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Apparent low-energy scale invariance in two-dimensional Fermi gases

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Recent experiments on a $2d$ Fermi gas find an undamped breathing mode at twice the trap frequency over a wide range of parameters. To understand this seemingly scale-invariant behavior in a system with a scale, we derive two exact results valid across the entire BCS-BEC crossover at all temperatures. First, we relate the shift of the mode frequency from its scale-invariant value to $\gamma_d \equiv (1+2/d)P - \rho(\partial P/\partial\rho)_s$ in d dimensions. Next, we relate γ_d to dissipation via a new low-energy bulk viscosity sum rule. We argue that $2d$ is special, with its logarithmic dependence of the interaction on density, and thus γ_2 is small in both the BCS and BEC regimes, even though $P - 2\varepsilon/d$, sensitive to the dimer binding energy that breaks scale invariance, is not.

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Systems exhibiting scaling symmetry or conformal invariance are very special. In all laboratory realizations, one needs to tune one or more physical parameters (temperature, chemical potential, coupling) to observe scale-invariant behavior, for instance, in the vicinity of a quantum critical point [1]. Another example is provided by strongly interacting Fermi gases in three spatial dimensions ($3d$), which display remarkable scale invariance properties at unitarity, where the s-wave scattering length diverges by tuning to the Feshbach resonance. This is manifested in universal thermodynamics [2], the vanishing of the d.c. bulk viscosity [3], and the entire bulk viscosity spectral function $\zeta(\omega, T)$ [4] at unitarity. There may also be tantalizing connections between the ratio of the shear viscosity η to the entropy density s of the unitary Fermi gas [5, 6] and the bound for η/s conjectured on the basis of gauge/gravity duality [7].

For the unitary gas in a $3d$ isotropic harmonic trap, scale invariance manifests itself most dramatically as an undamped monopole breathing mode oscillating at twice the trap frequency ω_0 independent of temperature [8, 9]. This mode corresponds to an isotropic *dilation* of the gas wherein the coordinates in the many-body wavefunction are scaled $\propto \cos(\omega t)$. Scale invariance implies that this wavefunction is an exact eigenstate of the Hamiltonian and oscillates at a frequency $2\omega_0$ without damping [8].

In a recent experiment [10], collective modes in a two-dimensional ($2d$) Fermi gas were measured over a broad range of temperatures and interaction strengths. Remarkably, the breathing mode was found to oscillate without any observable damping at $\simeq 2\omega_0$ for $0.37 \lesssim T/T_F \lesssim 0.9$ and $0 \lesssim \ln(k_F a_2) \lesssim 500$, where a_2 is the $2d$ scattering length. This observation is extremely surprising, given that there is no *a priori* reason to expect scale-invariant behavior in a system which has a scale, namely, the dimer binding energy in $2d$.

Our goal is to understand why the $2d$ Fermi gas appears to show nearly scale invariant behavior over a very broad range of parameters without the need for fine-tuning. Understanding this may give insight into related problems such as why, in some quantum field theories with conformal invariance broken by a mass term, the sound speed and bulk viscosity remain close to their conformal-limit values for a wide range of couplings [11].

We emphasize that this question is distinct from that of small deviations from scale invariance in weakly interacting $2d$ Bose gases. Quantum gases with an unregularized delta-function interaction have an SO(2,1) symmetry [12] and exhibit scale invariance. However, the cutoff essential to describe an actual short-range interaction leads to a violation of scale invariance (analogous to an anomaly in quantum field theory) and an interaction-dependent shift in the breathing mode frequency from $2\omega_0$ [13] in a $2d$ Bose gas. The $2d$ Bose gas experiments that see nearly scale-invariant behavior are in the weakly-interacting regime [14–16], where deviations are expected to be small. In contrast, the $2d$ Fermi case that we focus on is not weakly interacting and we must take into account strong interactions.

