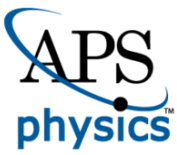

Condensation of Anyons in Frustrated Quantum Magnets

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Condensation of Anyons in Frustrated Quantum Magnets

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We derive the exact ground space of a family of spin-1/2 Heisenberg chains with uniaxial exchange anisotropy (XXZ) and interactions between nearest and next-nearest-neighbor spins. The Hamiltonian family, $\mathcal{H}_{\rm eff}(Q)$, is parametrized by a single variable Q. By using a generalized Jordan-Wigner transformation that maps spins into anyons, we show that the exact ground states of $\mathcal{H}_{\rm eff}(Q)$ correspond to a condensation of anyons with statistical phase $\phi = -4Q$. We also provide matrix-product state representations of some ground states that allow for the efficient computation of spin-spin correlation functions.

One-dimensional quantum magnets can realize exotic states of matter such as Luttinger liquids [1, 2], valence bond solids [3, 4], and spin supersolids [5]. A unique feature of these systems is that transmutations of particle statistics preserve the range of the interactions. This is the main reason behind the success of spin-fermion transformations for studying one-dimensional spin models [6, 7]. A simple generalization of the Jordan-Wigner transformation [6] allows us to map spins into anyons (particles that generalize the concept of fermions and bosons). By exploiting this generalization, we derive exact ground states of spin-1/2 frustrated magnets and provide an efficient method for computing spin-spin correlation functions. Remarkably, the resulting ground states are *anyon condensates* that spontaneously break the particle-number conservation symmetry. In contrast to the familiar Bose-Einstein condensates (BECs), the condensed particles satisfy anyonic statistics [8].

BECs are ubiquitous in many-body physics. Atomic gases, certain quantum magnets, and superconductors can be described as interacting gases of bosons that become BECs at low enough temperatures [12–16]. A BEC is characterized by the spontaneous breaking of the symmetry associated with the boson-number conservation. This symmetry breaking comes along with a macroscopic occupation of the single-particle ground state, suggesting that the notion of BEC cannot be generalized to other particle statistics. Indeed, *anyons*, which are identical particles obeying arbitrary statistics [17], are subject to a generalized Pauli exclusion principle that limits the occupancy of any given state. Therefore, the only way of generalizing the notion of BECs to *anyon condensates* (ACs) is by taking an adequate limit over the particle statistics that survives the exclusion principle. Like BECs, ACs spontaneously break the particle-number conservation symmetry.

Just as BECs can only exist at zero-temperature in low-dimensional systems with short range interactions [18], we will demonstrate that ACs are the ground states of a family of one-dimensional frustrated quantum magnets. The anyonic statistics is fixed by a phase ϕ that their wave function acquires after exchanging two anyons: $\Psi(\mathbf{r}_1,\mathbf{r}_2) = e^{i\phi}\Psi(\mathbf{r}_2,\mathbf{r}_1)$. This condition implies that anyons are in general "non-local" particles that interpolate continuously between bosons ($\phi = 0$) and fermions ($\phi = \pi$).

We will start by considering the dimerized $S = 1/2$ spin system of Fig. 1 A. Pairs of spin-1/2 particles in the same dimer interact via dominant antiferromagnetic exchange (double lines). Weaker inter-dimer exchange interactions are represented by single lines. In presence of an external field h, the low-energy single-dimer states are the triplet $|\uparrow\uparrow\rangle$ ($s^z = 1/2$) and the singlet $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$ ($s^z = -1/2$). The effective spin-1/2 XXZ model, $\mathcal{H}_{\rm eff}$, that results from projecting the original Hamiltonian into the low-energy subspace is [19–22] (see Fig. 1):

$$\mathcal{H}_{\rm eff} = \sum_{j;\nu=1,2} J_\nu \left[\Delta_\nu \left(s^z_{j+\nu}s^z_j - \frac{1}{4}\right) + s^x_{j+\nu}s^x_j + s^y_{j+\nu}s^y_j\right].$$

The eigenstates of s^z_j represent the singlet and the $S^z = 1$ triplet of the j-th dimer Hamiltonian.

