Spin-Gradient-Driven Light Amplification in a Quantum Plasma

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Spin gradient driven light amplification in a quantum plasma

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It is shown that the gradient "free-energy" contained in equilibrium spin vorticity can cause electromagnetic modes, in particular the light wave, to go unstable in a spin quantum plasma of mobile electrons embedded in a neutralizing ion background. For densities characteristic of both the solid state and very high density astrophysical systems, the growth rates are sufficiently high to overcome the expected collisional damping. Preliminary results suggest a powerful spin-inhomogeneity driven mechanism for stimulating light amplification.

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In this letter, we demonstrate that an inhomogeneous macroscopic spin field can induce instabilities in a variety of electromagnetic modes sustained by a spinning quantum plasma. In particular, we will show that the light wave branch of the standard plasma dispersion (in classical plasmas) gets profoundly affected by quantum modifications. The light wave is, generally, very stable, both in classical and in quantum plasmas, and it manages to stay so even in the presence of a broad set of spin inhomogeneities.

What is remarkable, however, is that there does seem to exist a class of inhomogeneities on which the light waves can feed, and grow by tapping the ambient free energy. Such inhomogeneities carry a non-vanishing spin/quantum vorticty. Constructed from the macroscopic spin field $\boldsymbol{S}$, the spin vorticity

$$\boldsymbol{\Omega}_q = \frac{\nabla S_i \times \nabla S_j}{S_k} \tag{1}$$

was recently introduced in the quantum plasma literature [1]. In Eq.(1), i, j, k are a cyclic permutation of the spin vector components; the spin vector has a unit modulus, $S_1^2 + S_2^2 + S_3^2 = 1$, with the concomitant constraint $S_1 \nabla S_1 + S_2 \nabla S_2 + S_3 \nabla S_3 = 0$. Note that, to insure a nonzero $\boldsymbol{\Omega}_q$, all components of the spin vector must be nonzero, and the system must have a variation in at least two directions.

The vortical formalism, developed in Ref. [1], is derivable from, and equivalent to previous formalisms [2, 3]. Its structure, however, leads to a "simpler" representation of the spin-plasma equations, allowing easier interpretation, classification, and manipulations pertaining to all incompressible motions. The identification of the quantum vorticity $\boldsymbol{\Omega}_q$ brings in conceptual depth as well as an enhanced capability to explore the rich physical content injected into quantum plasmas by a dynamical spin-field.

The subject of spin-created new complex possibilities [2, 3] in the linear as well as nonlinear waves supported by quantum plasmas has received considerable attention (see, for example, Refs. [4–15] and references therein). Much effort has been put into finding new instabilities driven by spin in low-frequency and electrostatic modes in the magnetized plasma regime [14, 16–19]. However, in most calculations (with a few exceptions [20–22]), spin is not considered as a dynamical variable. Instead, only its thermodynamical ensemble properties were used for the definition of the magnetization current.

The controlling role of quantum vorticity in the possible instability of the electromagnetic modes is, naturally, revealed in the recently introduced vortical formulation (in close analogy to the vortex dynamics of ideal fluids) of spinning quantum plasmas. The plasma dynamics, in this formulation [1], is contained in three vector equations: The standard spin evolution equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right) \boldsymbol{S} = \frac{2\mu}{\hbar} \boldsymbol{S} \times \left(\boldsymbol{B} + \frac{\hbar c}{2q} \nabla^2 \boldsymbol{S}\right) , \tag{2}$$

(where $\boldsymbol{v}$ is the velocity and $\boldsymbol{B}$ is the magnetic field), the Maxwell law

$$\nabla \times \boldsymbol{B} = \frac{4\pi}{c} q n \boldsymbol{v} + 4\pi \mu n \nabla \times \boldsymbol{S} + \frac{1}{c} \frac{\partial \boldsymbol{E}}{\partial t} , \tag{3}$$

(where $\boldsymbol{E}$ is the electric field), and the recently derived unified equation for the Grand generalized vorticity (GV)

$$\frac{\partial \boldsymbol{\Omega}_-}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{\Omega}_-) . \tag{4}$$

The GV, a combination of the canonical ($\boldsymbol{\Omega}_c = \boldsymbol{B} + (mc/q) \nabla \times \boldsymbol{v}$) and quantum ($\boldsymbol{\Omega}_q$) vorticities, is defined as

$$\boldsymbol{\Omega}_- = \boldsymbol{\Omega}_c - \frac{\hbar c}{2q} \boldsymbol{\Omega}_q . \tag{5}$$

In the preceding equations, q (m) is the particle charge (mass), $\mu = q\hbar/(2mc)$ is the elementary magnetic moment, $\hbar$ is the reduced Planck constant, and c is the speed of light. The plasma has been assumed to be incompressible (the fluid density n is constant).