Results— We begin by summarizing our approach and main results. We consider a dilute Fermi gas in $d = 2, 3$ dimensions with a short-range s-wave interaction, arising from a broad Feshbach resonance between two spin species, each with density $n/2$. The dimensionless interaction g_d is expressed as $g_3 = -1/k_F a_3$ in $3d$ and $g_2 = \log(k_F a_2)$ in $2d$; a_d is the s-wave scattering length that sets the dimer binding energy $\varepsilon_b = -1/ma_d^2$. The fermions have mass m, density $n \sim k_F^d$, and we set $\hbar = 1$.

The quantity of central interest in our analysis is

$$\gamma_d \equiv (1 + 2/d)P - \rho(\partial P/\partial\rho)_s, \tag{1}$$

which is the deviation of the adiabatic compressibility $\rho(\partial P/\partial\rho)_s$ from its value $(1+2/d)P$ in a scale-invariant system, where the pressure $P \propto \rho^{(1+2/d)}$. Here $s = S/N$ is the entropy per particle and $\rho = mn$ the mass density.

First, we show that γ_d governs the difference between the frequency ω_m of the hydrodynamic monopole breathing mode and $2\omega_0$ for a Fermi gas in an isotropic har-

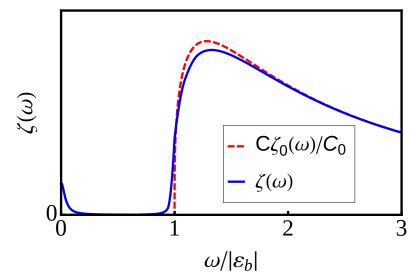

FIG. 1: Schematic plot of the bulk viscosity spectral function $\zeta(\omega)$ (solid line) and the scaled "vacuum" contribution $C\zeta_0(\omega)/C_0$ (dashed line) given by (6). The smallness of the subtracted sum rule (3) for $[\zeta(\omega) - C\zeta_0(\omega)/C_0]$ implies very little spectral weight below the dimer binding energy $|\varepsilon_b|$ for excitations that break scale invariance.

monic trap $V_{\rm ext}(r) = m\omega_0^2 r^2/2$. We find

$$\omega_m^2/4\omega_0^2 = 1 - \frac{d^2}{8}\int d^d\mathbf{r}\ \gamma_d(r) \Big/ \int d^d\mathbf{r}\, n(r) V_{\rm ext}(r). \quad (2)$$

Second, we show that γ_d is related to the exact sum rule for the bulk viscosity spectral function $\zeta(\omega)$:

$$(2/\pi)\int_0^\infty d\omega\,[\zeta(\omega) - C\zeta_0(\omega)/C_0] = \gamma_d. \quad (3)$$

Here, C is the "contact" [17, 18] with $\zeta_0(\omega)$ and C_0 the bulk viscosity and contact in the zero-density $n \to 0$ limit at $T=0$. The subtraction on the left-hand-side removes the large-ω tail of $\zeta(\omega)$ (see Fig. 1) and the sum rule thus measures the availability of *low-energy* ($\lesssim |\varepsilon_b|$) spectral weight for excitations that break scale invariance.

The experimental observations of Ref. [10] in $2d$ Fermi gases imply that $\gamma_2 \ll n\epsilon_F$. Our goal is to understand *why* γ_2 is so small for a wide range of interaction strengths, even though other measures of the departure from scale invariance, such as $P - 2\varepsilon/d$ (where ε is the energy density), are *not* small. γ_d strictly vanishes only at the unitary point $g_3 = 0$ in $3d$, and in the weak-coupling BCS limit $g_2 \to \infty$ in $2d$. However, we will argue that there is considerable evidence for an anomalously small γ_2 across the entire BCS-BEC crossover. Remarkably, within mean field theory [19] $\gamma_2^{\rm MF} = 0$ for all values of g_2 (and only in $d = 2$). In addition, the available $T=0$ quantum Monte Carlo (QMC) data in $2d$ [20] leads to an estimate of γ_2 that is consistent with zero over the entire crossover, except possibly near $g_2 = 0$. We reach the same conclusion at finite T using a scaling argument, and argue that this is due to the logarithmic dependence of g_2 on density. Using the sum rule (3), we will argue that a small γ_2 also gives insight into the negligible viscous damping of the monopole mode.

