Interacting One-Dimensional Fermionic Symmetry-Protected Topological Phases

Evelyn Tang and Xiao-Gang Wen

Phys. Rev. Lett. **109**, 096403 — Published 27 August 2012
DOI: 10.1103/PhysRevLett.109.096403

Interacting 1D fermionic symmetry protected topological phases

Evelyn Tang[1] and Xiao-Gang Wen[1, 2]

[1] *Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139*

[2] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2L 2Y5 Canada*

(Dated: Mar, 2012)

In free fermion systems with given symmetry and dimension, the possible topological phases are labeled by elements of only three types of Abelian groups, 0, $\mathbb{Z}_2$, or $\mathbb{Z}$. For example non-interacting 1D fermionic superconducting phases with S_z spin rotation and time-reversal symmetries are classified by $\mathbb{Z}$. We show that with weak interactions, this classification reduces to $\mathbb{Z}_4$. Using group cohomology, one can additionally show that there are only four distinct phases for such 1D superconductors even with strong interactions. Comparing their projective representations, we find all these four symmetry protected topological phases can be realized with free fermions. Further, we show that 1D fermionic superconducting phases with Z_n discrete S_z spin rotation and time-reversal symmetries are classified by $\mathbb{Z}_4$ when n = even and $\mathbb{Z}_2$ when n = odd; again, all these strongly interacting topological phases can be realized by non-interacting fermions. Our approach can be applied to systems with other symmetries to see which 1D topological phases can be realized by free fermions.

Symmetry protected topological (SPT) phases[1, 2] are short range entangled states with symmetry protected gapless edge excitations.[3–8] The Haldane phase on a spin-1 chain[9, 10] and 2D/3D topological insulators[11–16] are examples of SPT states. Using K-theory or topological terms, all free-fermion SPT phases can be classified[17, 18] for all 10 Altland-Zirnbauer symmetry classes[19] of *single-body* Hamiltonians. It turns out that different free fermion SPT phases are described by only three types of Abelian groups, 0, $\mathbb{Z}_2$, or $\mathbb{Z}$.

With interactions the classification is more varied, however we must first describe the symmetry differently. Instead of specifying the symmetry of single-body Hamiltonians, we treat the free fermion systems as many-body systems and specify the *many-body* symmetry of their *many-body* Hamiltonians. Only in this case can we accurately add interaction terms to the many-body Hamiltonian that preserve the many-body symmetry, and study their effect on the SPT phases of free fermions. A classification of various free-fermion gapped phases given their many-body symmetry can be found in Ref. 20.

Fidkowski and Kitaev (also Turner, Pollmann and Berg) studied interaction effects in one case: In their 1D time-reversal (TR) invariant topological superconductor[21–23] the $\mathbb{Z}$ classification in the free case breaks down to $\mathbb{Z}_8$ with interactions that preserve TR symmetry. Here we present another model beginning with a lattice Hamiltonian for a 1D superconductor with both TR and S_z spin-rotation symmetries, described by the $\mathbb{Z}$ classification in the free case. With the addition of weak interactions that preserve these symmetries, the classification reduces to $\mathbb{Z}_4$ (see Table I). Our interaction results are obtained by assuming that *edge degeneracy fully distinguishes each gapped phase*; e.g. all states without edge degeneracy belong to the same trivial phase.

We compare these four fermionic phases to the four phases predicted separately from group cohomology[4, 5, 24] (a method valid for strong interactions). We find each fermionic phase has a distinct projective representation[3, 4] and since group cohomology also gives rise to four and only four distinct phases,[7] we conclude that free fermions can realize all strongly interacting SPT phases in this case. We further study interaction effects on a 1D superconductor with Z_n discrete S_z spin rotation and TR symmetries. For this symmetry group, we find the SPT phases are classified by $\mathbb{Z}_4$ when n = even and $\mathbb{Z}_2$ when n = odd. Again, these results are separately obtained both from perturbing our fermionic lattice Hamiltonian and from the group cohomology classification for strong interactions – showing that again, all strongly interacting topological phases can be realized by non-interacting fermions.