In this Letter we derive the exact ground states of $\mathcal{H}_{\rm eff}(Q)$ for $\Delta_\nu = \cos(\nu Q)$ and $J_1 = -4J_2\cos Q$, with $0 \le Q < \pi$. If $Q = \pi/2$, the model reduces to two decoupled ferromagnetic Heisenberg chains whose exact solutions are known [23]. Consequently, if the number of effective spins in the ladder is L, the ground-space dimension is $d_{\rm gs} = (L/2+1)^2$ or $d_{\rm gs} = (L+3)(L+1)/4$ for even or odd L. We will provide a set of $d_{\rm gs}$ linearly independent ground states for each value of Q and a special choice of open boundary conditions.

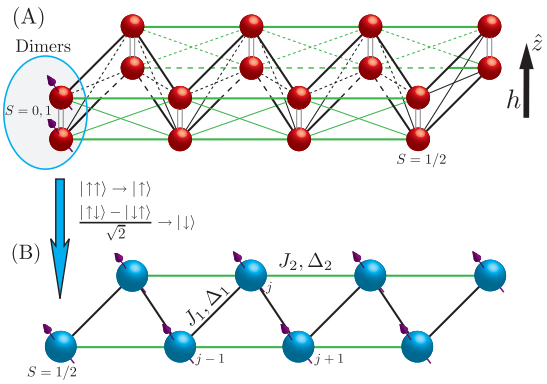

FIG. 1. Frustrated quantum magnet. (A) Original spin-1/2 system: pairs of spin-1/2 particles in the same dimer interact via dominant antiferromagnetic exchange (double lines). Weaker inter-dimer exchange interactions are represented by single lines. (B) Effective spin-1/2 XXZ model, $\mathcal{H}_{\rm eff}$, that results from projecting the original Hamiltonian into the low-energy subspace.

Some of the ground states are product states $|s_\theta(Q)\rangle$, in

which the spins are aligned according to a canted spiral of wave vector Q and arbitrary canting angle θ [24] (see Fig. 2 A). $|s_\theta(-Q)\rangle$ are also ground states due to spatial inversion symmetry: $\mathcal{H}_{\rm eff}(Q) = \mathcal{H}_{\rm eff}(-Q)$. The long-range spiral order is manifested by the spatial dependence of spin-spin correlations that oscillate with constant amplitude and wave-vector $\pm Q$ (see Fig. 2 B).

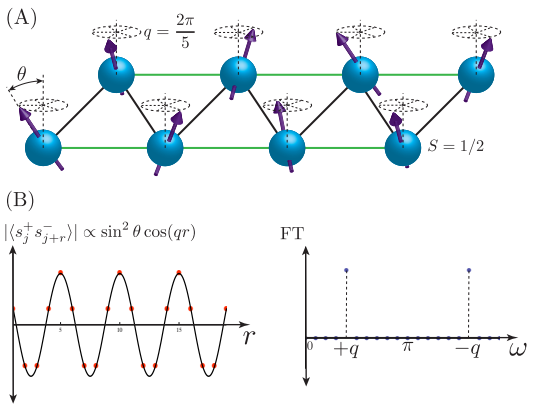

FIG. 2. Canted spiral order. (A) Spin-1/2 representation of the spiral states $|s_\theta(Q)\rangle$ for $Q = 2\pi/5$. (B) Spin-spin correlation as a function of distance r.

States with spiral order can be represented as BECs by mapping spin-1/2 operators into hard-core bosons via the Matsubara-Matsuda (MM) transformation [25]: $b_j^\dagger = s_j^+$. The operator $b_j^\dagger$ creates a hard-core boson at the j-th site of the ladder. The identity $(b_j^\dagger)^2 = 0$ excludes states with more than one boson per site. Since $\mathcal{H}_{\rm eff}(Q)$ is translationally invariant, it is useful to introduce the operators $\bar{b}_k^\dagger = \sum_{j=1}^L \exp(ikj) b_j^\dagger/\sqrt{L}$ that create bosons of momentum k. Contrary to real space bosons, $b_j^\dagger$, occupancy in momentum space is not subject to exclusion. The BEC states

$$|\psi_n(\pm Q)\rangle \propto (\bar{b}_{\pm Q}^\dagger)^n |\varnothing\rangle \tag{1}$$

span the ground space with spiral order [24]. $|\varnothing\rangle$ is the vacuum or empty state that corresponds to the fully-polarized state $|\downarrow \ldots \downarrow\rangle$ in the spin language and n is the number of single bosons with momentum $\pm Q$. $|\psi_n(Q)\rangle$ and $|\psi_n(-Q)\rangle$ are usually non-orthogonal but linearly independent. Finding all BEC states for $Q = 0$ requires to consider adequate linear combinations of $|\psi_n(Q)\rangle$ and $|\psi_n(-Q)\rangle$ before taking the limit $Q \to 0$. The single-boson excited eigenstates, $\bar{b}_k^\dagger|\varnothing\rangle$, have energy $\omega_k = 2J_2(\cos k - \cos Q)^2$ [24].