Monopole breathing mode— The normal mode solutions of the hydrodynamic equations with frequency ω are obtained from the Lagrangian [9]

$$\mathcal{L}[\boldsymbol{u}] = \omega^2\int d\mathbf{r}\ \rho_0 \boldsymbol{u}^2(\mathbf{r}) - \int d\mathbf{r}\Big[\rho_0^{-1}(\partial P/\partial\rho)_s\,(\delta\rho)^2 + 2\rho_0(\partial T/\partial\rho)_s\,\delta\rho\delta s + \rho_0\,(\partial T/\partial s)_\rho\,(\delta s)^2\Big], \quad (4)$$

describing quadratic fluctuations in entropy δs and density $\delta\rho$ about their equilibrium values, s_0 and ρ_0. The displacement field $\boldsymbol{u}(\mathbf{r}, t)$ is related to the velocity $\mathbf{v}$ by $\partial\boldsymbol{u}/\partial t = \mathbf{v}$. Conservation of density and entropy gives $\delta\rho = -\boldsymbol{\nabla}\cdot(\rho_0\boldsymbol{u})$ and $\delta s = -\boldsymbol{u}\cdot\boldsymbol{\nabla}s_0$. Eq. (4) is valid in both the normal as well as the superfluid phase, where it describes first sound with $\mathbf{v}_n = \mathbf{v}_s$ [9].

We obtain the result (2) for the breathing mode frequency using the scaling ansatz $\boldsymbol{u}(\mathbf{r}, t) = u\,\mathbf{r}\cos(\omega t)$ in (4), together with the Maxwell relation $(\partial P/\partial s)_\rho = \rho_0^2(\partial T/\partial\rho)_s$ and the equilibrium identities $\nabla P_0 = (\partial P/\partial\rho)_s\nabla\rho_0 + (\partial P/\partial s)_\rho\nabla s_0 = -n_0\nabla V_{\rm ext}$ for $V_{\rm ext} = m\omega_0^2 r^2/2$, and $\nabla T_0 = (\partial T/\partial\rho)_s\nabla\rho_0 + (\partial T/\partial s)_\rho\nabla s_0 = 0$. The above scaling ansatz provides a rigorous upper bound on the mode frequency [9]. Generalizing the variational ansatz to $\boldsymbol{u} = \mathbf{r}\sum_{n=0} u_n r^{2n}\cos(\omega t)$, it is easy to show that the corrections to (2) are governed by higher powers of γ_d. Thus, γ_d rigorously determines the deviation of the monopole frequency ω_m from $2\omega_0$.

We next relate γ_d to the contact C, given by $C = 2\pi m a_2(\partial\varepsilon/a_2)_s$ in $2d$ [21, 22] and $C = 4\pi m a_3^2(\partial\varepsilon/\partial a_3)_s$ in $3d$ [18]. We find $\gamma_2 = -[C + \frac{a_2}{2}(\partial C/\partial a_2)_s]/4\pi m$ and $\gamma_3 = -\left[C + a_3\left(\partial C/\partial a_3\right)_s\right]/36\pi m a_3$. This makes it clear that $\omega_m = 2\omega_0$ is strictly valid only for $a_2 \to \infty$, the BCS limit in $2d$, where $C \to 0$, and at unitarity in $3d$, where $|a_3| \to \infty$. On the other hand, the breathing mode frequency (2) is very sensitive to $\gamma_d \neq 0$ in both $2d$ and $3d$. Using $\int d^d\mathbf{r} n V_{\rm ext} \sim \mathcal{O}(N\epsilon_F)$, we estimate that a value of γ_d as small as $0.1 n\epsilon_F$ would give rise to a 5% shift in ω_m. The fact that no such shift is observed [10] in $2d$ indicates that we must understand why $\gamma_2 \ll n\epsilon_F$ for a wide range of g_2 and T.

