(2+1)-Dimensional Directed Polymer in a Random Medium: Scaling Phenomena and Universal Distributions

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The 2+1 Dimensional Directed Polymer in a Random Medium: Scaling Phenomena & Universal Distributions

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We examine numerically the zero-temperature 2+1 dimensional directed polymer in a random medium, along with several of its brethren via the Kardar-Parisi-Zhang equation. Using finite-size & KPZ scaling Ansätze, we extract the universal distributions controlling fluctuation phenomena in this canonical model of nonequilibrium statistical mechanics. Specifically, we study pt-pt, pt-line & pt-plane geometries, scenarios which yield higher dimensional analogues of the Tracy-Widom distributions of random matrix theory. Our analysis represents a robust numerical characterization of 2+1 KPZ Class universality.

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For the past 25 years, the celebrated work [1] of Kardar, Parisi & Zhang (KPZ) has held center-stage within the realm of nonequilibrium statistical mechanics. Invoking the spirit of Landau, these authors put forth a continuum description of kinetically roughened microscopic models of stochastic growth, capturing their essential universal scaling behaviors. In this context, KPZ consider a fluctuating height variable $h(\mathbf{x}, t)$, evolving as:

$$\partial_t h = \nu\nabla^2 h + \frac{1}{2}\lambda(\nabla h)^2 + \sqrt{D}\eta(\mathbf{x}, t)$$

where $\mathbf{x}$ is a d-dimensional vector in the substrate plane of growth, ν, λ & D are phenomenological parameters, the last setting the strength of the stochastic noise, which is uncorrelated in space & time and possessing variance $\langle\eta_{\mathbf{x}',t'}\eta_{\mathbf{x},t}\rangle = \delta^d(\mathbf{x}' - \mathbf{x})\delta(t' - t)$. As a diffusive Langevin equation supplemented by rotationally invariant nonlinearity, the KPZ equation has become a fundamental equation of 21^{st}-century theoretical physics. The simple Hopf-Cole transformation, $h(\mathbf{x}, t) = (2\nu/\lambda)\ln Z(\mathbf{x}, t))$, maps KPZ stochastic growth onto the equilibrium statistical physics of directed polymers in a random medium (DPRM); i.e., a Schrodinger equation with multiplicative disorder for the restricted partition function $Z(\mathbf{x}, t)$:

$$\partial_t Z = \nu\nabla^2 Z + (\lambda\sqrt{D}/2\nu)Z\eta$$

itself a paradigmatic model of ill-condensed matter physics. By contrast, the substitution $v = \nabla h$ takes one to the well-studied noisy Burgers equation, which in 1+1 dimensions, ties the KPZ growth & DPRM problems to the rather fertile & amply harvested field of driven lattice gases (DLG), beloved by the mathematics community. Consequently, investment in the KPZ triumvirate of stochastic growth, directed polymers, & driven lattice gases, pays off handsomely. Early analytical, numerical & experimental work on the KPZ class of problems, well-documented in reviews [2], focussed on scaling exponents, universal amplitude ratios [3], and a tantalizing glimpse [4] of full probability distributions (PDFs) of the 1+1 KPZ Class. The past twelve years have witnessed several spectacular advances within the realm of 1+1 KPZ, a consequence of mathematicians bringing a vast arsenal of analytical tools to bear on the matter [5]. Firstly, Johansson [6] revealed that the height fluctuations of the 1+1 single-step (SS) model [7] grown from a point seed were captured by the Tracy-Widom (TW) GUE distribution [8], an astounding, entirely unanticipated discovery that immediately connected the 1+1 KPZ triumvirate to a huge & distinguished university class of gaussian random matrices. Shortly thereafter, Prähofer & Spohn [9] observed that 1+1 polynuclear growth (PNG) mapped directly onto the Ulam problem regarding statistics of the longest increasing subsequence (LIS) of random permutations, also governed [10] by TW GUE. These authors made it quite clear, as well, that a different distribution, TW GOE, was relevant to 1+1 KPZ stochastic growth from a flat substrate IC. A decade later, tour de force exact solutions of the 1+1 KPZ/DPRM problem, including the full time evolution of the universal PDFs to their asymptotic TW forms was managed by independent researchers using complementary WASEP [11] & replica-theoretic DPRM approaches [12]. Simultaneously, in a series of beautiful experiments, Takeuchi & Sano [13] managed to observe both TW GOE & GUE statistics in a 1+1 KPZ kinetic roughening experiment down to a probabilities as small as 10^{-4}. The crucial role of KPZ scaling theory in deciphering these experiments, as well as providing a deeper understanding of the 1+1 exact solutions, has been much emphasized recently by Spohn [14].

