the following inequality has been used

$$
\left(2+\frac{\boldsymbol{l}^2}{\boldsymbol{k}^2}+\frac{\boldsymbol{k}^2}{\boldsymbol{l}^2}\right)=\left(\frac{|\,\boldsymbol{l}\,|}{|\,\boldsymbol{k}\,|}+\frac{|\,\boldsymbol{k}\,|}{|\,\boldsymbol{l}\,|}\right)^2\geq 2\left(\frac{|\,\boldsymbol{l}\,|}{|\,\boldsymbol{k}\,|}+\frac{|\,\boldsymbol{k}\,|}{|\,\boldsymbol{l}\,|}\right). \qquad (31)
$$

On the other hand, from Eq. (18),

$$
\begin{aligned}
&-\frac{dH^2}{dt}\leq\frac{V^2c^2\eta}{\pi}\sum_{\boldsymbol{k}}2\,|\,\boldsymbol{B}_{\boldsymbol{k}R}\,||\,\boldsymbol{B}_{\boldsymbol{k}I}\,||\,\boldsymbol{k}\,|\sum_{\boldsymbol{k}\neq 0}2\,|\,\boldsymbol{B}_{\boldsymbol{k}R}\,||\,\boldsymbol{B}_{\boldsymbol{k}I}\,|\,/\,|\,\boldsymbol{k}\,| \\
&\leq\frac{V^2c^2\eta}{\pi}\sum_{\boldsymbol{k}}\boldsymbol{B}_{\boldsymbol{k}}^2\,|\,\boldsymbol{k}\,|\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^2/\,|\,\boldsymbol{k}\,|=\frac{V^2c^2\eta}{\pi}\left[\sum_{\boldsymbol{k}\neq 0}\boldsymbol{B}_{\boldsymbol{k}}^4+\frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{l}\neq 0}^{\boldsymbol{k}\neq\boldsymbol{l}}\left(\frac{|\,\boldsymbol{l}\,|}{|\,\boldsymbol{k}\,|}+\frac{|\,\boldsymbol{k}\,|}{|\,\boldsymbol{l}\,|}\right)\boldsymbol{B}_{\boldsymbol{k}}^2\boldsymbol{B}_{\boldsymbol{l}}^2\right]. \qquad (32)
\end{aligned}
$$

Combining inequalities (30) and (32), we have inequality (8). Further, we see that equality in Eq. (8) is reached when the following four conditions are satisfied (i) $\boldsymbol{B}_0=0$, (ii) $\boldsymbol{B}_{\boldsymbol{k}R}$, $\boldsymbol{B}_{\boldsymbol{k}I}$, and $\boldsymbol{k}$ are perpendicular to each other, (iii) $|\boldsymbol{B}_{\boldsymbol{k}R}|=|\boldsymbol{B}_{\boldsymbol{k}I}|$, and (iv) all of the non-zero components will have the same $|\,\boldsymbol{k}\,|$,*i.e.*, $|\,\boldsymbol{k}\,|=\alpha$. Fourier decomposition of Eq. (1) verifies that these four conditions are necessary and sufficient for it to be satisfied. This completes the first proof that $d\left(QW-H^2\right)/dt\leq 0$, with equality upheld only for a plasma in a Woltjer-Taylor state.

We now demonstrate the second proof [36]. This proof does not require Fourier decomposition and is more elegant. The result for dW/dt given in Eq. (27) can also be written without using Fourier components as,

$$
\frac{dW}{dt}=-8\pi\eta\int_V\boldsymbol{j}\cdot\boldsymbol{j}d^3\boldsymbol{x}=-8\pi\eta\left\langle\boldsymbol{j},\boldsymbol{j}\right\rangle. \qquad (33)
$$

Similarly, in the Coulomb gauge, using the fact the electromagnetic field vanishes on the boundary, the result for dQ/dt given in Eq. (29) can be written as

$$
\frac{dQ}{dt}=-\frac{c^2\eta}{2\pi}\int_V\boldsymbol{B}\cdot\boldsymbol{B}d^3\boldsymbol{x}=-\frac{c^2\eta}{2\pi}\left\langle\boldsymbol{B},\boldsymbol{B}\right\rangle. \qquad (34)
$$

Then,

$$\frac{d}{dt}\left(QW-H^2\right)=2\eta\left[2c\left\langle\boldsymbol{A},\boldsymbol{B}\right\rangle\left\langle\boldsymbol{j},\boldsymbol{B}\right\rangle-\frac{c^2}{4\pi}\left\langle\boldsymbol{B},\boldsymbol{B}\right\rangle^2-4\pi\left\langle\boldsymbol{A},\boldsymbol{A}\right\rangle\left\langle\boldsymbol{j},\boldsymbol{j}\right\rangle\right]$$

$$=2\eta\left[-\left(\frac{c}{\sqrt{4\pi}}\left\langle\boldsymbol{B},\boldsymbol{B}\right\rangle-\frac{\left\langle\boldsymbol{A},\boldsymbol{B}\right\rangle\left\langle\boldsymbol{j},\boldsymbol{B}\right\rangle}{\left\langle\boldsymbol{B},\boldsymbol{B}\right\rangle}\sqrt{4\pi}\right)^2-4\pi\left(\left\langle\boldsymbol{A},\boldsymbol{A}\right\rangle\left\langle\boldsymbol{j},\boldsymbol{j}\right\rangle-\frac{\left\langle\boldsymbol{A},\boldsymbol{B}\right\rangle^2\left\langle\boldsymbol{j},\boldsymbol{B}\right\rangle^2}{\left\langle\boldsymbol{B},\boldsymbol{B}\right\rangle^2}\right)\right]. \tag{35}$$