Viscosity sum rules— The bulk viscosity ζ is the only transport coefficient that damps the scaling flow $\boldsymbol{u} \propto \mathbf{r}$ [23]. To gain insight into why it is small in $2d$, we derive a new bulk viscosity sum rule that relates γ_2 to the low-energy spectral weight for excitations that break scale-invariance symmetry.

The bulk viscosity spectral function $\zeta(\omega)$ is related by a Kubo formula to the transverse $\chi_T(\mathbf{q}, \omega)$ and longitudinal $\chi_L(\mathbf{q}, \omega)$ current correlators: $\zeta(\omega) = \lim_{q\to 0} m^2\omega\,[{\rm Im}\chi_L - (2 - 2/d){\rm Im}\chi_T]/q^2$. Generalizing Ref. [4] to arbitrary d, we obtain the exact sum rule

$$\frac{2}{\pi}\int_0^\infty d\omega\zeta(\omega) = -\left(2 - 2/d\right)X_T + X_L - \rho c_s^2. \quad (5)$$

Here, $X_{T(L)} = \lim_{q\to 0}\langle[\hat{j}^x_{-\mathbf{q}}, [\hat{H}, \hat{j}^x_{\mathbf{q}}]]\rangle_{T(L)}/q^2$ with the current $\hat{j}^x_{\mathbf{q}} = \sum_{\mathbf{k}\sigma}[(2\mathbf{k} + \mathbf{q})_x/2m]\hat{c}^\dagger_{\mathbf{k}\sigma}\hat{c}_{\mathbf{k}+\mathbf{q}\sigma}$. The subscript $T(L)$ denotes the transverse (longitudinal) $q \to 0$ limit [24], and $c_s \equiv (\partial P/\partial \rho)_s^{1/2}$ is the adiabatic sound speed. Evaluating the commutators in (5) for an isotropic pair potential with range r_0, we find the $2d$ result $(2/\pi)\int d\omega \zeta(\omega) = 2\varepsilon - \rho c_s^2 + \alpha C/m + \beta C \ln \Lambda/m$. Here, $\Lambda = 1/r_0$ is an ultraviolet (UV) cutoff, and α, β are constants. In $3d$ [4], the terms proportional to C are of the form $\alpha C/ma_3 + \beta C\Lambda/m$.

The key insight that allows us to obtain physical results independent of Λ is that an UV divergence of precisely the same form must arise in the two-body problem. The sum rule for ζ_0 has the same form as above, but with energy density and contact replaced by their zero-density, $T = 0$ values, ε_0 and C_0, while $c_s = 0$ for $n \to 0$. The exact solution ζ_0 of the two-body problem can then be used to regularize the divergence in the many-body problem. The same idea underlies the standard replacement of the bare interaction with the two-particle s-wave scattering length in the study of dilute gases [25].

The $T = 0$, zero-density limit $\zeta_0(\omega)$ of the viscosity spectral function has an *exact* representation in terms of the sum of all particle-particle ladder diagrams with two external current vertices. These are the well-known [26] Aslmazov–Larkin, Maki–Thompson, and self-energy diagrams. We thus obtain the $2d$ result [27]

$$\zeta_0(\omega) = \frac{C_0}{4m\omega} \frac{\Theta(\omega - |\varepsilon_b|)}{\ln^2(\omega/|\varepsilon_b| - 1) + \pi^2} \tag{6}$$

for $\omega > 0$, and $\zeta_0(-\omega) = \zeta_0(\omega)$. In $2d$, there is a bound state for all values of the scattering length [19]. Thus, in the zero-density (single dimer) and temperature limit, $\varepsilon_0 = \varepsilon_b = -1/ma_2^2$ and $C_0 = 4\pi/a_2^2$. The absence of spectral weight in ζ_0 below $|\varepsilon_b|$ is due to the fact that the only excitations in this limit involve pair disassociation with a gap $|\varepsilon_b|$ at $T = 0$.