Our purpose here is to unearth the rich universality of 2+1 KPZ, a problem for which there are no known exact results and where, furthermore, the higher dimensionality allows additional geometric possibilities. We focus attention on 7 distinct models within the 2+1 KPZ Class: a) 4 DPRM variations, b) RSOS- a classic model of KPZ kinetic roughening, c) a direct Eulerian integration of the 2+1 KPZ equation and, finally, d) 2d Driven Dimers- an octahedral KPZ deposition model due to Kelling & Ódor [15], which permits a direct interpretation in terms of a 2d DLG of dimers in the plane. Within the DPRM sector of the 2+1 KPZ Class, we have restricted ourselves to T=0, so our DPRM simulations amount to a transfer matrix calculation of the *globally optimal directed walk through a 3d lattice of random energy sites,* the total path energy being the sum of the site energies visited along the way. The three relevant 2+1 DPRM geometries are the pt-pt, pt-line, and pt-plane. In all instances, the directed walk commences at the origin, but in the pt-pt geometry the endpoint is also fixed, while the others constrain said endpoint to line or plane, respectively. Regarding the pdf from which the individual site energies are drawn, we stick to the main KPZ story line, considering disorder that is spatiotemporally uncorrelated, but pulled from uniform (u), gaussian (g), or exponential (e) distributions. There is freedom, too, in specifying the crystallographic nature of the 3d lattice (sc, fcc, or bcc), as well as the possible inclusion of a microscopic *elastic* energy cost, typically denoted γ [3], associated with transverse steps. Among our DP models, we consider: i) e3_{fcc} DPRM, in which trajectories emanate from the origin (0,0,0), proceed into the first octant, travel on average along the (1,1,1)-direction, collecting random site energies ε drawn from the exponential distribution p(ε)= $e^{-|\varepsilon|}$. Our e3_{fcc} DPRM generalizes to higher dimension the 1+1 DPRM in the wedge geometry [16] which, with exponentially distributed site energies, is strict counterpart to the 1+1 SS KPZ growth model & TASEP DLG, solved rigorously by Johannson [6], who made explicit the extraordinary connection of 1+1 KPZ to TW GUE random matrix statistics. For this reason alone, the e3_{fcc} DPRM plays a special role in our analysis, and is examined in all three geometries. We mention, too, that this particular model is the precise 2+1 DPRM analog of the 3d Corner Growth (CG) model examined by Olejarz *et al.* [17] whose focus was on the unknown macroscopic limit shape, rather than the height fluctuations which concern us here. Our own investigation of the *anisotropic line tension* of the 2+1 e3_{fcc} DPRM, relevant to the limit shape geometry of the CG model, will be discussed elsewhere. Other DP models include- ii) g4_{bcc}: body-centered cubic arrangement, with gaussian distributed site energies (zero mean, unit variance), iii) u5_{sc}: simple cubic lattice, site energies drawn uniformly from $(0, 1)$, with elastic energy cost γ=1, and finally, iv) g5_{sc}: simple cubic again, same γ, but with gaussian energies, this time zero mean, variance $\frac{1}{4}$. In the last instances, the transfer matrix code is built upon an update rule of the form- $E_{x',t+1}$=$\varepsilon+$Min$[E_{x_1,t}, E'_{x_2,t}, E'_{x_3,t}, E'_{x_4,t}, E'_{x_5,t}]$,where x_n locates the n^{th} nearest neighbor in the preceding plane (i.e., time-slice), $E'_{x_n,t}$=$E_{x_n,t}+\gamma$, and the TM calculation tracks, from one plane to the next, the entire collection of *locally* optimal DPRM paths via the above prescription; the *globally* extremal path being the least energetic member of this ensemble of locally optimal paths [2]. All DP models possess similar rules, though e3_{fcc} & g4_{bcc} have no elastic energy cost built in, so $\gamma = 0$.