Symmetry	Free classification	With interactions
$U(1) \times Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_4$
$Z_n \times Z_2^T$ (n even)	$\mathbb{Z}$	$\mathbb{Z}_4$
$Z_n \times Z_2^T$ (n odd and $n > 1$)	$\mathbb{Z}$	$\mathbb{Z}_2$

TABLE I. Symmetry groups described by the 1D Hamiltonian in Eq. 1 (where Z_2^T is time reversal), with their free fermion classification and how they reduce with interactions. The latter remains true with strong interactions so all such phases can be realized with free fermions.

Free fermion lattice model — We write a 1D Hamiltonian with a trivial and two non-trivial phases

$$H = -t \sum_{\langle ij \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} - 2\Delta_s \sum_j c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger + \text{h.c.} \pm i\Delta_p/2 \sum_j c_{j+1\uparrow}^\dagger c_{j\downarrow}^\dagger + c_{j+1\downarrow} c_{j\uparrow} + \text{h.c.} \qquad (1)$$

where the first term is typical nearest-neighbor hopping, the second term Δ_s represents on-site pairing and the last term with Δ_p pairs electrons on adjacent sites.

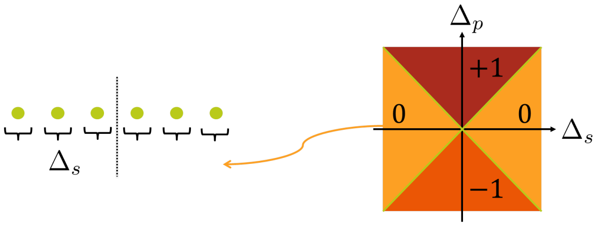

FIG. 1. (Color online). Phase diagram when varying the parameters Δ_s and Δ_p: The phase boundaries are $\Delta_s = \pm\Delta_p$ which separate three phases denoted by $N = +1, 0$ and -1. We can make Δ_s arbitrarily large without closing the gap: this limit describes on-site pairing where any cut cleanly separates the system in two without leaving edge states – allowing identification of this trivial $N = 0$ phase.

This Hamiltonian satisfies time-reversal T and S_z spin-rotation symmetries specified on $c_{i\sigma}^T = \{\hat{c}_{i\uparrow}, \hat{c}_{i\downarrow}\}$ as

$$\hat{T} c_{i\sigma} \hat{T}^{-1} = i\sigma_y c_{i\sigma}; \quad e^{i\theta \hat{S}_z} c_{i\sigma} e^{-i\theta \hat{S}_z} = \begin{pmatrix} e^{-i\theta/2} & \\ & e^{i\theta/2} \end{pmatrix} c_{i\sigma}$$

so that $\hat{T} H \hat{T}^{-1} = H$ and $e^{i\theta \hat{S}_z} H e^{-i\theta \hat{S}_z} = H$. As the bandgap closes to leave just the hopping component when $\Delta_s = \pm\Delta_p$, we obtain the phase diagram in Fig. 1.

We start by identifying the trivial phase $N = 0$: When $|\Delta_s| > |\Delta_p|$, we can arbitrarily increase the strength of Δ_s without closing the gap. In the limit of Δ_s much larger than the other terms, the Hamiltonian simply reduces to on-site pairing — any cut cleanly separates the system in two parts leaving no boundary states (see Fig. 1): this is the trivial $N = 0$ phase. Next, we look for ground-state degeneracy at the interface between this phase and its neighbors. This is most conveniently done in a low-energy continuum model, where the effective Hamiltonian becomes

$$H = -i \int dx \tilde{\Psi}^\dagger \left[(\sigma_z \otimes \mathbb{I}) \partial_x + \begin{pmatrix} & m \\ -m^T & \end{pmatrix} \right] \tilde{\Psi}$$

in a basis of right and left-moving fermion operators $\tilde{\Psi}^T = \left(\psi_{R\uparrow}, i\psi_{R\downarrow}^\dagger, i\psi_{L\downarrow}^\dagger, \psi_{L\uparrow}\right)$ close to the Fermi surface. Here $m = \Delta_p \mathbb{I} - \Delta_s \sigma_z$.

