Quench Dynamics of the Interacting Bose Gas in One Dimension

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We obtain an exact expression for the time evolution of the interacting Bose gas following a quench from a generic initial state using the Yudson representation for integrable systems. We study the time evolution of the density and noise correlation for a small number of bosons and their asymptotic behavior for any number. We show that for any value of the coupling, as long as it is repulsive, the system asymptotes towards a strongly repulsive gas, while for any value of an attractive coupling the long time behavior is dominated by the maximal bound state. This occurs independently of the initial state and can be viewed as an emerging "dynamic universality".

The interest in quantum systems out of equilibrium has been re-ignited with the advent of ultra-cooled atoms in laser traps or lithographic nano-devices, which allow finely controlled experiments to explore various aspects of non-equilibrium dynamics – quenching or non-linear transport being examples [1–4]. In parallel, advances in theoretical computational techniques now allow us to calculate the time evolution of various observables of interacting many body systems and study the underlying physics [5]. Many of the systems studied are governed, to a very close approximation, by integrable Hamiltonians [4, 6], opening the way to understanding their quantum dynamics in great detail.

We shall be concerned here with non-equilibrium *quench* dynamics – preparing the system in some initial state $|\Phi_0\rangle$ and following its evolution under the influence of a Hamiltonian H applied to it suddenly at time $t = 0$. For example, a Bose gas initially in a Mott state in a deep periodic trap, is time evolved under a superfluid Hamiltonian with the trap depth reduced, allowing the bosons to easily hop from site to site. The evolution can be formally obtained by expanding the initial state in terms of a complete set of orthonormal energy eigenstates $\{\lambda\}$,

$$|\Phi_0, t\rangle = e^{-iHt}|\Phi_0\rangle = \sum_{\{\lambda\}} e^{-iE_\lambda t}|\lambda\rangle\langle\lambda|\Phi_0\rangle. \tag{1}$$

For an exactly solvable model, these eigenstates $|\lambda\rangle$ are given by the Bethe Ansatz. This approach has been effective in obtaining the full spectrum (and thermodynamics) of a number of important models [7–9]. However, for time evolution, in addition to the spectrum, one also needs to compute the overlap of the eigenstates with the initial state and carry out the summation – difficult tasks due to the complexity of the Bethe Ansatz eigenstates. In spite of this, some progress has been made [10–12].

In this letter, we address this question generalizing a "contour integral" approach introduced by V. Yudson for the purpose of studying superradiance effects in the context of the Dicke model [13]. The contour representation does away with both the above steps, and replaces the summation over eigenvalues with an integral along contours in the complex plane, appropriately chosen to faithfully represent the initial state, capturing the relevant overlaps via residues of poles in the two particle S-matrix of the Bethe Ansatz eigenstate.

We show that the framework has general applicability and use it to understand the quench dynamics of an interacting gas of bosons, with short range attractive or repulsive interactions,

$$H = \int_x \partial b^\dagger(x)\partial b(x) + cb^\dagger(x)b(x)b^\dagger(x)b(x). \tag{2}$$

We shall obtain the time evolution of an arbitrary initial state, and calculate the evolution of the density and noise correlations.

The model (2) was solved by Lieb and Liniger [7]. Its N-particle eigenstates, labeled by the momenta $\{\lambda\}$, are:

$$|\lambda\rangle = \int_y \prod_{i<j} \mathcal{S}_y\big(Z^y_{ij}(\lambda_i - \lambda_j)\prod_j e^{i\lambda_j y_j}\big)b^\dagger(y_j)|0\rangle, \tag{3}$$