For each 2+1 DPRM model, we study initially the *full free-energy PDF* associated with the globally minimal trajectory documenting, as well, its rms fluctuation, skewness s, and kurtosis k. For the u5_{sc} DPRM, our gold

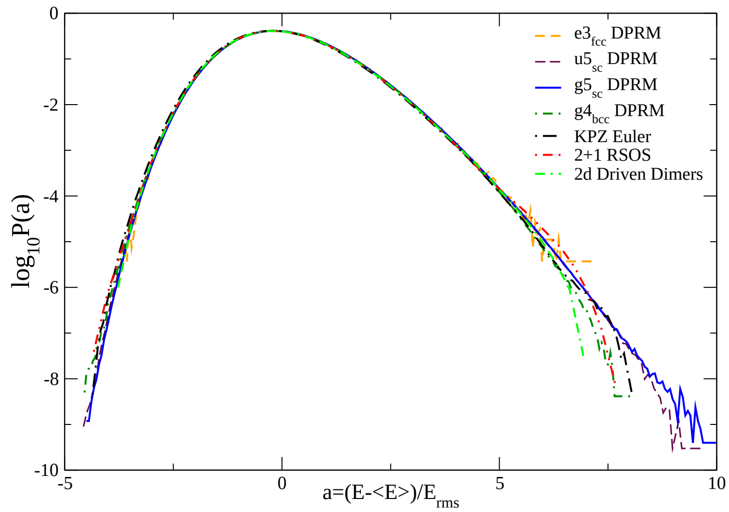


FIG. 1: Fluctuation PDFs: 2+1 KPZ Class Models.

standard, we simulated polymers of length t=1000, in an LxL system of transverse size L=5000, averaging over nr=1880 realizations of the random energy landscape, representing some 47 billion data points in our binned construction of the free energy PDF, shown in Figure 1, a semi-log plot. Similarly, g5_{sc}:(L, nr, t)=$(10^4, 432, 10^3)$; g4_{bcc}:(L, nr, t)=$(10^4, 48, 10^3)$; e3_{fcc}:(nr, t)=$(10^7, 500)$ all comply to a universal 2+1 KPZ data collapse. Here, the abscissa $a \equiv (E - \langle E \rangle)/E_{rms}$, the deviation scaled by the rms fluctuation, is the key dimensionless quantity. We also include in the figure our 2+1 KPZ Euler integration, with $(L, nr, t) = (10^4, 102, 2000)$, a quite substantial numerical investment, well beyond [18]; parameter values chosen were (λ,ν,D)=(20,$\frac{1}{2}$,1), with white noise, and integration time step δt=0.02. We show too, in Fig 1, the height fluctuation PDFs of the 2+1 RSOS & 2d Driven Dimer models; again $L = 10^4$, with averaging over 32 & 12 runs, resp. For these two kinetic roughening models, as well as the KPZ equation itself, $a \equiv (h - \langle h \rangle)/h_{rms}$, representing the *scaled height fluctuation,* as h and F are cognate variables in the stochastic growth and DPRM contexts. In any case, the severe data collapse for these 7 distinct members of the 2+1 KPZ Class provide us with strong evidence, indeed, of universality in this higher dimension, well into the tails.

To dig deep to the core of 2+1 KPZ universality, and lay the groundwork for the remainder of this paper, we now discuss in greater detail the full machinery of KPZ scaling theory, which rests upon a careful determination of the characteristic KPZ nonlinearity λ, as well as the static amplitude A, defined via the fixed-time height-height k-space correlator: $\langle|h(\mathbf{k})|^2\rangle$=$Ak^{-2-2\chi}$. The essential ideas, laid out already for 1+1 KPZ [3], with additional helpful details in [5], focusses on the dimensionless time θt, where θ=$A^{1/\chi}\lambda$, with χ the *steady-state* KPZ critical index. The basic KPZ narrative involves a bump on the surface of height $\xi_\perp$, lateral dimension

