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Complete Tenth-Order QED Contribution to the Muon $g - 2$

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We report the result of our calculation of the complete tenth-order QED terms of the muon $g - 2$. Our result is $a_{\mu}^{(10)} = 753.29 \pm 1.04$ in units of $(\alpha/\pi)^5$, which is about 4.5 s.d. larger than the leading-logarithmic estimate $663 (20)$. We also improved the precision of the eighth-order QED term of $a_{\mu}$, obtaining $a_{\mu}^{(8)} = 130.8794 (63)$ in units of $(\alpha/\pi)^4$. The new QED contribution is $a_\mu(QED) = 116 584 718 951 (80) \times 10^{-14}$, which does not resolve the existing discrepancy between the standard-model prediction and measurement of $a_\mu$.

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The anomalous magnetic moment \( a_\mu \) of the muon has been studied extensively both experimentally and theoretically since it provides one of the promising paths in exploring possible new physics beyond the standard model. For this purpose it is crucial to know the prediction of the standard model as precisely as possible.

On the experimental side the current world average of the measured \( a_\mu \) is \([1, 2]\):

\[
a_\mu(\text{exp}) = 116\,592\,089\, (63) \times 10^{-11} \quad [0.5 \text{ ppm}].
\]

New experiments designed to improve the precision further are being prepared at Fermilab \([3]\) and J-PARC \([4]\).

In the standard model, \( a_\mu \) can be divided into electromagnetic, hadronic, and electroweak contributions

\[
a_\mu = a_\mu(\text{QED}) + a_\mu(\text{hadronic}) + a_\mu(\text{electroweak}).
\]

At present \( a_\mu(\text{hadronic}) \) is the largest source of theoretical uncertainty. The uncertainty comes mostly from the \( O(\alpha^2) \) hadronic vacuum-polarization (v.p.) term, \( \alpha \) being the fine-structure constant. The lattice QCD simulations have attempted to evaluate this contribution \([5–10]\). At present, most accurate evaluations must rely on the experimental information. Three types of measurements are available for this purpose: (1) \( e^+e^- \rightarrow \text{hadrons} \), (2) \( \tau^\pm \rightarrow \nu + \pi^\pm + \pi^0 \), (3) \( e^+e^- \rightarrow \gamma + \text{hadrons} \). These processes have been investigated intensely by many groups \([11–13]\). We list here one of them \([13]\):

\[
a_\mu(\text{had. v.p.}) = 6949.1 \, (37.2)_{\exp} (21.0)_{\text{rad}} \times 10^{-11},
\]

which overlaps other values based on the \( e^+e^- \) data \([11, 12]\) and makes the standard-model prediction closest to the experiment (1). The next-to-leading-order (NLO) hadronic vacuum-polarization contribution is also known \([13]\):

\[
a_\mu(\text{NLO had. v.p.}) = -98.4 \, (0.6)_{\exp} (0.4)_{\text{rad}} \times 10^{-11}.
\]

The hadronic light-by-light scattering contribution \( (l-l) \) is of similar size as \( a_\mu(\text{NLO had. v.p.}) \), but has a much larger theoretical uncertainty \([14–17]\)

\[
a_\mu(\text{had. } l-l) = 116 \, (40) \times 10^{-11},
\]

where the uncertainty \( 40 \times 10^{-11} \) covers almost all values obtained in different publications.

The electroweak contribution has been calculated up to 2-loop order \([18–21]\):

\[
a_\mu(\text{weak}) = 154 \, (2) \times 10^{-11}.
\]

Since this uncertainty is 30 times smaller than the experimental precision of (1), it can be regarded as known precisely.

The primary purpose of this letter is to report the complete numerical evaluation of all tenth-order QED contribution to \( a_\mu \). It leads to a sizable reduction of the uncertainty of the previous estimate by the leading-log approximations \([22, 23]\). We have also improved the numerical precision of the eighth-order QED contribution including the newly evaluated tau-lepton contribution. Together they represent a significant reduction in the theoretical uncertainty of the QED part of \( a_\mu \).