The UV divergences can now be removed by looking at the difference between the sum rule for the interacting many-body system and that for the $T = 0$, $n \to 0$ limit, scaled by (C/C_0). In $2d$, we find $(2/\pi)\int d\omega\, [\zeta(\omega) - C\zeta_0(\omega)/C_0] = 2\varepsilon - \rho c_s^2 - 2C\varepsilon_0/C_0$. Using $C = 4\pi m(P - \varepsilon)$ and $\varepsilon_0/C_0 = -1/(4\pi m)$, we obtain (3) in $2d$. The same methodology can be also be used to obtain corresponding results for the bulk viscosity in $3d$ as well as the shear viscosity in any d [27].

Our main focus will be on (3), which quantifies the *low-energy* spectral weight in $\zeta(\omega)$ with the high-energy tail $C\zeta_0(\omega)/C_0$ subtracted out; see Fig. 1. However, we can also obtain the total spectral weight in $\zeta(\omega)$:

$$S_{2d} \equiv \frac{2}{\pi}\int_0^\infty d\omega \zeta(\omega) = 3P - \varepsilon - \rho c_s^2 = -\frac{1}{8\pi m}\left(\frac{\partial C}{\partial g_2}\right)_s. \tag{7}$$

$S_{2d} \geq 0$ for all g_2 [21], as required by $\zeta(\omega) \geq 0\ \forall\omega$ [4]. In Fig. 2, we plot S_{2d} as a function of $g_2 = \log(k_F a_2)$ using $T = 0$ QMC data [20] to evaluate the right-hand-side of (7). Both S_{2d} and its $3d$ counterpart S_{3d} [4] (inset of Fig. 2 using the QMC data of Ref. [28]) are $\ll n\epsilon_F$ in the BCS region $g_d \gtrsim 1$, but become significantly larger on the BEC side.

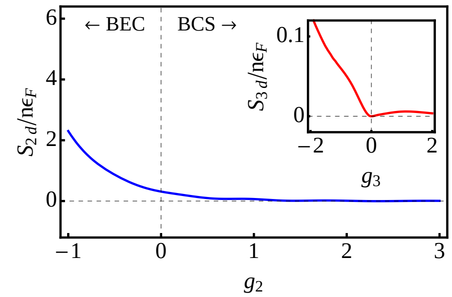

FIG. 2: *Bulk viscosity sum rule:* The $2d$ sum rule S_{2d} at $T = 0$ in units of $n\epsilon_F$ plotted as a function of the coupling $g_2 = \ln(k_F a_2)$. Inset: The corresponding $3d$ result [4] as a function of $g_3 = -1/(k_F a_3)$. In both $2d$ and $3d$, $g_d \to +\infty$ the BCS limit while $g_d \to -\infty$ is the BEC limit.

Apparent scale invariance—We now have all the results in hand to discuss deviations from scale invariance. So far, we have shown that γ_d controls the deviation of the monopole ω_m from $2\omega_0$ and also governs the availability of low-energy bulk viscosity spectral weight $\zeta(\omega)$. We can intuitively understand the exact relation (3) between a ζ-sum rule and the shift in the mode frequency as a Kramers–Kronig transform of the d.c. bulk viscosity $\zeta(0)$ that damps the monopole mode.