Smoothly varying our mass term $m(x)$ across an interface, we set $\Delta_p(x) = \frac{1}{2}(1 + \tanh x)$ and $\Delta_s(x) = \frac{1}{2}(1 - \tanh x)$. This has the zero-energy solution

$$\hat{\psi}_{0+} = \int dx \mathrm{sech}(x) (\hat{\psi}_{R\uparrow} + i\hat{\psi}_{L\downarrow}^\dagger) \tag{2}$$

This complex fermion operator ($\hat{\psi}_{0+} \neq \hat{\psi}_{0+}^\dagger$) with energy $E = 0$ contains a double degeneracy (empty or filled) that allows labelling of $\Delta_p > |\Delta_s|$ as the non-trivial $N = 1$ phase. This mode transforms under symmetry as

$$\hat{T} \begin{pmatrix} \hat{\psi}_{0+} \\ \hat{\psi}_{0+}^\dagger \end{pmatrix} \hat{T}^{-1} = -\sigma_y \begin{pmatrix} \hat{\psi}_{0+} \\ \hat{\psi}_{0+}^\dagger \end{pmatrix},$$
$$e^{i\theta \hat{S}_z} \hat{\psi}_{0+} e^{-i\theta \hat{S}_z} = e^{-i\theta/2} \hat{\psi}_{0+} \tag{3}$$

Since the two degenerate states differ by $S_z = 1/2$ and are related by time-reversal, each state carries quantum number of $S_z = \pm 1/4$ respectively.

Using the symmetry relations in Eq. 3, we check if any perturbations in the Hamiltonian can shift the energy of this mode. We find density terms $\delta H = c\hat{\psi}_{0+}^\dagger \hat{\psi}_{0+}$ are forbidden by TR, hence our ground-state degeneracy is protected by system symmetries — this $N = 1$ phase is stable against perturbations.

To find the $N = -1$ phase, we change $\Delta_p(x) \to -\Delta_p(x)$ and upon repeating our procedure, find a different zero mode solution that we label

$$\hat{\psi}_{0-} = \int dx \mathrm{sech}(x) (i\hat{\psi}_{R\downarrow}^\dagger - \hat{\psi}_{L\uparrow}) \tag{4}$$

and instead transforms as

$$\hat{T} \begin{pmatrix} \hat{\psi}_{0-} \\ \hat{\psi}_{0-}^\dagger \end{pmatrix} \hat{T}^{-1} = \sigma_y \begin{pmatrix} \hat{\psi}_{0-} \\ \hat{\psi}_{0-}^\dagger \end{pmatrix},$$
$$e^{i\theta \hat{S}_z} \hat{\psi}_{0-} e^{-i\theta \hat{S}_z} = e^{-i\theta/2} \hat{\psi}_{0-} \tag{5}$$

This state has stable ground-state degeneracy as $\delta H = c\hat{\psi}_{0-}^\dagger \hat{\psi}_{0-}$ is also forbidden by TR, indicating $\Delta_p < |\Delta_s|$ is a non-trivial phase as well.

Is it meaningful to label this second non-trivial phase $N = -1$? We examine what happens upon stacking two chains both with non-trivial phases but the first with $\Delta_p > |\Delta_s|$ and the second with $\Delta_p < |\Delta_s|$. (The first chain would have the zero mode $\hat{\psi}_{0+}$ and the second $\hat{\psi}_{0-}$.) We find the coupling $\delta H = c\hat{\psi}_{0+}^\dagger \hat{\psi}_{0-} + \text{h.c.}$ is allowed within system symmetries and makes the ground state nondegenerate. So two chains with two distinct zero modes (labelled + and −) combine to become trivial, indicating the two phases *should* be labelled with opposite index. Naturally then the phase with $\hat{\psi}_{0-}$ would be the $N = -1$ phase, so this model indeed gives three symmetry protected phases $N = -1, 0$ and $+1$.