The QED contribution to \( a_\mu \) can be evaluated by the perturbative expansion in \( \alpha/\pi \):

\[
a_\mu(\text{QED}) = \sum_{n=1}^{\infty} \left( \frac{\alpha}{\pi} \right)^n a_\mu^{(2n)},
\]

where \( a_\mu^{(2n)} \) is finite thanks to the renormalizability of QED and can be written as

\[
a_\mu^{(2n)} = A_1^{(2n)} + A_2^{(2n)}(m_\mu/m_e) + A_3^{(2n)}(m_\mu/m_\tau) + A_4^{(2n)}(m_\mu/m_e, m_\mu/m_\tau) + A_5^{(2n)}(m_\mu/m_e, m_\mu/m_\tau).
\]

\( A_1^{(2n)} \) is independent of mass and universal for all leptons. \( A_1^{(2)} \), \( A_1^{(4)} \) and \( A_1^{(6)} \) are known exactly \([24–27]\). Mass dependence is known analytically for \( A_2^{(2n)} \) and \( A_3^{(2n)} \) for \( n = 2, 3 \) \([28–32]\). We reevaluated them using the latest values of the muon-electron mass ratio \( m_\mu/m_e = 206.768\, 2843 \) \((52)\) and/or the muon-tau mass ratio \( m_\mu/m_\tau = \)

\[
\text{(to be continued)}.
\]
FIG. 1. Vertex diagrams representing 13 gauge-invariant subsets contributing to the lepton $g - 2$ at the eighth-order. Solid and wavy lines represent lepton and photon lines, respectively.

5.946 49 (54) x $10^{-2}$ [33]. In the same order of terms as shown on the right-hand-side of (8), the results are summarized as follows:

$$a^{(2)}_\mu = 0.5,$$

$$a^{(4)}_\mu = -0.328 478 965 579 \ldots + 0.094 258 312 0 (83)$$
$$+ 0.780 79 (15) \times 10^{-4}$$
$$= 0.765 857 425 (17),$$

$$a^{(6)}_\mu = 1.181 241 456 \ldots + 22.868 380 04 (23)$$
$$+ 0.360 70 (13) \times 10^{-3} + 0.527 76 (11) \times 10^{-3}$$
$$= 24.050 509 96 (32).$$

(9)

The value of $a^{(8)}_\mu$ has been obtained mostly by numerical integration [34–36]. They arise from 13 gauge-invariant sets whose representative diagrams are shown in Fig. 1. We have reevaluated some of them for further check and improvement of numerical precision. The results for the mass-dependent terms are summarized in Table I.

From the data listed in Table I and the value of $A^{(8)}_1$ from Refs. [35–37], we obtain the following value for the eighth-order QED contribution $a^{(8)}_\mu$:

$$a^{(8)}_\mu = -1.9106 (20) + 132.685 2 (60)$$
$$+ 0.042 34 (12) + 0.062 72 (4)$$
$$= 130.879 6 (63).$$

(10)

<table>
<thead>
<tr>
<th>group</th>
<th>$A^{(8)}<em>1(m</em>\mu/m_e)$</th>
<th>$A^{(8)}<em>2(m</em>\mu/m_e)$</th>
<th>$A^{(8)}<em>3(m</em>\mu/m_e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I(a)</td>
<td>7.74547 (42)</td>
<td>0.000032 (0)</td>
<td>0.002209 (0)</td>
</tr>
<tr>
<td>I(b)</td>
<td>7.58201 (71)</td>
<td>0.000252 (0)</td>
<td>0.002611 (0)</td>
</tr>
<tr>
<td>I(c)</td>
<td>1.624307 (40)</td>
<td>0.000737 (0)</td>
<td>0.001807 (0)</td>
</tr>
<tr>
<td>I(d)</td>
<td>-0.22982 (37)</td>
<td>0.000368 (0)</td>
<td>0</td>
</tr>
<tr>
<td>II(a)</td>
<td>-2.77888 (38)</td>
<td>-0.007329 (1)</td>
<td>0</td>
</tr>
<tr>
<td>II(b)</td>
<td>-4.55277 (30)</td>
<td>-0.002036 (0)</td>
<td>-0.000908 (1)</td>
</tr>
<tr>
<td>II(c)</td>
<td>-9.34180 (83)</td>
<td>-0.005246 (1)</td>
<td>-0.019642 (2)</td>
</tr>
<tr>
<td>III</td>
<td>10.7934 (27)</td>
<td>0.04504 (14)</td>
<td>0</td>
</tr>
<tr>
<td>IV(a)</td>
<td>123.78551 (44)</td>
<td>0.038513 (11)</td>
<td>0.083739 (36)</td>
</tr>
<tr>
<td>IV(b)</td>
<td>-0.4170 (37)</td>
<td>0.006106 (31)</td>
<td>0</td>
</tr>
<tr>
<td>IV(c)</td>
<td>2.9072 (44)</td>
<td>-0.01823 (11)</td>
<td>0</td>
</tr>
<tr>
<td>IV(d)</td>
<td>-4.43243 (58)</td>
<td>-0.015868 (37)</td>
<td>0</td>
</tr>
</tbody>
</table>
FIG. 2. Self-energy-like diagrams representing 32 gauge-invariant subsets contributing to the lepton $g - 2$ at the tenth order. Solid lines represent lepton lines propagating in a weak magnetic field.