The MM transformation provides then a simple alternative representation of the spiral ground states in terms of hard-core bosons. Remarkably, a generalization of the particle statistics will allow us to find exact expressions for *all* ground states of $\mathcal{H}_{\rm eff}(Q)$. The generalized JW transformation $a_j^\dagger = \exp\{i\phi\sum_{l<j}(s_l^z + 1/2)\}s_j^+$ maps spins into anyons. The operator $a_j^\dagger$ creates a hard-core anyon $[(a_j^\dagger)^2 = 0]$ at the j-th site of the ladder. The MM and JW transformations are recovered for $\phi = 0$ (hard-core bosons) and $\phi = \pi$ (spinless fermions), respectively. The anyonic statistics becomes manifest in the commutation relations $a_j^\dagger a_l^\dagger = e^{-i\phi} a_l^\dagger a_j^\dagger$, when $j < l$. The states

$$|\psi_{n,m}(Q)\rangle = \lim_{\phi\to -4Q} c_\phi (\bar{a}_Q^\dagger)^n (\bar{a}_{-Q}^\dagger)^m |\varnothing\rangle \ , \tag{2}$$

with $0 \le n, m \le L/2$, provide a natural generalization of the BEC states in Eq. (1). $\bar{a}_k^\dagger = \sum_{j=1}^L \exp(ikj) a_j^\dagger/\sqrt{L}$ is the operator that creates an anyon with momentum k and c_ϕ is the normalization constant. The limit in Eq. (2) is necessary to avoid a trivial cancellation due to a generalized Pauli exclusion principle: $(\bar{a}_k^\dagger)^p = 0$ for $\phi = 2\pi/p$ and p an integer greater than 1. The statistical phase does not play a role if $n = 0$ or $m = 0$ because $\lim_{\phi\to -4Q} c_\phi (\bar{a}_Q^\dagger)^n |\varnothing\rangle \propto (\bar{b}_Q^\dagger)^n |\varnothing\rangle$. However, the same limiting procedure leads to states that are qualitatively different from a double-Q BEC, $(\bar{b}_Q^\dagger)^n (b_{-Q}^\dagger)^m |\varnothing\rangle$, when $n, m \neq 0$. We note that the generalized JW transformation was previously used to solve one-dimensional anyonic models in the continuum [11].

$|\psi_{n,m}(Q)\rangle$ span the full ground space of $\mathcal{H}_{\rm eff}(Q)$ – see below. As the total number of particles is not preserved for a general ground state, the U(1) symmetry of $H_{\rm eff}(Q)$ is spontaneously broken, i.e. ground states with different $n + m$ belong to different irreducible representations of the U(1)-symmetry group. Therefore, we refer to the physical phenomenon characterized by such states as *anyonic condensation* and to $|\psi_{n,m}(Q)\rangle$ as *AC states*. The physical properties of ACs are better understood if we first consider the action of $a_j^\dagger$. Besides polarizing the j-th spin, such operator creates a "phase kink": it rotates the spins on the *left* of j by an angle ϕ along the $\hat{z}$ axis. The phase kink gives rise to the anyonic statistics that is absent in the MM transformation. Consequently, the ground state $\bar{a}_Q^\dagger |s_\theta(-Q)\rangle$ corresponds to a phase soliton attached to a boson that propagate with momentum Q across the spiral $|s_\theta(-Q)\rangle$ – see Fig. 3 A. In general, the ground states $(\bar{a}_Q^\dagger)^n |s_\theta(-Q)\rangle$ will correspond to a condensation of soliton-boson bound states.