Model	$f_\infty(v_\infty)$	λ	A	χ	θ	β	$\langle\xi\rangle$	$\langle\xi^2\rangle_c$	s	k
u5$_{sc}$ Kim DP	0.38390	-0.1585	1.1978	0.389	0.2518	0.2402	-0.714	0.250	-0.422	0.343
g5$_{sc}$ DPRM	-0.55336	-0.2182	1.74215	0.381	0.9363	0.2425	-0.675	0.211	-0.433	0.356
e3$_{fcc}$ DPRM	-2.64381	-0.1439	21.03	0.387	375.3	0.248	-0.754	0.208	-0.435	0.362
g4$_{bcc}$ DPRM	-1.80949	-0.5014	2.8248	0.380	7.7198	0.235	-0.851	0.240	-0.412	0.320
KPZ Euler	0.17606	20	0.02295	0.388	1.192x10^{-3}	0.2408	-0.679	0.236	-0.423	0.354
2+1 RSOS	0.31270	-0.414	1.2005	0.383	0.66144	0.2422	-0.737	0.233	-0.430	0.357
2d Driven Dimers	0.34141	-0.6094	1.2201	0.375	1.0359	0.2415[a]	-0.830	0.256	-0.414	0.338

[a]ref. [15]

TABLE I: 2+1 KPZ Class Parameters, *pt-plane* DPRM geometry; equivalently, KPZ stochastic growth from a flat substrate.

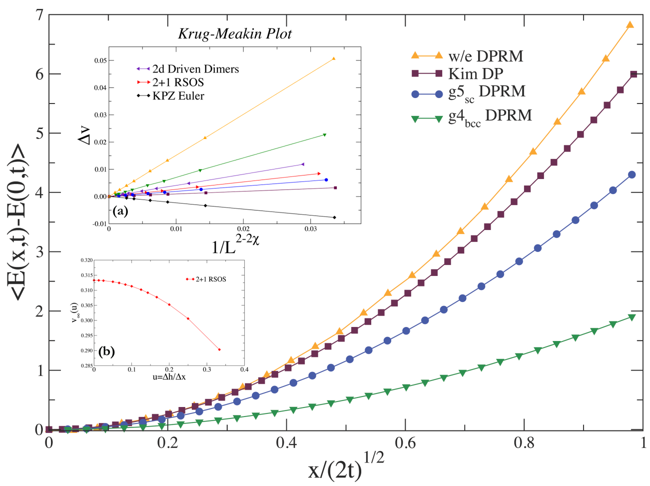


FIG. 2: Disorder-averaged, *parabolic* DPRM free energy profile. Insets: a) Summary Krug-Meakin plot for 2+1 KPZ Class models, b) *quadratic* KPZ tilt-dependent growth velocity [23] for the 2+1 RSOS model.