where $\mathcal{S}_y$ is a symmetrizer. The factor, $Z^y_{ij}(z) = \frac{z-ic\,\mathrm{sgn}(y_i-y_j)}{z-ic}$, incorporates the S-matrix, $S_{ij}(\lambda_i-\lambda_j) = \frac{\lambda_i-\lambda_j+ic}{\lambda_i-\lambda_j-ic}$, that describes the scattering of two bosons with momenta λ_i, λ_j. The corresponding eigenenergies are $E_\lambda = \sum_{j=1}^N \lambda_j^2$ with the momenta being real-valued for repulsion ($c > 0$), or complex conjugate pairs (signifying bound states) for attraction. One usually proceeds by imposing periodic boundary conditions to determine the values of the $\{\lambda\}$'s and subsequently studies the infinite volume limit. Instead, we shall work directly in the infinite volume limit with the momenta $\{\lambda\}$ unconstrained, allowing us to integrate over them instead of summing over discrete values.

A given state $|\Phi_0\rangle = \int_x \Phi_0(x_1\cdots x_N)\prod_{j=1}^N b^\dagger(x_j)|0\rangle$ (Φ_0 symmetric) can be directly time evolved if it is expressed in terms of the eigenstates, eq. (3). It can be shown that, (we denote $\theta(x_1 > \cdots > x_N) = \theta(\vec{x})$),

$$|\Phi_0\rangle = \int_{x,y,\lambda} \theta(\vec{x})\Phi_0(\vec{x})\big(\mathcal{S}_y\prod_j e^{i\lambda_j(y_j-x_j)} \times \prod_{i<j} Z^y_{ij}(\lambda_i-\lambda_j)\big)b^\dagger(y_j)|0\rangle \tag{4}$$

where the momenta integration contours are determined by the interaction. In the repulsive case, all $\{\lambda_j\}$ have to be integrated along the real line, while in the attractive case, the λ_j have to be integrated along lines parallel to the real axis that are separated in the imaginary direction by a little more than $|c|$. This separation captures the bound states that appear in the spectrum of this model. The proof of the statement involves studying ordered initial states, $|\vec{x}\rangle = \theta(\vec{x})\prod_j b^\dagger(x_j)|0\rangle$ which, when represented in the form above, have zero residues for the chosen integration contours. It can then be shown that each λ_j integration reduces to $\delta(y_j - x_j)$ proving the validity of (4). For details of the formalism as applied to the interacting Bose gas, please see [14].

In the rest of this article, we discuss the quenching of a system of bosons initially in a Mott state (a deep periodic trap) or in a condensate (a parabolic trap) with the trap removed at $t = 0$ and the system evolving subsequently under the action of the interacting Hamiltonian (2). We shall be interested in the effects of interaction, attractive and repulsive, on the evolution.

We first discuss the case of bosons trapped in a periodic potential $|\Phi_{\rm latt}\rangle = \prod_j \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}}\int_x e^{-\frac{(x_j+(j-1)a)^2}{2\sigma^2}} b^\dagger(x_j)|0\rangle$. If we assume that the wave functions of neighboring bosons do not overlap significantly, i.e., $e^{-\frac{a^2}{\sigma^2}} \ll 1$, then an ordering of the initial particles (required in the Yudson representation), with $x_j = (1-j)a$, is induced by the non-overlapping support. To illustrate this, we find the exact two particle finite time wave function:

$$|\Phi_{\rm latt}, t\rangle_2 = \frac{1}{4\pi i t}\int_{x,y} e^{-\frac{x_1^2}{\sigma^2}-\frac{(x_2+a)^2}{\sigma^2}} e^{i\frac{(y_1-x_1)^2}{4t}+i\frac{(y_2-x_2)^2}{4t}}$$
$$\left[1 - c\sqrt{\pi i t}\,\theta(y_2-y_1)e^{\frac{i}{8t}\alpha(t)^2}\,\mathrm{erf}\left(\frac{i-1}{4}\frac{i\alpha(t)}{\sqrt{t}}\right)\right] b^\dagger_{y_1}b^\dagger_{y_2}|0\rangle,$$