To demonstrate that $|\psi_{n,m}(Q)\rangle$ is a ground state of $\mathcal{H}_{\rm eff}(Q)$, we rewrite the $\mathcal{H}_{\rm eff}(Q)$ as

$$\mathcal{H}_{\rm eff}(Q) = \sum_{j=2}^{L-1} (J_2 + J_1^2/(8J_2)) \Pi_j(Q) \ , \tag{3}$$

where the $\Pi_j(Q)$ are projectors acting on triangular plaquettes [24]. These projectors are

$$\Pi_j(Q) = |\xi_j^\uparrow\rangle\langle\xi_j^\uparrow| + |\xi_j^\downarrow\rangle\langle\xi_j^\downarrow| \ ,$$

where $|\xi_j^\uparrow\rangle \propto (2\cos Q s_{j-1}^+ s_{j+1}^+ - s_j^+ s_{j+1}^+ - s_{j-1}^+ s_j^+)|\downarrow\downarrow\downarrow\rangle$ and $|\xi_j^\downarrow\rangle \propto (2\cos Q s_j^+ - s_{j-1}^+ - s_{j+1}^+)|\downarrow\downarrow\downarrow\rangle$ are plaquette states. Note that Eq. (3) refers to a particular choice of open boundary conditions. Since the fully-polarized spin states $|\downarrow \ldots \downarrow\rangle$ and $|\uparrow \ldots \uparrow\rangle$ are orthogonal to $|\xi_j^\sigma\rangle$, they are zero-energy ground states of $\mathcal{H}_{\rm eff}(Q)$. This observation implies that $\mathcal{H}_{\rm eff}(Q)$ are

parent Hamiltonians, also known as frustration-free Hamiltonians in the quantum-information community [26], although they are still frustrated in a physical sense.

The AC states satisfy

$$|\psi_{n,m}(Q)\rangle = \lim_{\phi\to -4Q} c_\phi (\bar{a}^\dagger_Q)^n (\bar{a}^\dagger_{-Q})^m |\varnothing\rangle$$
$$\propto \sum_{\substack{j_1<\ldots<j_n \\ l_1<\ldots<l_m}} \prod_{q=1}^{n} e^{iQj_q} a^\dagger_{j_q} \prod_{r=1}^{m} e^{-iQl_r} a^\dagger_{l_r} |\varnothing\rangle \ . \quad (4)$$

The limit in Eq. (4) removes the prefactors that arise from exchanging two anyons with the same momentum. This is necessary to avoid a trivial cancellation of $(\bar{a}^\dagger_{\pm Q})^n$ caused by the generalized exclusion principle. In particular, the limit implies that $|\psi_{n,0}(Q)\rangle \propto (\bar{b}^\dagger_Q)^n|\varnothing\rangle$ and $|\psi_{0,m}(Q)\rangle \propto (\bar{b}^\dagger_{-Q})^m|\varnothing\rangle$. Let $v_p = \{p-1, p, p+1\}$. Then, if $j_q \notin v_p$ and $l_r \notin v_p$ for all q, r,

$$\Pi_p(Q) \prod_{q=1}^{n} a^\dagger_{j_q} \prod_{r=1}^{m} a^\dagger_{l_r} |\varnothing\rangle = \prod_{q=1}^{n} a^\dagger_{j_q} \prod_{r=1}^{m} a^\dagger_{l_r} \Pi_p(Q)|\varnothing\rangle = 0.$$

We now consider the terms in Eq. (4) for which a single $j_i \in v_p$ and $l_r \notin v_p$ for all r. We fix all the indexes except for j_i and consider the sum over the terms that have j_i in v_p. The resulting state satisfies

$$\Pi_p(Q) \sum_{j_i=p-1}^{p+1} \prod_{q=1}^{n} e^{iQj_q} a^\dagger_{j_q} \prod_{r=1}^{m} e^{-iQl_r} a^\dagger_{l_r} |\varnothing\rangle \propto$$
$$\prod_{q\neq i} e^{iQj_q} a^\dagger_{j_q} \prod_{r=1}^{m} e^{-iQl_r} a^\dagger_{l_r} \Pi_p(Q) \sum_{j_i=p-1}^{p+1} e^{iQj_i} a^\dagger_{j_i} |\varnothing\rangle = 0 \ .$$