Over the period of more than nine years we have numerically evaluated all 32 gauge-invariant sets of diagrams that contribute to $a_{\mu}^{(10)}$ [22, 37–40], whose representative diagrams are shown in Fig. 2. The results for mass-dependent terms are summarized in Table II. Some simple diagrams were analytically evaluated [41–44]. The results are consistent with our numerical ones.

From the data listed in this Table and the value of $A_{1}^{(10)}$ from Ref. [37], we obtain the complete tenth-order result:

$$a_{\mu}^{(10)} = 9.168 (571) + 742.18 (87) - 0.068 (5) + 2.011 (10) = 753.29 (1.04).$$

(11)

The uncertainty 1.04 is attributed entirely to the statistical fluctuation in the Monte-Carlo integration of Feynman amplitudes by VEGAS [45]. This is 20 times more precise than the previous estimate, 663 (20), obtained in the leading-logarithmic approximation [22]. This is mainly because we had underestimated the magnitude of the contribution of the Set III(a). Note also that (11) is about 4.5 s.d. larger than the leading-log estimate. The numerical values of $(\alpha/\pi)^{n}a_{\mu}^{(2n)}$ for $n = 1, 2, \cdots, 5$ are summarized in Table III.

In order to evaluate $a_{\mu}(\text{QED})$ using (7), a precise value of $\alpha$ is needed. At present, the best non-QED $\alpha$ is the one obtained from the measurement of $h/m_{\text{Rb}}$ [46], combined with the very precisely known Rydberg constant and $m_{\text{Rb}}/m_{e}$ [33]:

$$\alpha^{-1}(\text{Rb}) = 137.035 \, 999 \, 049 \, (90) \quad [0.66 \, \text{ppb}].$$

(12)

Actually, we have a more precise value of $\alpha$ which is derived from the measurement [47, 48] and theory of the electron
TABLE II. Tenth-order mass-dependent contribution to the muon $g - 2$ from 31 gauge-invariant subsets shown in Fig. 2. The mass-dependence of $A_3^{(10)}$ is $A_3^{(10)}(m_e/m_e, m_e/m_e)$.