While two chains containing zero modes with opposite index become trivial, we further consider the stability of two chains containing zero modes with the same positive (or negative) index. This may generalize to larger integers in the $\mathbb{Z}$ group, so now we examine the stacking of two chains with similar index more systematically.

A generic coupling term (see Fig. 2) is $\delta H = c\hat{\psi}_{+a}^\dagger M_{ab} \hat{\psi}_{+b} + \text{h.c.}$. Here a and b are indices running over the chain number 1,2... e.g. $\hat{\psi}_{+1}$ denotes a zero mode from the $N = 1$ phase in the first chain; and M_{ab} is any generic coupling between these two operators. We examine the simplest case of $a = 1$ and $b = 2$. Terms

FIG. 2. We stack two chains in the same non-trivial phase with positive (or negative) index to see if their edge states are stable. With the first chain $a = 1$ and the second $b = 2$, M_{ab} is any coupling between them. We find all possible couplings are forbidden by our system symmetries so two similar modes are stable and form the $N = 2$ phase.

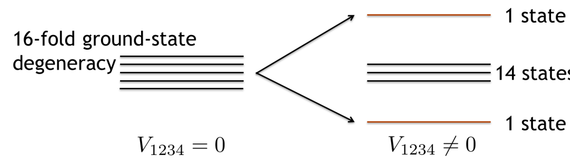

FIG. 3. Without interactions $V_{1234} = 0$ the ground-state degeneracy for four zero modes is $2^4 = 16$. With interactions $V_{1234} \neq 0$ two states are split by $\delta E = \pm|V_{1234}|$, making the ground state nondegenerate and this $N = 4$ phase trivial. As $N = 4$ is now smoothly connected to the trivial $N = 0$ phase, our classification reduces from $\mathbb{Z}$ to $\mathbb{Z}_4$ with interactions.

of the form $\delta H = c\hat{\psi}^\dagger_{+1} M_{12} \hat{\psi}_{+2} + \text{h.c.}$ are forbidden by TR symmetry as specified in Eq. 3, while fermion pairing terms such as $\delta H = c\hat{\psi}^\dagger_{+1} M_{12} \hat{\psi}^\dagger_{+2} + \text{h.c.}$ violate S_z spin rotation symmetry. As there are no other quadratic fermion terms, the stacking of two chains is stable against perturbations and combine to give an $N = 2$ phase.

Hence adding a number of 1D chains with positive index gives a positive integer in the $\mathbb{Z}$ group. The negative numbers are obtained simply by stacking chains with $\hat{\psi}_{0-}$. As we showed earlier that a pair of $\hat{\psi}_{0+}$ and $\hat{\psi}_{0-}$ coupled together become trivial, the integer N in our $\mathbb{Z}$ group is the difference between all positive and negative zero modes. Then each phase labelled by N has $2^{|N|}$ degenerate ground-states.

Interaction effects — Now we allow couplings with an arbitrary number of fermion operators. We look at terms with four and two operators which take the general form

$$\delta H = V_{abcd} \hat{\psi}^\dagger_{+a} \hat{\psi}_{+b} \hat{\psi}^\dagger_{+c} \hat{\psi}_{+d} + W_{ab} \hat{\psi}^\dagger_{+a} \hat{\psi}_{+b} + \text{h.c.} \quad (6)$$

δH is compatible with both TR and S_z spin-rotation symmetry when V_{abcd} and W_{ab} satisfy certain conditions.

A possible term couples four separate chains through an interaction with only $V_{1234} \neq 0$. This δH is invariant under both TR and S_z spin-rotation symmetry and couples two states $|0101\rangle$ and $|1010\rangle$ in our four-mode basis (0 and 1 denote unoccupied and occupied respectively for each of the four chains). Without interactions we have a ground-state degeneracy of $2^4 = 16$; with interactions two of these 16 states split in energy by $\delta E = \pm|V_{1234}|$, see Fig. 3. This makes the ground-state nondegenerate and the phase $N = 4$ trivial.