$\xi_\parallel$, which evolves according to the KPZ nonlinear term as $\dot{\xi}_\parallel \approx \lambda(\xi_\perp/\xi_\parallel)$. With the transverse fluctuation scaling as $\xi_\perp \sim A\xi_\parallel^\chi$, consistency demands $\xi_\parallel \sim (A^2\lambda t)^{1/z}$ while $\xi_\perp \sim (\theta t)^{\beta=\chi/z}$, with θ as above, and dynamic index z=2-χ given by the KPZ identity. With the key scaling parameter θ known, universal KPZ amplitudes can be extracted for each model and compared across the 2+1 KPZ spectrum. Motivated by the exact 1+1 results of Sasamoto & Spohn [11], we conjecture that the solution of the 2+1 KPZ equation for 3d wedge (i.e.,"conical") IC centered at the origin has the form: $h(\mathbf{x},t) = -x^2/2\lambda t + v_\infty t + (\theta t)^\beta \xi$ with the understanding that the statistics of the random variable ξ has become the focus, and, *its PDF the definitive expression of 2+1 KPZ universality.* We have determined the KPZ early-time exponent β independently for each model; our DPRM, RSOS, & KPZ Euler results in fine accord with both revered [19], 0.240, and more recent [15] blue-chip estimates for this index. To pin down $P(\xi)$, and reveal its universal nature, we sift, anew, through the large data sets underlying the fluctuation PDFs of Figure 1, recasting the analysis in terms of $\xi = (h - v_\infty t)/(\theta t)^\beta$, where, in the KPZ context, $v_\infty = \langle dh/dt \rangle$ is the asymptotic instantaneous growth velocity; analogously, $f_\infty = \langle dF/dt \rangle$, the DPRM free energy per unit length. It is the distribution $P(\xi)$ which lies at the heart of 2+1 KPZ Class universality, and the matter demands, in addition to knowledge of θ, a precise determination of KPZ/DPRM v_∞/f_∞. To this end, we have relied heavily upon a Krug-Meakin [20] finite-size scaling analysis which, by virtue of a truncated Fourier sum over modes, reveals that the KPZ growth velocity in a system of finite size L suffers a small shift from its true asymptotic value: $\Delta v \equiv \langle dh/dt \rangle - v_\infty = -\frac{1}{2}A\lambda/L^{2-2\chi}$; for the DPRM problem, the corresponding *free energy shift* Δf represents an ill-condensed matter manifestation of the Casimir Effect [21]. In Figure 2, we show results for our seven 2+1 KPZ Class models- in fact, the first pass involves a 3-parameter fit, yielding v_∞, the product $A\lambda$, and χ; knowing v_∞ and χ, see Table I for values, allows construction of a summary Krug-Meakin plot of Δv vs $1/L^{2-2\chi}$, including all 7 models, with slopes set by $-\frac{A\lambda}{2}$, see Fig 2(a). Via diverse procedures, it is also possible to extract the KPZ nonlinearity λ directly; for the DPRM systems, we rely upon the *disorder-averaged quadratic free-energy profile,* an insight that dates back to Parisi & Mezard [22], but is implicit in our Sasamoto-Spohn conjecture above. Ultimately, it follows from the fact that at early times with conical IC, the KPZ nonlinearity dominates, generating Cole-Hopf paraboloids with small superposed distortions arising from the additive KPZ noise term. While such noise is visible for each individual run, ensemble averaging produces a smooth parabolic profile- see Fig 2, proper, which follows from 10^4 realizations of our DPRM random energy landscape. Alternatively, for the KPZ stochastic growth models, such as 2+1 RSOS, we study the tilt-dependent growth velocity [23], Fig 2(b). For 2d Driven Dimers, A is known [15], so we get λ directly from the Krug-Meakin plot. Recall all DPRM models, by default, have $\lambda < 0$; by contrast, our KPZ Euler integration was done with positive λ=20. With θ and f_∞ extracted for each of our models, refer to Table I, we can now craft a full portrait of 2+1 KPZ universality for the *pt-plane* DPRM geometry- see Figure 3. In this much more stringent test of universality, we note several key points gleaned from Figure 3 and associated Table I, where the final four columns record our model estimates for the mean, variance, skewness and kurtosis of $P(\xi)$: i) Firstly, the existence of a unified 2+1 KPZ Class, *per*