where $\alpha(t) = 2ct - i(y_1 - x_1) - i(y_2 - x_2)$. In the attractive case with the same initial conditions we find the same result with erfc (the complementary error function) replacing erf. This induces a significant change in behavior, as can be seen from the evolution of the density. Fig. 1 shows the evolution of the density measured at $x = 0$ for free, attractive and repulsive bosons. While the repulsive, and non-interacting cases are nearly identical the attractive case shows oscillations with period $T \sim \frac{1}{c^2}$, due to the competition between diffusion of the bosons and the action of the attractive potential. This can be seen in the time evolution of the wave-function for two attractive bosons, (Fig. 2). As time evolves, the initial peaks coalesce and increase in height. At longer times, this central peak "breathes" as it diffuses. The oscillations arise due to the formation of a bound state that appears as a contribution from a pole of the S-matrix [15]. These effects are measurable in time-of-flight experiments in both cases. We expect the results to be qualitatively similar for any number of particles.

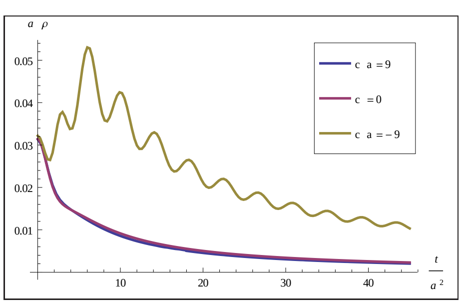

FIG. 1. $\langle\rho(x=0,t)\rangle$ vs. t, after the quench from $|\phi_{\rm latt}\rangle$. $\sigma/a \sim 0.3$

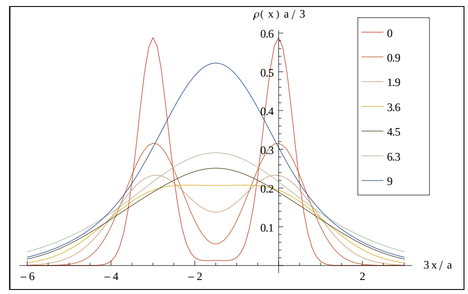

FIG. 2. $\langle\rho(x,t)\rangle$ vs. x of two attractive bosons after a quench from $|\phi_{\rm latt}\rangle$ plotted for different times. The two separate peaks coalesce and form a bound state as $t\to\infty$. The key shows the times (in units of a^2).

Exact integrations beyond two particles are difficult. However, the contour approach allows us to extract the asymptotic behavior of the evolved wavefunctions. Consider repulsive bosons and carry out the scaling $\lambda \to \lambda\sqrt{t}$. Then, $Z^y_{ij}(\lambda_i-\lambda_j) \to \mathrm{sgn}(y_i-y_j)[1-\frac{1}{c\sqrt{t}}(\lambda_i-\lambda_j)+\ldots]$, yielding to leading order,

$$|\Phi_0, t\rangle \to \int_x\int_y\int_\lambda \theta(\vec{x})\Phi_0(\vec{x})$$
$$\times\prod_j e^{-i\lambda_j^2+i\lambda_j(y_j-x_j)/\sqrt{t}}\prod_{i<j}\mathrm{sgn}(y_i-y_j)b^\dagger(y_j)|0\rangle$$
$$= e^{-iH_0^f t}\int_y \mathcal{A}_y\theta(\vec{y})\Phi_0(\vec{y})\prod_j c^\dagger(y_j)|0\rangle,$$

$c^\dagger(y)$ being fermionic creation operators replacing the "fermionized" bosonic operators, $\prod_j c^\dagger(y_j) = \prod_{i<j}\mathrm{sgn}(y_i-y_j)b^\dagger(y_j)$. $H_0^f = \int_x \partial c^\dagger(x)\partial c(x)$ is the free fermionic Hamiltonian. $\mathcal{A}_y$ is an anti-symmetrizer acting on y. Thus, the repulsive Bose gas for *any* value of $c > 0$, is governed in the long time by the $c = \infty$ hard core boson limit (or its fermionic equivalent) [16, 17], and the system equilibrates with an asymptotic momentum distribution, $n_k = \langle\tilde{\Phi}_0|c_k^\dagger c_k|\tilde{\Phi}_0\rangle$, determined by the antisymmetric wavefunction $\tilde{\Phi}_0(\vec{y}) = \mathcal{A}_y\theta(\vec{y})\Phi_0(\vec{y})$ and the total energy, $E_{\Phi_0} = \langle\Phi_0|H|\Phi_0\rangle$.