By the Cauchy-Schwartz inequality, $\left\langle\boldsymbol{A},\boldsymbol{A}\right\rangle\left\langle\boldsymbol{B},\boldsymbol{B}\right\rangle\geq\left\langle\boldsymbol{A},\boldsymbol{B}\right\rangle^2$ and $\left\langle\boldsymbol{j},\boldsymbol{j}\right\rangle\left\langle\boldsymbol{B},\boldsymbol{B}\right\rangle\geq\left\langle\boldsymbol{j},\boldsymbol{B}\right\rangle^2$, implying that the last term in Eq. (35) is negative, *i.e.*,

$$-\left\langle\boldsymbol{A},\boldsymbol{A}\right\rangle\left\langle\boldsymbol{j},\boldsymbol{j}\right\rangle+\frac{\left\langle\boldsymbol{A},\boldsymbol{B}\right\rangle^2\left\langle\boldsymbol{j},\boldsymbol{B}\right\rangle^2}{\left\langle\boldsymbol{B},\boldsymbol{B}\right\rangle^2}\leq 0\,. \tag{36}$$

Thus, Eqs. (35) and (36) verify inequality (8), and equality holds if and only if $\boldsymbol{B}=\alpha\boldsymbol{A}$ for constant α. This completes the second proof of inequality (8), with equality upheld only if the plasma is in a Woltjer-Taylor state.

In the derivation of Eqs. (5), (27) and (29), we have assumed that the variation of magnetic energy due to the Lorentz force is small compared with that due to the resistivity. This approximation has essentially been adopted by Taylor and other researchers [10, 15, 16, 27] when Eq. (5) is used. Here we discuss the validity of this assumption. First of all, it is reasonable to argue that the Lorentz force term $i\boldsymbol{k}\times(\boldsymbol{v}\times\boldsymbol{B})_{\boldsymbol{k}}\cdot\boldsymbol{B}^*_{\boldsymbol{k}}=\frac{4\pi}{c}\sum_{\boldsymbol{l}}\boldsymbol{B}_{\boldsymbol{l}}\times\boldsymbol{j}_{-\boldsymbol{k}}\cdot\boldsymbol{v}_{\boldsymbol{k}-\boldsymbol{l}}$ in Eq. (22) is nonlinear and responsible for turbulence generation, and should therefore not be ignored. On the other hand, if we consider the kinetic energy equation [Eq. (25)] in the low β limit, it is clear that the variation of magnetic energy due to the Lorentz force is completely converted into kinetic energy. This is of course not surprising. If we assume that the kinetic energy in the system is smaller than the magnetic energy for all time, it follows that the work done by the Lorentz force *(i.e.*, the energy exchange between the magnetic and kinetic energies) must be smaller than the magnetic energy itself. Thus, any substantial magnetic energy variation has to be caused by resistivity. We have further assumed that this fact is true at large scales. We emphasize that this assumption does not imply that the nonlinearity of the Lorentz force is not as important as other nonlinearities in the relaxation process (such as the convection term $\boldsymbol{v}\cdot\nabla\boldsymbol{v}$). Rather, it simply means that the total magnetic energy *dissipation* is mainly due to resistivity. This assumption is actually independent from other assumptions that one may wish to adopt in developing a theory of plasma relaxation and we believe that it is also a crucial component of Taylor's theory [*e.g.*, Eq. (5)]. This further corroborates the view that the Woltjer-Taylor state will not be that to which a plasma relaxes at high β or when the kinetic energy is comparable to the magnetic energy [23–26].

It is also necessary to point out that as a theoretical model certain simplifications have been adopted in order to make progress. For example, as in previous analyses carried out by Taylor [15, 16], Schnack [27], and Bhattacharjee [17, 18, 20–22], we have assumed that resistivity is constant, even though the resistivity in laboratory discharge experiments can vary significantly between the center and edge of the plasma. Nevertheless, we believe the theoretical understanding enabled by these simplifications does provide valuable insights into the complex plasma relaxation process. To bring our understanding to the next level and include more realistic effects (*e.g.,* inhomogeneous resistivity, pressure and density gradients) further investigations are certainly necessary, probably with new and refined theoretical tools and methods.

To summarize, in our new theory of plasma relaxation, the relaxed Woltjer-Taylor state is reached as the non-negative quantity $QW - H^2$ evolves towards zero. In contrast to Taylor's theory, which is only valid for relaxation dominated by short wavelength fluctuations, our theory is valid for an arbitrary perturbation spectrum. The new theory can be tested by experiment and numerical simulation. The predictions of our theory, specifically the inequality (8) and the variation of H^2 and QW (as illustrated in Fig. 1), can be verified using magnetic fluctuation spectrum data. Testing the validity of this new relaxation theory is one of the scientific objectives of the Keda Toroidal eXperiment (KTX), a RFP device that is currently being constructed at the University of Science and Technology of China.

This research was supported by the ITER-China Program (2010GB107001, 2011GB106000, 2011GB105003), National Natural Science Foundation of China (NSFC-11075162), and U.S. Department of Energy (AC02-76CH03073).

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