What, if anything, is special about $2d$ that leads to the strong experimental signatures [10] of scale invariance? We begin by addressing this question at $T = 0$ and then generalize to finite temperatures. The first clue comes from mean field theory (MFT), which in $2d$ has a transparent solution [19] across the entire $T = 0$ BCS-BEC crossover: $\varepsilon = n\epsilon_F/2 - n|\varepsilon_b|/2$. This leads to $P = n\epsilon_F/2$ and thus $\gamma_2^{\rm MFT} \equiv 0$ for *all* couplings g_2. Contrast this with the $2d$ MFT result $P - \varepsilon = n|\varepsilon_b|/2$, which is very small in the BCS regime but very large on the BEC side. This is our first hint of something we will see again: γ_d is small in part because it does not involve physics on the scale of the dimer binding energy, whereas $P - \varepsilon$ does.

To understand how quantum fluctuations beyond MFT affect the result for γ_2, we use $T = 0$ QMC [20]. We find that the QMC-derived γ_2 is vanishingly small in both BCS and BEC regimes, and even for $g_2 \sim 0$, $\gamma_2 \sim 0$ (within large error bars) as shown in Fig. 3. We also see from this figure that the $2d$ result is quite different from the $3d$ case. The QMC estimate for γ_3 (using data from Ref. [28]), though quite small on the BCS side of the crossover, is large in the BEC region in $3d$.

We now show that the difference between $2d$ and $3d$ is tied to the form of the dimensionless couplings $g_2 = \log(k_F a_2)$ and $g_3 = -1/k_F a_3$. In the BCS limit ($g_d \gg 1$), the equation of state has the form $\varepsilon = (n\epsilon_F/2)[1 + A/g_d + B/g_d^2 + \ldots]$ in both $2d$ and $3d$. The perturbative Hartree

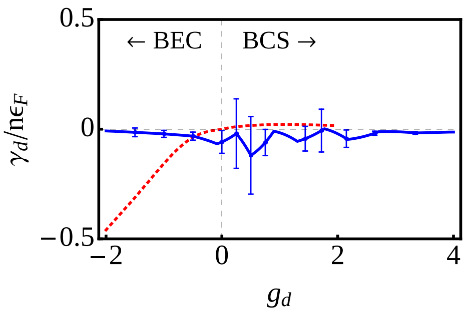

FIG. 3: γ_2 (solid blue line) and γ_3 (dashed red line) shown in units of $n\epsilon_F$, where n and ϵ_F are the two and three dimensional density and Fermi energy, respectively. The error bars on γ_2 are associated with numerical derivatives of QMC data (with errors) [20]. The coupling g_d is $\ln(k_F a_2)$ for $d = 2$ and $-1/(k_F a_3)$ for $d = 3$.

plus "Fermi liquid" corrections are larger than the pairing contribution not shown. (The only qualitative difference is that A is negative in $3d$ but positive in $2d$ [29].) Both γ_d, calculated using (1), and $P - \varepsilon = (a_d/d)\,(\partial\varepsilon/\partial a_d)$, are small in the BCS limit. In the BEC limit ($g_d < 0$ and $|g_d| \gg 1$), we get $\varepsilon = -n|\varepsilon_b|/2 + \ldots$, which is the energy density of $n/2$ dimers with perturbative corrections in powers of $1/|g_d|$. The key difference between $2d$ and $3d$ is in the g_d-dependence of the binding energy $|\varepsilon_b|$, which $\sim \exp(|g_2|)$ in $2d$ and $\sim 1/|g_3|^2$ in $3d$.

To understand the effects of finite temperature, we write the pressure and energy density, related by $P = n(\partial\varepsilon/\partial n)_s - \varepsilon$, in the scaling forms $P = n\epsilon_F \mathcal{F}(g_d, s)$ and $\varepsilon = n\epsilon_F \mathcal{E}(g_d, s)$. There is a qualitative difference between the g_2-dependence of the scaling functions $\mathcal{F}$ and $\mathcal{E}$ in $2d$. The pressure does not have a contribution on the scale of the dimer binding energy $|\varepsilon_b| = 1/ma_2^2$; i.e., it does not have a potentially exponentially large contribution in $g_2 = \log(k_F a_2)$, while the energy density does. We have already seen this in the $T = 0$ MFT results, and the same is also observed in the $2d$ virial expansion [30]. We conjecture that the scaling function $\mathcal{F}$ is a slowly varying function of g_2 at all temperatures in $2d$ (except in the immediate vicinity of a weak singularity at T_c). The equation of state is then $P \sim n^2$ up to logarithmic corrections, leading to a small γ_2.