It is, perhaps, obvious that the construction of $\boldsymbol{\Omega}_-$, obeying the basic vortex equation (4), was motivated by a desire to eliminate the quantum force (that destroys the canonical vortex structure) in the evolution equation of the canonical $\boldsymbol{\Omega}_c$. Evidently, (4) insures the conservation of the associated helicity $h_- = \int d^3\boldsymbol{x}\ \boldsymbol{\Omega}_- \cdot \left(\nabla^{-1} \times \boldsymbol{\Omega}_-\right)$.

We begin our investigation of the linear wave propagation (in a flow-free plasma without any external magnetic field) by expanding the general perturbations in terms of Fourier modes: $\boldsymbol{Q}_1 = (Q_x\hat{e}_x + Q_y\hat{e}_y + Q_z\hat{e}_z)\exp[i(\boldsymbol{k}\cdot\mathbf{x} - \omega t)]$, where ω and $\boldsymbol{k}$ are, respectively, the wave frequency and the wave vector, and $\boldsymbol{Q}_1$ is the generic linear perturbation: the magnetic ($\mathbf{B}_1$), the velocity ($\boldsymbol{v}_1$), and the spin ($\boldsymbol{\Sigma}$). Throughout this paper, the subindex 1 (0) labels the perturbed (equilibrium) quantities (in order to avoid ambiguity of notation, we label the spin perturbation as $\boldsymbol{\Sigma}$ instead of $\boldsymbol{S}_1$).

Because the plasma will have equilibrium spin gradients (for a nonzero quantum vorticity), we must resort to a local analysis for which the usual requirement is that the scale length L of the spin inhomogeneity must be much larger than the wavelength k^{-1} (k is the modulus of the wave vector) of the mode. Specifically, $|\nabla \boldsymbol{S_0}|/S_0 \sim 1/L \ll k \sim |\nabla\boldsymbol{\Sigma}|/|\boldsymbol{\Sigma}|$, i.e., $1/kL \sim \epsilon \ll 1$. Thus, ϵ measures the strength of the spin gradients. We will find that it is the equilibrium quantum vorticity, $\boldsymbol{\Omega}_{q0} = \nabla S_{0_1} \times \nabla S_{0_2}/S_{0_3}$ (of order ϵ^2), that is responsible for creating imaginary parts in the dispersion relation (S_{0_j} are the components of $\boldsymbol{S}_0$).

One of the equilibrium conditions requires that $\boldsymbol{B}_0 = a\boldsymbol{S}_0 + \nabla\varphi$, and, for simplicity, we choose $\varphi = 0$. An equilibrium with a spin field, then, must necessarily have an intrinsic equilibrium magnetic field (even when the external field is zero). Keeping this fact in mind, the normalized perturbed equations, written in terms of just one characteristic parameter $a = \hbar\omega_p/(2mc^2)$, spell out as

$$
\begin{aligned}
-i\omega\boldsymbol{\Sigma} + (\boldsymbol{v}_1\cdot\nabla)\boldsymbol{S}_0 &= \boldsymbol{S}_0\times\boldsymbol{B}_1 - a(1+K^2)\boldsymbol{S}_0\times\boldsymbol{\Sigma} \\
&\quad + a\,\boldsymbol{\Sigma}\times\nabla^2\boldsymbol{S}_0\,, \qquad (6)\\
\boldsymbol{B}_1 + i\boldsymbol{K}\times\boldsymbol{v}_1 - a\boldsymbol{\Omega}_{q1} &= -\frac{a}{\omega}\boldsymbol{v}_1(\boldsymbol{S}_0 - \boldsymbol{\Omega}_{q0})\cdot\boldsymbol{K} \\
&\quad - \frac{ai}{\omega}(\boldsymbol{v_1}\cdot\nabla)\boldsymbol{S}_0\,, \qquad (7)\\
-F\boldsymbol{B}_1 &= i\boldsymbol{K}\times\boldsymbol{v}_1 - a\boldsymbol{K}(\boldsymbol{K}\cdot\boldsymbol{\Sigma}) \\
&\quad + aK^2\boldsymbol{\Sigma}\,, \qquad (8)
\end{aligned}
$$

where $F = \omega^2 - K^2$, $K = kc/\omega_p$ is the normalized wave number, and all frequencies and length scales are normalized to ω_p and $\lambda_s \equiv c/\omega_p$, respectively ($\omega_p$ is the plasma frequency). The velocity is normalized to the speed of light, and the magnetic field is normalized to $\hbar\omega_p/(2\mu)$. Note the appearance of a slew of terms proportional to $\nabla\boldsymbol{S}_0$ reflecting the spatial dependence of the spin field. We have kept terms to order ϵ^2 only.