The corrections to the asymptotics, can be obtained using the stationary phase approximation to carry out the λ integrations. We have for the repulsive case,

$$|\Phi_{latt},t\rangle \to \int_y \prod_{i<j} Z_{ij}(\xi_i-\xi_j) \prod_j \frac{e^{it\xi_j^2-i\xi_j x_j}}{\sqrt{4\pi i t}} b^\dagger(y_j)|0\rangle$$

where $\xi_y = \frac{y}{2t}$ and we drop $\frac{x}{2t}$ since the initial conditions have finite spatial extent. Asymptotically, the evolution of the density is not instructive, so we compute the N-boson density-density (noise) correlation function, $\rho_2(z,z') = \langle \rho(z)\rho(z')\rangle$. At long times,

$$\rho_2(z,z';t) \to \rho_2(\xi_z,\xi_{z'}) = \frac{N^2\sigma^2}{4\pi t^2} e^{-(\xi_z^2+\xi_{z'}^2)\sigma^2}\Bigg[1+\frac{2}{N^2}\mathrm{Re}S(\xi_z-\xi_{z'})e^{ia(\xi_z-\xi_{z'})} \times \frac{N(1-e^{ia(\xi_z-\xi_{z'})}g_{zz'})+e^{iaN(\xi_z-\xi_{z'})}g_{zz'}^N-1}{(1-g_{zz'}e^{ia(\xi_z-\xi_{z'})})^2}\Bigg]$$

with $g_{zz'} = 1-2c\sqrt{\pi}\sigma S(\xi_z-\xi_{z'}-ic)\big[e^{(c+i\xi_z)^2\sigma^2}\,\mathrm{erfc}((c+i\xi_z)\sigma)+e^{(c-i\xi_{z'})^2\sigma^2}\,\mathrm{erfc}((c-i\xi_{z'})\sigma)\big]$ and $\xi_z \equiv z/2t$. This quantity was also studied by Lamacraft [18] for the repulsive model. However, our result differs from his.

An interesting way to observe the evolution of repulsive bosons to free fermions is the Hanbury-Brown Twiss (HBT) effect. Consider the normalized spatial noise correlations, $\frac{\rho_2(z,z')}{\rho(z)\rho(z')}-1 \equiv C_2(z,z')$. In the non-interacting case $S(\xi) = 1$ and $g_{zz'} = 1$ and we recover the HBT result for $N = 2$ [19], $C_2^0(\xi_z,\xi_{z'}) = \frac{1}{2}\cos(a(\xi_z-\xi_{z'}))$. At any finite c, we can see a sharp fermionic dip appear (anti-correlation) that gets broader with increasing c as shown in figure 3. For higher particle number, as shown

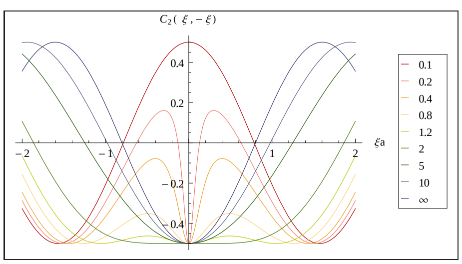

FIG. 3. Normalized noise correlation function $C_2(\xi,-\xi)$. Fermionic correlations develop on a time scale $\tau \sim c^{-2}$, so that for any c we get a sharp fermionic peak near $\xi = 0$, i.e., at large time. The key shows values of ca.

in fig. 4, we see "interference fringes" that get narrower and more numerous with an increase in number of particles. However, the asymptotic fermionic character is still visible.