Since four chains with all positive (or negative) index are equivalent to the trivial phase, we can smoothly connect the $N = 3$ phase to the $N = -1$ phase by adding four chains with all negative index. So with only three distinct non-trivial phases, the $\mathbb{Z}$ integer classification for free fermions reduces to $\mathbb{Z}_4$ in the presence of interactions.

Four-fermion interaction terms also reduce the ground-state degeneracy in the $N = 2$ phase from $2^2 = 4$ to a two-fold degeneracy. The term

$$\delta H = V_{1122}(\hat{\psi}^\dagger_{+1}\hat{\psi}_{+1} - \hat{\psi}_{+1}\hat{\psi}^\dagger_{+1})(\hat{\psi}^\dagger_{+2}\hat{\psi}_{+2} - \hat{\psi}_{+2}\hat{\psi}^\dagger_{+2})$$

causes two states $|00\rangle$ and $|11\rangle$ to shift in energy by V_{1122} while two other states $|01\rangle$ and $|10\rangle$ shift by $-V_{1122}$. As we still have doubled ground-state degeneracy, the state $N = 2$ remains non-trivial. To summarize, interaction effects reduce our degeneracy leaving three non-trivial phases each with a two-fold ground state degeneracy.

Four distinct projective representations — Our results demonstrate the stability of free fermion phases with weak interactions. This method may not capture all possible interacting phases as strongly interacting topological phases may not adiabatically connect to free fermion phases. Or, different phases from weak interactions may become the same phase with strong interactions. To address these issues, we illustrate a distinct projective representation[25] for each phase which corresponds to a different 1D SPT phase.

Using the symmetry operations defined in Eqs. 3 and 5, we write their matrix representation on the degenerate subspace. In the $N = 1$ phase with basis $|0_+\rangle$ and $|1_+\rangle$

$$U_\theta \to M(U_\theta) = \begin{pmatrix} 1 & \\ & e^{-i\theta/2} \end{pmatrix}, \quad \tilde{T} \to M(\tilde{T})K = \sigma_x K$$

where $\tilde{T} = U_{-\pi}T$, a rotated TR operator we can introduce since U_θ and T commute; and K is the anti-unitary operator corresponding to complex conjugation. Then

$$M(\tilde{T})KM(U_\theta) = e^{i\theta/2} M(U_\theta) M(\tilde{T})K \quad (7)$$

and $M(\tilde{T})KM(\tilde{T})K = 1$. This is a projective representation as the phase in Eq. 7 cannot be removed by adding any phase factor to $M(U_\theta)$.

Moving to the $N = -1$ phase with ground states $|0_-\rangle$ and $|1_-\rangle$, we have the same representation for $M(U_\theta)$ while $M(\tilde{T})K = e^{-i\sigma_z \pi/2}\sigma_x K$ in this case. Eq. 7 remains true but now $M(\tilde{T})KM(\tilde{T})K = -1$ so we have a different projective representation.

In the $N = 2$ phase, our ground states are four-fold: $|0_+0_+\rangle, |0_+1_+\rangle, |1_+0_+\rangle$ and $|1_+1_+\rangle$. Here

$$M(U_\theta) = \begin{pmatrix} e^{i\theta/4} & \\ & e^{-i\theta/4} \end{pmatrix} \otimes \begin{pmatrix} e^{i\theta/4} & \\ & e^{-i\theta/4} \end{pmatrix},$$
$$M(\tilde{T})K = \sigma_x \otimes \sigma_y K. \quad (8)$$

While $M(\tilde{T})K$ and $M(U_\theta)$ commute this time, $M(\tilde{T})KM(\tilde{T})K = -1$ again making a third non-trivial projective representation.

As each non-trivial phase has a distinct non-trivial projective representation, they remain distinct phases even when interactions are strong. We can compare our results to the unperturbative bosonic classification in 1D obtained by group cohomology[4, 7, 24] as we can bosonize our fermionic model. The resulting bosonic model would have the same symmetry $U(1) \times Z_2^T$ with phases classified by $\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e. four distinct projective representations there correspond to four different strongly interacting phases. Our fermionic results similarly contain all four phases with these distinct projective representations, so this model realizes all possible non-trivial phases with strong interactions.