The absence of high energy contributions on the scale of the dimer binding energy to P and the compressibility is also consistent with γ_2 being related to the low energy spectral weight as shown by our sum rule. Once high energy excitations on the scale of $|\varepsilon_b|$ are excluded, low energy phonons (with a near scale-invariant dispersion $\omega_{\mathbf{q}}(n) \sim \sqrt{n}q$), for instance, dominate the equation of state leading to $P \sim n^2$ and a small γ_2. Unlike γ_2, however, $P - \varepsilon$ is not small, as it involves high energy contributions on the scale of $|\varepsilon_b|$ in the BEC regime.

Another way to characterize the deviation from scale invariance, analogous to the "trace anomaly" in quantum field theory, is to rewrite (1) as as $\gamma_d = -(\partial P/\partial g_d)\beta(g_d)/d$, where $\beta(g_d) \equiv k_F(\partial g_d/\partial k_F)$ describes the scaling of the coupling g_d with respect to the momentum scale k_F. We see that $\beta(g_2) = 1$ while $\beta(g_3) = g_3$, reflecting the difference between the logarithmic and power-law dependence on the density in $2d$ and $3d$ respectively. In both the BCS ($g_d \gg 1$) and BEC ($g_d \ll -1$) regions, the $2d$ beta function is much smaller than its $3d$ counterpart.

Finally, using the sum rules (7) and (3), we discuss the damping of the monopole mode, controlled by $\zeta(0)$. Although a small value for the sum rule by itself does not rigorously upper-bound ζ, any physically reasonable functional form for the spectral function (e.g., a Drude form for $\omega \lesssim |\varepsilon_b|$; see Fig. 1) would lead to a very small value for $\zeta(0)$. We see from Fig. 2 that in the BCS regime $g_d \gtrsim 1$, the sum rule $\int d\omega\zeta(\omega) \ll n\epsilon_F$ in both $2d$ and in $3d$. We would thus expect a very small $\zeta \ll n$ here in both $2d$ and $3d$. This sum rule is quite large on the BEC side and does not lead to any restriction on ζ. From the low-energy sum rule (3), however, we see that the large value of S_{2d} in the BEC limit is entirely dominated by the high-energy tail on scales larger than the dimer binding energy. Once this is subtracted out, the low-energy integrated spectral weight, equal to γ_2, is very small even in the BEC regime (see Fig. 1). Thus in $2d$, we expect the bulk viscosity ζ to be very small both in the BCS and in the BEC regimes.

Conclusions—We have shown that the parameter γ_d controls the deviation of the breathing mode frequency ω_m from its scale-invariant value $2\omega_0$ and also quantifies the low-energy spectral weight for excitations that break scale invariance, using an exact sum rule. We argue that $2d$ is special, with a coupling that depends logarithmically on density, leading to a very small γ_2, even in the BEC regime where scale invariance is strongly broken by the large dimer binding energy (and hence $P-\varepsilon$ is large). The small γ_2 also implies, via the $2d$ sum rule, weak damping of the monopole mode in both the BCS and BEC regimes. The regime very near $g_2 = 0$ deserves further theoretical and experimental investigation, but the available evidence suggests that γ_2 might be small there as well.

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Note added in proof—While completing this manuscript we became aware of Ref. [31], which analyzes the experiment of Ref. [10] from the different perspective of quantum anomalies at $T = 0$. In the only area of substantial overlap, our general result (2) for the breathing mode reduces to that of Ref. [31] if we assume a polytropic equation of state.

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