Dotting (6) with $\boldsymbol{S}_0$ and (7) with $\boldsymbol{K}$, we obtain the two constraints $\boldsymbol{S}_0\cdot[(\boldsymbol{v}_1\cdot\nabla)\boldsymbol{S}_0] = 0$ and $\boldsymbol{K}\cdot[(\boldsymbol{v}_1\cdot\nabla)\boldsymbol{S}_0] = 0$. Whereas the former condition is satisfied trivially (recall that $|\boldsymbol{S}_0\cdot\boldsymbol{S}_0| = 1$), the only physically meaningful way to satisfy the latter is to assume $\boldsymbol{K}\cdot\boldsymbol{S}_0 = 0$. However, since $\boldsymbol{S}_0$ varies in space, it is clear that this condition can only be fulfilled locally.

The linearized mode equations (6)-(8), after being decomposed in a convenient basis ($\boldsymbol{e}_s \equiv \boldsymbol{S}_0$, $\hat{\boldsymbol{k}} \equiv \boldsymbol{K}/K$ and $\boldsymbol{e}_s\times\hat{\boldsymbol{k}}$), are manipulated (to $\mathcal{O}(\epsilon^2)$) to obtain the general dispersion relation

$$
AD - BC = 0, \qquad (9)
$$

where

$$
\begin{aligned}
A &= -\Big(\omega(1-F) - \frac{a}{K}F\,\hat{\boldsymbol{k}}\cdot(\nabla\times\boldsymbol{e}_s) - aK\,\hat{\boldsymbol{k}}\cdot(\nabla\times\boldsymbol{e}_s)\Big) + i\frac{ak_z}{S_3}(\boldsymbol{e}_s\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\Omega}_{q0})\,,\\
B &= a\left((1-F)(1+K^2+\boldsymbol{e}_s\cdot\nabla^2\boldsymbol{e}_s) - K^2 - \frac{a}{\omega K}(\omega^2+FK^2)\hat{\boldsymbol{k}}\cdot(\nabla\times\boldsymbol{e}_s) + \frac{(\boldsymbol{e}_s\times\hat{\boldsymbol{k}})_z}{S_3}\boldsymbol{e}_s\times\hat{\boldsymbol{k}}\cdot\boldsymbol{\Omega}_{q0}\right),
\end{aligned}
$$

$$
\begin{aligned}
C &= \frac{\omega}{K}F\hat{\boldsymbol{k}}\cdot(\nabla\times\boldsymbol{e}_s) - a(1+K^2+\boldsymbol{e}_s\cdot\nabla^2\boldsymbol{e}_s)\,,\\
D &= \omega - \frac{a}{K}(\omega^2+FK^2)\hat{\boldsymbol{k}}\cdot(\nabla\times\boldsymbol{e}_s)\,,
\end{aligned}
$$

with $k_z = \boldsymbol{e}_z\cdot\hat{\boldsymbol{k}}$, $(\boldsymbol{e}_s\times\hat{\boldsymbol{k}})_z = \boldsymbol{e}_z\cdot(\boldsymbol{e}_s\times\hat{\boldsymbol{k}})$. Note the striking fact that, though there are many gradient-dependent terms, the only imaginary term in (9) is proportional to the equilibrium spin vorticity $\boldsymbol{\Omega}_{q0}$. The dispersion relation for the light wave branch allows an analytic approximation,

$$
\omega \approx \omega_* - \frac{a(1+2K^2)}{2\omega_* K\sqrt{1+K^2}}\hat{\boldsymbol{k}}\cdot(\nabla\times\boldsymbol{e}_s) + i\Gamma\,, \qquad (10)
$$

where $\omega_* = \sqrt{1+K^2+a^2K^2/(1-a^2(1+K^2))}$, the dominant part of the mode frequency, is mostly classical with modifications from quantum effects through a.