Modifying symmetry from $U(1)$ *to* Z_n *spin-rotation* — As our fermion model respects S_z spin-rotation and TR symmetry, it naturally contains Z_n discrete spin-rotation as well. We can replace $U(1)$ spin-rotation by Z_n spin-rotation, i.e. rotation by an arbitrary angle is now constrained to values of $\theta = 2\pi/n$ and our new symmetry group has generators time-reversal T and discrete S_z rotation $R = e^{\mathrm{i}\, S_z \frac{2\pi}{n}}$ satisfying

$$T^2 = (-)^{N_F}, \quad R^n = (-)^{N_F}, \quad TR = RT. \tag{9}$$

Here $(-)^{N_F}$ is the fermion number parity operator. When n = even, this group $G(T, Z_n)$ is generated by R and $\tilde{T} = R^{n/2}T$, so $G(T, Z_n) = Z_{2n} \times Z_2^{\tilde{T}}$. When n = odd, we find that $\tilde{R} = RT$ alone generates this group $G(T, Z_n) = Z_{4n}^T$[26].

For $n \geq 2$, no new fermion bilinear terms are allowed so the free fermion classification does not change from $\mathbb{Z}$. In the case of $n = 1$, new quadratic terms of the form $\delta H = c\hat{\psi}^\dagger_{+1}\hat{\psi}^\dagger_{+2} + \text{h.c.}$ are permitted. This term couples two chains forming the $N = 2$ phase to make the ground-state nondegenerate. The $N = 2$ phase becomes trivial and the classification for $n = 1$ reduces to $\mathbb{Z}_2$.

Similarly for higher n, we can always add interacting terms with $2n$ $\hat{\psi}_+$ operators similar to the term in the $n = 1$ case above. For $n = 2$ for instance, this term is $\delta H = c\hat{\psi}^\dagger_{+1}\hat{\psi}^\dagger_{+2}\hat{\psi}^\dagger_{+3}\hat{\psi}^\dagger_{+4} + \text{h.c.}$. Such interactions couple $2n$ zero modes each in the $N = 1$ phase to render the ground-state nondegenerate. In effect, Z_n spin-rotation symmetry allows interactions that reduce the classification to $\mathbb{Z}_{2n}$.

We had established that under $U(1)$ spin-rotation symmetry, interactions reduce the classification to $\mathbb{Z}_4$. Including more interactions as allowed by Z_n spin-rotation further reduce the classification to $\mathbb{Z}_{2n}$. Taken together, we find there is no effect on even n which remains $\mathbb{Z}_4$ since $2n$ is a multiple of 4. Odd n, however, reduces to a $\mathbb{Z}_2$ classification (as 2 becomes the largest common denominator between $2n$ and 4).

The number of non-trivial phases can be compared to and matches with the group cohomology prediction $\mathbb{Z}_2 \times \mathbb{Z}_2$ for even n and $\mathbb{Z}_2$ for odd n[26]. We find that different symmetry groups with the same free fermion classification reduce to various results (here $\mathbb{Z}_4$ or $\mathbb{Z}_2$ are examples) in the presence of interactions (summary in Table I). As verified by comparison of these phases with group cohomology, all possible strongly-interacting phases can be realized by free fermions in this model.

Lastly, we note that our classification is protected only by system symmetries of spin-rotation and TR. As shown earlier, without such symmetry a term $\delta H = c\hat{\psi}^\dagger_{0+}\hat{\psi}_{0+}$ would be permitted which renders the ground-state nondegenerate and the classification trivial ($\mathbb{Z}_1$).

Discussion — We study the SPT phases of 1D fermionic superconductors with TR and S_z spin-rotation symmetries. If fermions do not interact, their classification is given by the $\mathbb{Z}$ group; with weak interactions this reduces to a $\mathbb{Z}_4$ classification. As each of our four fermion phases have distinct projective representations, they correspond to four distinct phases by comparison with group cohomology, which predicts four and only four different gapped phases even with strong interactions.