This equality follows from $\sum_{j_i=p-1}^{p+1} e^{iQj_i} a^\dagger_{j_i}|\varnothing\rangle$ being orthogonal to $|\xi^\sigma_p\rangle$. A similar reasoning applies if a single l_r is in v_p and $j_q \notin v_p$ for all q. Finally, we analyze the case for which only two indexes of $\{j_q\}_q \cup \{l_r\}_r$ are in v_p. If the two indexes are either in $\{j_q\}_q$ or $\{l_r\}_r$, we can use a similar argument as the one above to show that $\Pi_p(Q)$ acts trivially in the particular sum of states. We then analyze the case for which one of the indexes in v_p is in $\{j_q\}_q$ and the other one is in $\{l_r\}_r$. We label these indexes by j_i and l_h, respectively. If we fix all the remaining indexes and consider the sum of terms over the six possible values of (j_i, l_h), the resulting state satisfies

$$\Pi_p(Q) \sum_{(j_i,l_h)} \prod_{q=1}^{n} e^{iQj_q} a^\dagger_{j_q} \prod_{r=1}^{m} e^{-iQl_r} a^\dagger_{l_r} |\varnothing\rangle \propto$$
$$\prod_{q\neq i} e^{iQj_q} a^\dagger_{j_q} \prod_{r\neq h} e^{-iQl_r} a^\dagger_{l_r} H_p(Q) \sum_{(j_i,l_h)} e^{iQ(j_i-l_h)} a^\dagger_{j_i} a^\dagger_{l_h} |\varnothing\rangle \ .$$

The state $\sum_{(j_i,l_h)} e^{iQ(j_i-l_h)} a^\dagger_{j_i} a^\dagger_{l_h}|\varnothing\rangle$ is orthogonal to $|\xi^\sigma_p\rangle$ when

$$e^{iQ} + e^{i(3Q+\phi)} - \cos Q(1 + e^{i(4Q+\phi)}) = 0 \ ,$$

where ϕ is the anyonic statistical phase. This condition is satisfied for $\phi = -4Q$.

For even L, $\mathcal{H}_{\text{eff}}(\pi/2)$ corresponds to two decoupled ferromagnetic Heisenberg chains of length $L/2$. The ground space is spanned by the linearly independent states $|\psi_{n,m}(\pi/2)\rangle$, with $0 \le n, m \le L/2$, and the ground space dimension is $(L/2+1)^2$. Because the AC states are analytic in Q, the set $\{|\psi_{n,m}(Q)\rangle\}_{n,m}$ remains linearly independent and the dimension of the corresponding subspace does not change with Q. (Numerical calculations suggest that these are all possible ground states.) The states $|\psi_{n,m}(Q)\rangle$ and $|\psi_{m,n}(-Q)\rangle$ become equal for $Q \to 0$. However, it is possible to obtain two linearly independent states by taking a proper limit of linear combinations of $|\psi_{n,m}(Q\rangle$ and $|\psi_{m,n}(Q\rangle$. For instance,

$$\lim_{Q\to 0} \frac{1}{Q} \left[|\psi_{n,0}(Q)\rangle - |\psi_{0,n}(Q)\rangle\right] \propto$$
$$\propto \sum_{j_1<\ldots<j_n} (j_1 + \cdots + j_n) b^\dagger_{j_1} \ldots b^\dagger_{j_n} |\downarrow \ldots \downarrow\rangle$$

is also a ground state of $\mathcal{H}_{\text{eff}}(0)$.

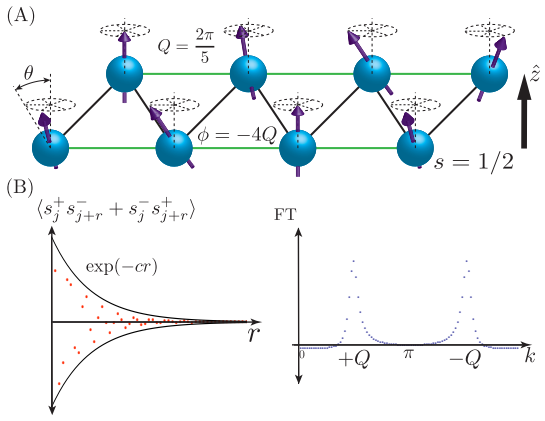

FIG. 3. Phase soliton. (A) A topological defect results from the creation of an anyon on the spiral state, i.e. $a^\dagger_j|s_\theta(-Q)\rangle$ (Fig. 2 A). (B) Exponential decay of spin-spin correlations as a function of distance r for an AC. This correlation function was obtained by using a matrix-product state representation. The Fourier transform (FT) of the correlation function is peaked at the momentum of the AC. $c > 0$, $Q = 2\pi/5$, and $L = 100$.