<table>
<thead>
<tr>
<th>set</th>
<th>$A_3^{(10)}(m_e/m_e)$</th>
<th>$A_3^{(10)}(m_e/m_e)$</th>
<th>$A_3^{(10)}(m_e/m_e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I(a)</td>
<td>22.166 973 (3)</td>
<td>0.060 038 (0)</td>
<td>0.017 312 (1)</td>
</tr>
<tr>
<td>I(b)</td>
<td>30.667 091 (3)</td>
<td>0.000 269 (0)</td>
<td>0.020 179 (1)</td>
</tr>
<tr>
<td>I(c)</td>
<td>54.141 395 (1)</td>
<td>0.000 397 (0)</td>
<td>0.002 330 (0)</td>
</tr>
<tr>
<td>I(d)</td>
<td>8.8921 (11)</td>
<td>0.000 380 (0)</td>
<td>0.024 487 (2)</td>
</tr>
<tr>
<td>I(e)</td>
<td>25.0312 (24)</td>
<td>0.000 232 (0)</td>
<td>0.002 370 (0)</td>
</tr>
<tr>
<td>I(f)</td>
<td>3.685 049 (90)</td>
<td>0.002 162 (0)</td>
<td>0.023 390 (2)</td>
</tr>
<tr>
<td>I(g)</td>
<td>2.607 87 (7)</td>
<td>0.001 698 (0)</td>
<td>0.002 729 (1)</td>
</tr>
<tr>
<td>I(h)</td>
<td>-0.5768 (11)</td>
<td>0.000 163 (1)</td>
<td>0.001 976 (3)</td>
</tr>
<tr>
<td>I(i)</td>
<td>0.0871 (59)</td>
<td>0.000 024 (0)</td>
<td>0</td>
</tr>
<tr>
<td>I(j)</td>
<td>-1.2617 632 (14)</td>
<td>0.000 168 (1)</td>
<td>0.000 110 (5)</td>
</tr>
<tr>
<td>I(k)</td>
<td>-70.4717 (38)</td>
<td>-0.018 882 (8)</td>
<td>-0.290 853 (85)</td>
</tr>
<tr>
<td>I(l)</td>
<td>-34.7715 (26)</td>
<td>-0.035 615 (20)</td>
<td>-0.127 369 (60)</td>
</tr>
<tr>
<td>I(m)</td>
<td>-5.385 75 (99)</td>
<td>-0.016 348 (14)</td>
<td>-0.040 800 (51)</td>
</tr>
<tr>
<td>I(n)</td>
<td>0.4972 (65)</td>
<td>-0.007 673 (14)</td>
<td>0</td>
</tr>
<tr>
<td>I(o)</td>
<td>3.265 (12)</td>
<td>-0.038 066 (13)</td>
<td>0</td>
</tr>
<tr>
<td>I(p)</td>
<td>-77.465 (12)</td>
<td>-0.207 23 (73)</td>
<td>-0.502 95 (68)</td>
</tr>
<tr>
<td>I(q)</td>
<td>0.8193 (33)</td>
<td>0.283 000 (32)</td>
<td>0.891 40 (44)</td>
</tr>
<tr>
<td>I(r)</td>
<td>11.9367 (45)</td>
<td>0.143 600 (10)</td>
<td>0</td>
</tr>
<tr>
<td>I(s)</td>
<td>7.37 (15)</td>
<td>0.1999 (28)</td>
<td>0</td>
</tr>
<tr>
<td>IV(a)</td>
<td>-38.79 (17)</td>
<td>-0.4357 (25)</td>
<td>0</td>
</tr>
<tr>
<td>IV(b)</td>
<td>629.141 (12)</td>
<td>0.246 10 (18)</td>
<td>2.3590 (18)</td>
</tr>
<tr>
<td>IV(c)</td>
<td>181.1285 (51)</td>
<td>0.096 522 (93)</td>
<td>0.194 76 (26)</td>
</tr>
<tr>
<td>IV(d)</td>
<td>-36.58 (12)</td>
<td>-0.2001 (28)</td>
<td>-0.5018 (89)</td>
</tr>
<tr>
<td>IV(e)</td>
<td>-7.92 (60)</td>
<td>0.0818 (17)</td>
<td>0</td>
</tr>
<tr>
<td>IV(f)</td>
<td>-4.32 (14)</td>
<td>-0.035 94 (32)</td>
<td>-0.1122 (24)</td>
</tr>
<tr>
<td>IV(g)</td>
<td>-38.16 (15)</td>
<td>0.034 477 (85)</td>
<td>0.0659 (31)</td>
</tr>
<tr>
<td>IV(h)</td>
<td>6.96 (48)</td>
<td>-0.044 51 (96)</td>
<td>0</td>
</tr>
<tr>
<td>IV(i)</td>
<td>-8.55 (23)</td>
<td>0.084 65 (46)</td>
<td>0</td>
</tr>
<tr>
<td>IV(j)</td>
<td>-27.34 (12)</td>
<td>-0.003 45 (33)</td>
<td>-0.0027 (11)</td>
</tr>
<tr>
<td>IV(k)</td>
<td>-25.505 (20)</td>
<td>-0.011 49 (33)</td>
<td>-0.016 03 (58)</td>
</tr>
<tr>
<td>IV(l)</td>
<td>97.123 (62)</td>
<td>0.092 17 (16)</td>
<td>0</td>
</tr>
</tbody>
</table>

$g - 2$ [37]:

$$\alpha^{-1}(a_e) = 137.035\;999\;1736 \times 10^{-11},$$

where the first three uncertainties are due to the eighth-order term, tenth-order term, and the hadronic and electroweak terms, involved in the evaluation of $a_e$. The fourth uncertainty comes from the measurement of $a_e$. At present the difference between (12) and (13) is much smaller than the current uncertainty in the measurement of $a_e$ so that one may use either one of these two. However, some caution must be exercised to employ $\alpha^{-1}(a_e)$ to calculate $a_e$, when more accurate experiment of $a_e$ becomes available, because theoretical calculation of $a_e$ is strongly correlated with that of $a_e$.