Hence all distinct symmetric gapped phases with strong interactions are realized by non-interacting fermions in this case. Fermion parity is part of our $U(1)$ symmetry which cannot be spontaneously broken. Therefore this model does not have the fermion parity symmetry broken phases corresponding to Majorana topological modes[24]. The edge states in our 1D superconductors are described by complex fermions and it is unsurprising that our interacting classification is half of the result from Kitaev and Fidkowski's Majorana model[21–23].

We further studied the SPT phases of 1D superconductors with TR and Z_n discrete S_z spin-rotation symmetries, to find they are classified by $\mathbb{Z}_4$ when n = even and $\mathbb{Z}_2$ when n = odd. Again, as phases in our fermionic model matches with the group cohomology prediction, all gapped phases of these 1D fermionic superconductors are also realized by non-interacting fermions.

Interactions on different symmetry groups with the same free fermion classification give rise to varied results (Table I). Here perturbing from a free fermion model gives all strongly interacting phases, however in other cases such phases may not be realizable with free fermions. Lastly, the effectiveness of this method remains open especially in higher dimensions where additional tools may be needed. Further study of different symmetry groups or in higher dimensions would be worthwhile.

Towards the completion of this paper, we noted the work of A. Rosch (arXiv:1203.5541) which shows "a topological insulator made of four chains of superconducting spinless fermions characterized by four Majorana edge states can adiabatically be deformed into a trivial band insulator" via "interactions to *spinful* fermions", which has some relation to our $\mathbb{Z}_4$ classification of 1D fermionic superconducting phases with TR and S_z spin-rotation symmetries.

We thank Zheng-Xin Liu, Andrew Potter and Xie Chen for helpful discussions. This work is supported by NSF Grant No. DMR-1005541 and NSFC 11074140.

[1] Z.-C. Gu and X.-G. Wen, Phys. Rev. B **80**, 155131 (2009), arXiv:0903.1069.
[2] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **82**, 155138 (2010), arXiv:1004.3835.
[3] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. **81**, 064439 (2010), arXiv:0910.1811.
[4] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011), arXiv:1008.3745.
[5] N. Schuch, D. Perez-Garcia, and I. Cirac(2011), arXiv:1010.3732.
[6] X. Chen, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B **84**, 235141 (2011), arXiv:1106.4752.
[7] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen(2011), arXiv:1106.4772.
[8] Z.-C. Gu and X.-G. Wen(2012), arXiv:1201.2648.
[9] F. D. M. Haldane, Physics Letters A **93**, 464 (1983).
[10] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Commun. Math. Phys. **115**, 477 (1988).
[11] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005), cond-mat/0411737.
[12] B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).
[13] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005), cond-mat/0506581.
[14] J. E. Moore and L. Balents, Phys. Rev. B **75**, 121306 (2007), cond-mat/0607314.
[15] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007), cond-mat/0607699.
[16] X.-L. Qi, T. Hughes, and S.-C. Zhang, Phys. Rev. B **78**, 195424 (2008), arXiv:0802.3537.
[17] A. Kitaev, the Proceedings of the L.D.Landau Memorial Conference “Advances in Theoretical Physics”(2008), arXiv:0901.2686.
[18] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B **78**, 195125 (2008), arXiv:0803.2786.
[19] A. Altland and M. R. Zirnbauer, Phys. Rev. **55**, 1142 (1997), arXiv:cond-mat/9602137.
[20] X.-G. Wen, Phys. Rev. B **85**, 085103 (2011), arXiv:1111.6341.
[21] L. Fidkowski and A. Kitaev, Phys. Rev. B **81**, 134509 (2010), arXiv:0904.2197.
[22] A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B **83**, 075102 (2011), arXiv:1008.4346.
[23] L. Fidkowski and A. Kitaev, Phys. Rev. B **83**, 075103 (2011), arXiv:1008.4138.
[24] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **84**, 235128 (2011), arXiv:1103.3323.
[25] More accurate math terminology is “equivalence class of projective representations” where we merely write “projective representation” so as not to distract readers from the physics. This is also true in some subsequent instances, e.g. “different projective representation”..
[26] See Supplemental Material.