The scaling argument fails for the attractive gas as the λs are integrated on parallel trajectories, and as t increases, their separation also grows. The asymptotic behavior of attractive bosons, as we shall see (and already observed earlier), is dominated by a maximally bound state. Since the contours of integration spread out in the

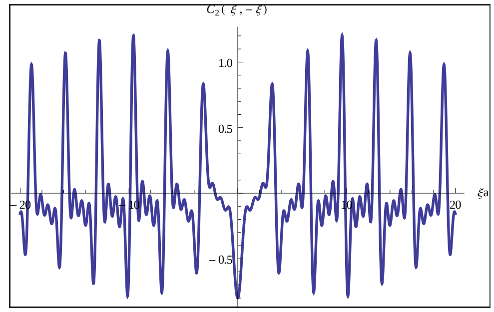

FIG. 4. Normalized noise correlation function $C_2(\xi,-\xi)$ for five repulsive bosons released from a Mott-like state for $ca = 2$.

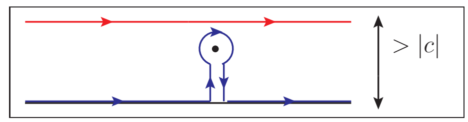

FIG. 5. Contribution from stationary phase and pole at large time in the attractive model. The blue curve represents the shifted contour.

imaginary direction, in addition to the stationary phase contributions at large time, we also need to take into account the contributions from the poles. Figure 5 shows an example of how this works. Taking into account all the pole contributions, we get a sum over several terms given by,

$$|\Phi_{\rm latt},t\rangle = \int_y \sum_{\xi_j^*=\{\xi_j,\xi_{i<j}^*+ic\}} \prod_{i<j} Z_{ij}(\xi_i^*-\xi_j^*) \times \prod_j (4\pi i t)^{-1/2} e^{-it(\xi_j^*)^2+i\xi_j^*(2t\xi_j-x_j)} b^\dagger(y_j)|0\rangle \tag{5}$$

with $Z_{ij}(-ic) \equiv c\sqrt{t}\theta(y_j > y_i)$. Note that while the asymptotic dynamics of the repulsive model is given purely in terms of the "lightcone" variables $\xi_j \equiv \frac{y_j}{2t}$, this is not the case in the attractive model where observables acquire explicit time dependence. Also, important to note is that the bound state contributions which appear when one or more of the S-matrix poles are picked up, are higher order in t as compared to the non-bound states. Asymptotically therefore, the larger bound states dominate. In fig. 6 (obtained by numerically integrating the expression for the noise correlation) we see interference fringes similar to the repulsive case with the central peak increasing positively indicating the dominance of the maximally bound state.

We now consider the evolution of the Bose gas after a quench from an initial state where all the bosons are in a ground state of a harmonic trap, $|\Phi_{\rm cond}\rangle = \int_x \prod_j \frac{1}{(\pi\sigma^2)^{\frac{1}{4}}} e^{-x_j^2/\sigma^2} b^\dagger(x_j)|0\rangle$. In order to use the Yudson representation, we have to rewrite the initial state as a sum over all different orderings of the coordinates using the symmetry of the initial wavefunction Φ_0. For

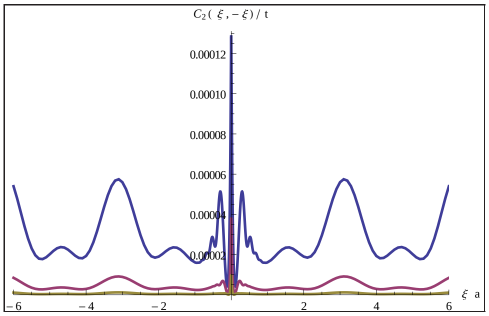

FIG. 6. $C_2(\xi, -\xi)$ for three particles in the attractive case plotted for three different times. At larger times, the correlations away from zero fall off. $ta^2 = 20, 40, 60$ for blue, magenta and yellow respectively.