The difference between ACs and BECs is evident from their single-particle correlation functions. The two-point correlator exhibits long-range order, $\lim_{r\to\infty} |\langle s^+_j s^-_{j+r}\rangle| = \lim_{r\to\infty} |\langle b^\dagger_j b_{j+r}\rangle| \neq 0$, for a BEC (Fig. 1 B). In the anyonic language, the operator $s^+_j s^-_{j+r}$ maps to $\exp\{i\phi N_{jr}\} a^\dagger_j a_{j+r}$, where $N_{jr} = \sum_{j\le l<j+r} a^\dagger_l a_l$ counts the number of anyons between sites j and $j+r$. Then, the large fluctuations of N_{jr} present in an AC translate into large fluctuations of the relative azimutal phase between spins j and $j+r$ – see Fig. 3 A. If we assume that the probabilities of having anyons in different sites are roughly independent of each other, we obtain $\langle s^+_j s^-_{j+r}\rangle \sim e^{-iQr}(1-\rho(1-e^{i\phi}))^r$ for large r and $\rho = n/L$.

This simple analysis suggests that the two-point correlator in the ACs decays exponentially in r – see Fig. 3 B.

Spin-spin correlations can be efficiently computed by introducing matrix-product state representations for the ground states that are invariant under translation [27, 28]. These states are represented as

$$|\psi(Q)\rangle \propto \sum_{\sigma_1,\ldots,\sigma_L} \mathrm{tr}\left[A^{\sigma_1}(Q)\ldots A^{\sigma_L}(Q)\right]|\sigma_1\ldots\sigma_L\rangle, \quad (5)$$

where each $A^{\sigma_j}(Q)$ is a matrix associated with the j-th spin in the ladder and $\sigma_j \in \{\uparrow,\downarrow\}$. Correlation functions can be computed analytically or numerically in this representation by a contraction of a tensor network [29]. The translational-invariant ground states of the matrix-product form of Eq. (5) satisfy

$$\Pi_p(Q)|\psi(Q)\rangle = 0 \;\forall\, p\,. \quad (6)$$

The action of $\Pi_p(Q)$ on $|\psi(Q)\rangle$ outputs another matrix-product state whose three-spin tensor, associated with v_p, is

$$\begin{aligned} &B^{\tau_{p-1},\tau_p,\tau_{p+1}}(Q) = \quad (7)\\ &= \sum_{\sigma_{p-1},\sigma_p,\sigma_{p+1}} A^{\sigma_{p-1}}(Q)A^{\sigma_p}(Q)A^{\sigma_{p+1}}(Q)\times\\ &\times\langle\tau_{p-1},\tau_p,\tau_{p+1}|\Pi_p(Q)|\sigma_{p-1},\sigma_p,\sigma_{p+1}\rangle, \end{aligned}$$

and $\tau,\sigma \in \{\uparrow,\downarrow\}$. Equation (7) follows from the contraction of a tensor network [29]. $B^{\tau_{p-1},\tau_p,\tau_{p+1}}(Q) = 0$ for all $\tau_{p-1},\tau_p,\tau_{p+1}$ is a sufficient condition to satisfy Eq. (6). It follows from the definition of $|\xi_p^\sigma\rangle$ that:

$$\begin{aligned} &A^\uparrow(Q)A^\downarrow(Q)A^\downarrow(Q) + A^\downarrow(Q)A^\downarrow(Q)A^\uparrow(Q) -\\ &\quad - 2\cos(Q)A^\downarrow(Q)A^\uparrow(Q)A^\downarrow(Q) = 0\,,\\ &A^\downarrow(Q)A^\uparrow(Q)A^\uparrow(Q) + A^\uparrow(Q)A^\uparrow(Q)A^\downarrow(Q) -\\ &\quad - 2\cos(Q)A^\uparrow(Q)A^\downarrow(Q)A^\uparrow(Q) = 0\,. \end{aligned} \quad (8)$$

These equations are satisfied if $A^\uparrow(Q)A^\downarrow(Q) = e^{\pm iQ}A^\downarrow(Q)A^\uparrow(Q)$. Equations (8) are also satisfied for $Q = 2\pi/3$ if $A^{\uparrow,\downarrow}(2\pi/3)$ are the matrices that correspond to a ground state of the well-known Majumdar-Ghosh model [30]. Note, however, that the ground-space degeneracy of $\mathcal{H}_{\rm eff}(Q)$ and the Majumdar-Ghosh model is very different implying that both models are not unitarily related.