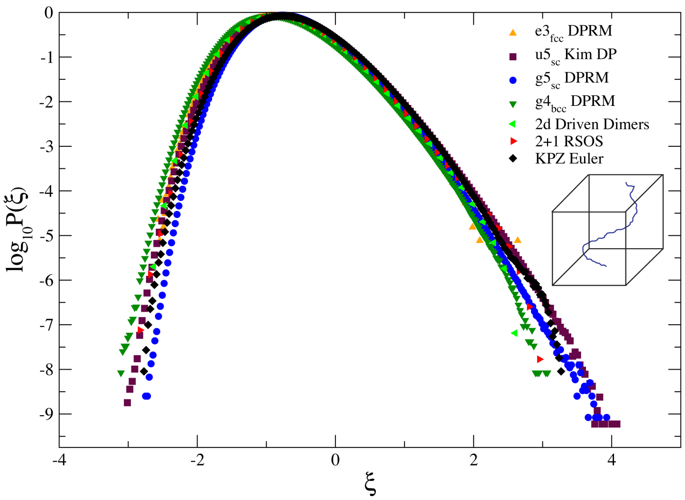


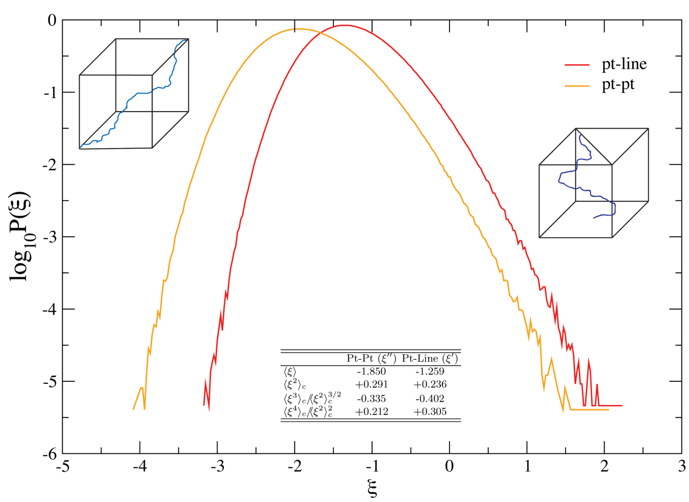
FIG. 3: Universal PDF, 2+1 KPZ Class: DPRM *pt-plane* geometry; KPZ stochastic growth with flat interface IC.

se, is manifest, ii) As in the case of 1+1 KPZ [24], there is a small, but persistent, dispersion of among the models owing to a stubborn approach to asymptopia by the first moment $\langle\xi\rangle$; see, esp, g4$_{bcc}$ DPRM, iii) Skewness & kurtosis of 2+1 KPZ pt-plane problem are larger than TW GOE; an average over our models yields $\overline{s}$=-0.426(6), $\overline{k}$=0.35(1), $\overline{\langle\xi^2\rangle}_c$=0.232(18) & $\overline{\langle\xi\rangle}/\sqrt{\overline{\langle\xi^2\rangle}_c}$=-1.46(9); even so, the iv) Long right tail, as measured precisely by us for the g5$_{sc}$ DPRM, has an exponent 1.495(10) nearly indistinguishable from the exact Airy value $\frac{3}{2}$, characteristic of Tracy-Widom 1+1 KPZ. We wonder whether, intuition to the contrary, the 2+1 KPZ Class may possess some vestigial link to the Painlève system & determinantal point processes, leaving open, perhaps, the possibility of a fulcrum, rational critical index.

Finally, in Figure 4, we plot up the challenging distributions associated with the *pt-line* & *pt-pt* geometries, studied by us with the wedge e3$_{fcc}$ DPRM. In the 2+1 KPZ kinetic roughening context, these PDFs dictate height fluctuations for self-similar growth initiated, resp., from point seed and 1d groove initial conditions, the former corresponding to TW GUE, the latter having no lower dimensional analog. We note here, particularly, that s & k increase in magnitude as one progresses from pt-pt, pt-line, and pt-plane problems, whereas the first and second moments trend in the opposite fashion- see Fig 4, table insert. We expect the pt-pt geometry, with extremal trajectories connecting far corners of the cube, will witness the first analytical advance; in its semi-discrete Poissonized DPRM form, it is germane to the generalized random permutation & PNG problems [9].

In summary, we have presented a complete, multi-faceted portrait of 2+1 KPZ Class universality, extracting the three characteristic PDFs associated with distinct DPRM pt-plane, pt-line, & pt-pt geometries. This represents a robust numerical solution of the 2+1 KPZ problem. It is our hope that this work will inspire experimentalists as well as anchor future analytical efforts, providing a target for physicists & mathematicians alike to shed light on the kinship of these distributions, and reveal its underlying superuniversal fixed point structure. Given the replica-theoretic interpretation of the 2+1 DPRM, explicit calculation of any one of our numerical PDFs would not only reveal much regarding KPZ stochastic growth phenomena & DLG interacting particle systems, but also the foundational statistical mechanical problem of attractive bosons in two dimensions.

We are grateful to J. Krug & E. Brézin for illuminating, very helpful discussions regarding our work.



	Pt-Pt (ξ'')	Pt-Line (ξ')
$\langle\xi\rangle$	-1.850	-1.259
$\langle\xi^2\rangle_c$	+0.291	+0.236
$\langle\xi^3\rangle_c/\langle\xi^2\rangle_c^{3/2}$	-0.335	-0.402
$\langle\xi^4\rangle_c/\langle\xi^2\rangle_c^2$	+0.212	+0.305

FIG. 4: Universal PDFs: 2+1 DPRM pt-pt & pt-line geometries, left & right, resp. Table insert: Distribution moments.

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