The principal result of this paper, the growth rate,

$$\Gamma = -\frac{ak_z}{2\omega_* S_3\sqrt{1+K^2}}(\boldsymbol{e}_s \times \hat{\boldsymbol{k}}) \cdot \boldsymbol{\Omega}_{q0}\,, \tag{11}$$

determined by the zeroth order quantum vorticity, however, has no classical analogue.

To explore the nature of these modes in detail, the dispersion relation (9) must be solved numerically. For simplicity, we assume the x (y) component of the equilibrium spin S_1 (S_2) to vary only in y (x) direction so that the equilibrium quantum vorticity Ω_{q0} lies in the $-z$ direction (note that this means that $\Gamma > 0$). Then, for fixed S_3, the imaginary part reaches its maximum at maximum $k_z = (\boldsymbol{e}_s \times \hat{\boldsymbol{k}})_z$. The maximization condition, when combined with the fact that the vector $\boldsymbol{S}$ has a unit modulus, enforces $k_z^2 = \frac{1}{2}(1 - S_3^2)$, and leads to $k_z^2/S_3^2 = 1$ for an isotropic spin distribution ($S_1^2 = S_2^2 = S_3^2 = 1/3$). An anisotropic spin distribution with $S_3^2 < 1/3$ would imply $k_z^2/S_3^2 > 1$. However, there is no reason to assume the spin distribution to be very strongly anisotropic. These relations simplify Γ and subsequent calculations.

The dispersion relation admits five different modes, two of which are the positive and the negative light waves. The other three, all new quantum branches (with possibly unstable roots), though interesting in their own right, will be dealt with in a follow-up paper.

The primary focus of this paper is the light wave. As indicated in the introduction, both branches of the light wave have a purely positive growth rate, peaking for small K. We display the pertinent real and imaginary parts (for the latter, both analytic and numerical results) in Figs. 1 and 2. We have chosen two distinct density regimes to study: the comparatively lower density, $n \approx 10^{23}\mathrm{cm}^{-3}$, typical to the electron gas in metals (to be called a solid state plasma), and the higher density, $n \approx 10^{30}\mathrm{cm}^{-3}$, corresponding to the degenerate electron gas in a white dwarf (compact star). In a neutron star, the electron densities (about 1% of the neutron density) can go as high as $10^{34} - 10^{35}\mathrm{cm}^{-3}$. The high-density systems will be called astrophysical plasmas.

The real part of the frequency, in both cases, is readily understandable; the mode tends to be more and more light like ($F = 0$, $\omega = cK$) as K increases. The principal interest, however, is in the imaginary part of the mode (normalized to the electron plasma frequency). This is in the range of $\sim 2.4 \times 10^{-7}$ for the solid state relevant plasmas (see Fig. 1), and of $\sim 8 \times 10^{-4}$ in astrophysical scenarios (Fig. 2). Note that the analytic approximation is almost indistinguishable from the numerical result.

We must, of course, examine if these growth rates are high enough to compete with other plasma processes, i.e., if they are, for example, large enough to overcome damping by collisions. In order to estimate the effect of collisions, we resort to the literature on collisional quantum plasmas [23]. The normalized collision frequency (to the plasma frequency ω_p), relevant to the regime of solid state plasmas (density around $10^{23}\mathrm{cm}^{-3}$, temperature $T = 30K$) has been estimated to be

$$\frac{\nu_{ee}}{\omega_p} = \frac{1}{g_Q^{1/2}}\left(\frac{T}{T_F}\right)^2 \simeq 3.3 \times 10^{-8}, \tag{12}$$

where T_F (E_F) is the Fermi temperature (energy) and $g_Q \equiv E_{\mathrm{int}}/E_F \sim (\hbar\omega_p/E_F)^2$ is the quantum coupling parameter that measures the interaction energy $E_{\mathrm{int}} = 4\pi e^2 n^{1/3}$ versus the average kinetic energy.