Solutions for $Q = 2\pi/p$, with integer p, are given by $A^\uparrow(Q) = Z_p$ and $A^\downarrow(Q) = (aX_p + bX_p^\dagger)Z_p^\dagger$. The complex constants a and b are arbitrary and the corresponding states span the full translational-invariant ground subspace. Z_p and X_p are p-dimensional unitary matrices that satisfy $Z_pX_p = e^{-iQ}X_pZ_p$. This condition resembles the anyonic nature of our solutions whose statistical phase is $\phi = -4Q$. For example, Z_p is the diagonal matrix and X_p is the cyclic permutation that generate the finite Heisenberg or generalized Pauli group of order p^2 [31]:

$$Z_p = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0\\ 0 & \omega & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \cdots & \vdots\\ 0 & 0 & 0 & \cdots & \omega^{p-1}\end{pmatrix},\; X_p = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 0 & 0 & \vdots & 0\end{pmatrix},$$

and $\omega = \exp(i2\pi/p)$. Few-spin correlators can be numerically obtained with a low computational cost that is polynomial in p [28]. For instance,

$$\begin{aligned} \langle\psi(Q)|s_j^+ s_{j+r}^-|\psi(Q)\rangle = \mathrm{tr}\left[E^{j-1}\times(A^\downarrow(Q)\otimes(A^\uparrow(Q))^*)\times\right.\\ \left.\times E^{r-1}\times(A^\uparrow(Q)\otimes(A^\downarrow(Q))^*)\times E^{L-r}\right]\,, \end{aligned} \quad (9)$$

and $E = A^\uparrow(Q)\otimes(A^\uparrow(Q))^* + A^\downarrow(Q)\otimes(A^\downarrow(Q))^*$. The normalization constant is $\langle\psi(Q)|\psi(Q)\rangle = \mathrm{tr}[E^L]$. We used Eq. (9) and Gram-Schmidt orthogonalization to compute the exact spin-spin correlator in Fig. 3 B and confirm the expected exponential decay for ACs.

Finally, we note that the particular ground state solutions (1) imply that the scattering length of two bosons with momenta k and k' diverges asymptotically for $k,k' \to Q$ or $k,k' \to -Q$. However, the boson-boson scattering length remains finite if $k \to Q$ and $k' \to -Q$. In contrast, our general ground state solutions given in Eq. (2) imply that the scattering length of two *anyons* with momenta k and k' diverges asymptotically in both cases: $(k,k') \to (Q,Q)$ or $(k,k') \to (-Q,-Q)$ *and* $(k,k') \to (Q,-Q)$. Therefore, the low-energy excitations on top of the condensate are quasi-free anyons.

In summary, we have introduced the notion of anyon condensates as a generalization of BECs. By using a natural extension of the Jordan-Wigner transformation, we have shown that that the AC states (2) are the exact ground states of a family one-dimensional XXZ magnets with nearest and next-nearest-neighbor exchange interactions. ACs spontaneously break the particle-number conservation symmetry and correspond to a condensation of topological defects. We have also provided a matrix-product representation for the translational-invariant ground states, which allows for the efficient computation of correlation functions. These states also satisfy an *area law*, which is rather common in frustrated systems, and results in an upper bound on the entanglement entropy [32]. The exponential decay obtained for the single-particle correlator in the AC states is a natural consequence of the topological nature of anyons. Since each anyon is a bound state of a boson and a phase soliton, the large fluctuations in the anyon number that characterize an AC state lead to large phase fluctuations. This behavior is qualitatively different from the bosonic limit (BEC) in which particle number and phase are conjugate variables and large number fluctuations lead to long-range phase coherence.

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[8] There is a point of view which maintains that statistics is not a sound concept in 1+1 dimensions because the particle statistics can be absorbed by a short-range interaction potential. As it is explained in Ref. [9], this argument is not a valid objection against statistics *per se*. It is well-known that the particle statistics can also be modified in 2+1 dimensions by introducing a singular gauge potential to attach a flux to each particle (Chern-Simons construction). However, the notion of fractional particle statistics is still a useful concept because it simplifies the description of certain states of matter. Here we adopt this criterion for 1+1 dimensional systems. A similar criterion has been previously adopted in many works on one-dimensional models [10, 11].
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