Substituting (9), (10), and (11) in Eq. (7) and using (12), we obtain

$$a_e(QED, Rb) = 116\;584\;718\;951\;9(19)/(7)(77) \times 10^{-14},$$

where the uncertainties are from the lepton mass ratios, the eighth-order term, the tenth-order term, and the value of $\alpha$ in (12), respectively. If we use the value of $\alpha$ in (13) instead, we get

$$a_e(QED, a_e) = 116\;584\;718\;845\;9(19)/(7)(30) \times 10^{-14}.$$

TABLE III. Contributions to muon $g - 2$ from QED perturbation term $a_e^{(2n)}(\alpha/\pi)^n \times 10^{11}$. They are evaluated with two values of the fine-structure constant determined by the Rb experiment and by the electron $g - 2$ ($a_e$).

<table>
<thead>
<tr>
<th>order</th>
<th>with $\alpha^{-1}(Rb)$</th>
<th>with $\alpha^{-1}(a_e)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>116 140 973 318 (77)</td>
<td>116 140 973 212 (30)</td>
</tr>
<tr>
<td>4</td>
<td>413 217 6291 (90)</td>
<td>413 217 6284 (89)</td>
</tr>
<tr>
<td>6</td>
<td>30 141 902 48 (41)</td>
<td>30 141 902 39 (40)</td>
</tr>
<tr>
<td>8</td>
<td>381 008 (19)</td>
<td>381 008 (19)</td>
</tr>
<tr>
<td>10</td>
<td>5.0938 (70)</td>
<td>5.0938 (70)</td>
</tr>
<tr>
<td>$a_e(QED) \times 10^{11}$</td>
<td>116 584 718 951 (80)</td>
<td>116 584 718 845 (37)</td>
</tr>
</tbody>
</table>
Note that the uncertainties of the lepton mass ratios, the eighth-order term, the tenth-order terms, and $\alpha(a_e)$ are improved by factors 1.7, 1.3, 20, and 1.5, respectively, compared with $a_\mu$(QED, $a_e$) given in Eq. (99) of Ref. [49].

The difference between (14) and (15) is less than $1.2 \times 10^{-12}$ so that we may use either one as far as comparison with the current experimental data is concerned.

In view of the rather large value of $A_2^{(10)}(m_\mu/m_\mu)$ one might wonder how large $A_2^{(12)}(m_\mu/m_\mu)$ might be. As a matter of fact it is not difficult to estimate its size. For this purpose note that the dominant contribution to $A_2^{(12)}(m_\mu/m_\mu)$ comes from the Group IV(a) and the dominant contribution to $A_2^{(10)}(m_\mu/m_\mu)$ comes from the Set VI(a). Both are integrals obtained by inserting several second-order vacuum-polarization loops $\Pi_2$ into the virtual photon lines of the sixth-order diagram $A_2^{(6)}(m_\mu/m_\mu; l-l)$ which contains a light-by-light scattering electron loop. Analogously the leading contribution to the twelfth-order term will come from insertion of three $\Pi_2$’s in $A_2^{(6)}(m_\mu/m_\mu; l-l)$, namely,

$$
A_2^{(12)}(m_\mu/m_\mu) \sim A_2^{(6)}(m_\mu/m_\mu; l-l)
$$

$$
\times \left( \frac{2}{3} \ln \left( \frac{m_\mu}{m_\mu} \right) - \frac{5}{9} \right)^3 \times 10
$$

and

$$
A_2^{(12)}(m_\mu/m_\mu) \times \left( \frac{\alpha}{\pi} \right)^6 \sim 0.8 \times 10^{-12},
$$

noting that $A_2^{(6)}(m_\mu/m_\mu; l-l) \sim 20$ and the factor 10 accounts for the possible ways of insertion of $\Pi_2$. Including the contribution of other diagrams, the size of the 12th-order term might be as large as $10^{-12}$. This is larger than the uncertainty of the 10th-order term in (14) so that it would be desirable to obtain at least a crude evaluation of this term.

Adding (3), (4), (5), (6), and (14), and using $\alpha$ from (12), the theoretical value of $a_\mu$ in the standard model is given by

$$
a_\mu(\text{SM}) = 116 \ 591 \ 840 \ (59) \times 10^{-11}.
$$

We have therefore

$$
a_\mu(\text{exp}) - a_\mu(\text{SM}) = 249 \ (87) \times 10^{-11}.
$$

The size of discrepancy between theory and experiment has not changed much, since the tenth-order QED contribution is not a significant source of theoretical uncertainties. Let us emphasize, however, that the complete calculation of $a_\mu^{(10)}$ enables us to concentrate on improving the precision of the hadronic contributions.

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