For solid state quantum plasmas, a comparison of the estimate (12) with the growth rates (also normalized to ω_p) reveals that, in the low-K range, the growth rates are greater than the collision frequency, and in the intermediate-K range, the two become comparable. One could conclude that, at least for not too high K, damp-

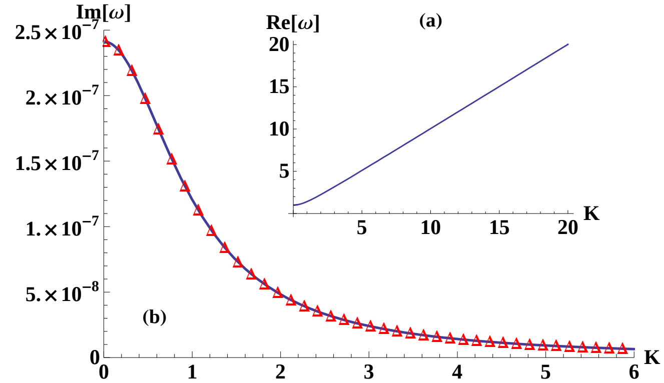

FIG. 1: Real and imaginary parts of the electromagnetic modes for density $n = 10^{23}\mathrm{cm}^{-3}$ (solid state plasmas), $a \approx 10^{-5}$ and $\epsilon = 0.1$. (a) The real part of the dispersion relation (9). It is almost unchanged in the range shown. (b) The imaginary part of the dispersion relation (9) (blue line) and analytical approximation (11) (in triangles). Note the good agreement.

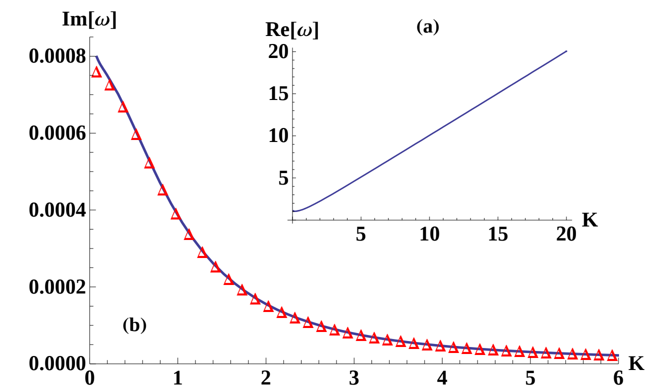

FIG. 2: Real and imaginary parts of the electromagnetic modes for density $n = 10^{30}\mathrm{cm}^{-3}$ (astrophysical plasmas), $a \approx 4 \times 10^{-2}$ and $\epsilon = 0.1$. (a) The real part of the dispersion relation (9). It is almost unchanged in the range shown. (b) The imaginary part of the dispersion relation (9) (blue line) and analytical approximation (11) (in triangles).

ing of the modes via dissipation may not seriously contend with the spin-gradient induced growth in solid state plasmas. For astrophysical parameters (densities around $n = 10^{30}\text{cm}^{-3}$), the collisional damping turns out to be negligible compared with the growth rate. The normalized growth rates are considerably greater than the normalized growth rates for the solid state systems.

The behavior of the modes in the two distinct density regimes is qualitatively similar. For large K, the light wave interacts strongly with one of the quantum branches; this interaction induces the appearance of a secondary peak in the growth rate. This, however, happens outside the range of K displayed in the figures. Details will be given in a follow-up paper.

The polarization of the light wave is calculated from Eqs. (6)-(8) in terms of the perturbed magnetic field components along and perpendicular to the ambient spin field, $\boldsymbol{B}_1 = b_s \boldsymbol{e}_s + b_p \boldsymbol{e}_s \times \hat{\boldsymbol{k}}$. The ratio $b_s/b_p \approx \epsilon(-1 + ia\sqrt{1+K^2})/(aK\sqrt{1+K^2})$ predicts the components to be, only, slightly out of phase because the imaginary part is much smaller than the real part. The ratio of the two components is quite different for the two examples we have worked out: for solid state plasmas, the magnetic field is mainly orientated along the ambient spin, but, for astrophysical plasmas, the perpendicular component is dominant. The polarization is found to be, essentially, independent of the isotropic ambient spin field.

The primary result of this paper is the creation of a theoretical framework for a very exciting, and plausible light amplification mechanism. It is shown that, for a broad range of densities, the light wave propagating in a quantum plasma can be driven unstable with growth rates large enough to overcome intrinsic collisional damping. The waves feed on the free energy inherent in the inhomogeneous macroscopic spin field. We find that the mere existence of gradients is not sufficient. The spin field variation has to be complex enough to guarantee a non-vanishing equilibrium spin vorticity. It is the spin vorticity that conspires to make the gradient free-energy "available" for wave growth.

Our results were obtained for a quantum plasma with no background magnetic field. If the plasma is, however, embedded in an external magnetic field, kinetic theory predicts instabilities [15] when the spin distribution is not in thermodynamic equilibrium. It will be interesting to compare and contrast the characteristics of the (spin vorticity driven) fluid instability and the kinetic one.

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