the repulsive model, the stationary phase contribution is all that appears:

$$|\Phi_{\text{cond}}, t\rangle = \int_{x,y} \prod_{i<j} Z^y_{ij} \left(\frac{y_i - y_j - x_i + x_j}{2t} \right) \prod_j \frac{1}{\sqrt{2\pi i t}} e^{i\frac{(y_j - x_j)^2}{4t} - \frac{x_j^2}{\sigma^2}} b^\dagger(y_j)|0\rangle. \tag{6}$$

For the attractive case, as shown in eq. (5), the bound states dominate, with the larger bound states (more pole contributions) dominating at larger time. This holds independently of the initial conditions.

Figure 7 shows the noise correlation for two and three repulsive bosons starting from a condensate. We see the characteristic fermionic dip develop. The plots for three attractive bosons are shown in fig. 8. The oscillations arising from the interference of particles separated spatially (HBT) do not appear. The attractive case however does show the oscillations near the central peak that are visible in the case when we start from a Mott insulator.

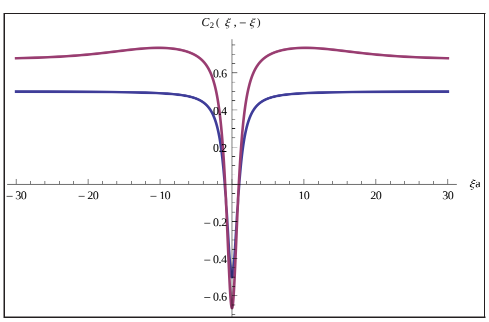

FIG. 7. $C_2(\xi, -\xi)$ for two (blue) and three repulsive bosons starting from a condensate. Unlike the attractive case, there is no explicit time dependence asymptotically. $ca = 3$

We conclude that although the details of the time evolution depend on the initial state we quench from, the asymptotics show universal features given in terms of a "dynamic RG picture" with the asymptotic dynamics

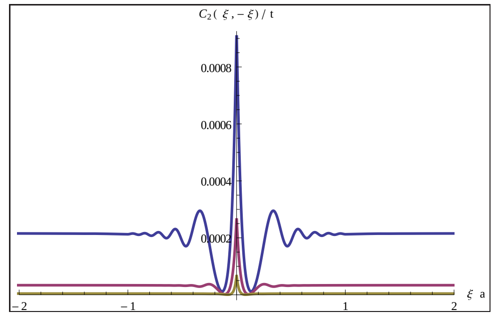

FIG. 8. $C_2(\xi, -\xi)$ for three attractive bosons starting from a condensate. Note that the side peak structure found in fig. 6 is missing due to the initial condition. We show the evolution at three times. As time increases, the oscillations near the central peak die out. Times from top to bottom $tc^2 = 20, 40, 60$

being controlled by $c = \pm\infty$ Hamiltonians for the repulsive and attractive interactions, respectively, with time playing the role of the successively reduced cut-off in an RG procedure, projecting out the effective low energy physics. This picture suggests "basins of attraction" of perturbed Hamiltonians around the Lieb-Liniger Hamiltonian (e.g., a short range potential replacing the delta function), whose long time evolutions are given by the $c = \pm\infty$ limit. We therefore expect our results, beyond their theoretical significance, to also provide experimental predictions.

In summary, we have used the contour integral representation for both the attractive and repulsive Lieb-Liniger model and shown that we can extract the asymptotic wave function and observables analytically. The representation overcomes some of the major difficulties involved in using the Bethe-Ansatz to study the dynamics of some integrable systems. The results indicate the emergence of universal dynamic features. The application to other models is under consideration.

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